11. Contagious structure in projection theory

Thur March 20

Suppose that $X \subset \mathbb{F}_q$. Recall that the exceptional directions are directions θ so that $\pi_\theta(X)$ is very small. In this class we explore the algebraic structure of the set of exceptional directions. A basic example is that X is a square grid. In this case, the exceptional directions are rational numbers with small numerator / denominator. (The smaller the height of the rational number, the smaller $|\pi_\theta(X)|$ is. Notice that this set of exceptional directions has a lot of algebraic structure: the sum or product of two exceptional directions is also (pretty) exceptional. We call this contagious structure. Using combinatorial number theory, we show that for any set X, the set of exceptional directions has contagious structure. This idea builds on work of Edgar-Miller and was developed by Bourgain-Katz-Tao.

This technique will play an important role in the proof of the Bourgain-Katz-Tao projection theorem.

11.1. Contagious Structure Lemma.

Lemma 11.1. *If Z is an abelian group and $A \subset Z$, and*

$$|A - tA| \leq K|A| \text{ and } |A - t_2 A| \leq K|A|,$$

then $|A - (t_1 \cdot t_2)A| \leq K^2 A$.

Proof. Note that $|A - t_2A| \leq K|A|$ implies that $|t_1A - t_1t_2A| \leq K|A|$. Let $\bar{B} = A$, $\bar{C} = t_1t_2A$, $\bar{A} = A$ in Rusza's inequality, so

$$|t_1A||A - t_1t_2A| \leq |t_1A - A||t_1 - t_1t_2A|.$$

Thus, $|A||A - t_1t_2A| \leq K^2|A|$. □

Lemma 11.2. *If $|A + tA| \leq K|A|$ then $|A - tA| \leq K^2|A|$.*

Proof. By Rusza's inequality, $|A||A - tA| \leq |A + A||A + tA|$. By Plunnecke's inequality, $|A + A| \leq K^2|A|$. Thus, $|A||A - tA| \leq K^3|A|$. □

Lemma 11.3. *If $|A + t_1A| \leq |A|$ and $|A + t_2A| \leq K|A|$, then*

$$|A + (t_1 + t_2)A| \leq K^5|A|.$$

Proof. Note that $|A + (t_1 + t_2)A| \leq |A + t_1A + t_2A|$. By the main lemma, there exists $X_1 \subset A$ so for any C, we have $|X_1 + C + t_1A| \leq |X_1 + C|$ and there exists $X_2 \subset A$

so for any C, we have $|X_2 + C + t_1 A| \leq |X_2 + C|$.

$$\begin{aligned} |A + t_1 A + t_2 A| &\leq |x_1 + x_2 + A + t_1 A + t_2 A| \\ &\leq K|x_1 + x_2 + A + t_2 A| \\ &\leq K^2|x_1 + x_2 + A| \\ &\leq K^2|A + A + A| \\ &\leq K^5|A| \end{aligned}$$

□

Our goal today is to prove the following theorem.

Theorem 11.4. *If $A \subset \mathbb{F}_p$, $|A| = p^{s_A}$, $D \subset \mathbb{F}_p$, $|D| = p^{s_D}$, $0 < s_A, s_p < 1$. Then, there exists $\epsilon(s_A, s_D) > 0$, $\max(s_A, s_D) > 0$, $\max(|A + tA|) \geq p^{s_A + \epsilon(s_A, s_D)}$.*

Corollary 11.5. $|A + A \cdot A| \geq p^{s_A + \epsilon}$.

Now, let's recall double counting result.

Lemma 11.6. *(Double Counting)*
Suppose $X \subseteq \mathbb{F}_p^2$, and $D \subseteq \mathbb{F}_p$, then

$$\max_{t \in D} |\pi_t(X)| \gtrsim \min(|X|, |D|).$$

Note that if $s_D > s_A$, then double counting implies theorem 11.4, so the hard cases are the cases in which $0 < s_D < s_A$. Let's also recall a corollary from the previous section.

Lemma 11.7. *If $0 < s < t < 1$, then there exists $k = k(s, t)$ so if $A \subseteq \mathbb{F}_p$, $|A| = p^s$ then $|poly_k(A)| \geq p^t$.*

The proof idea is to use lemma 11.7 to increase s_D to be bigger than s_A by taking sums and products and then use the contagious structure.

Proof. By lemma 11.7, there exists $K(s_A, s_D)$ so $|poly_k(D)| > p^{s_A + r}$. By double counting there exists $u \in poly_k(D)$ so $|A + uA| > p^{s_A + r}$. But if $\max_{t \in D} |A + tA| \leq K|A|$, then the contagious structure says that

$$\max_{u \in poly_k(D)} |A + uA| \leq K^{c(k)}|A| = K^{c(s_A, s_D)}|A| = K^c p^s.$$

However, this would imply that $K^c \geq p^r$ which would imply that $p^{r/c} = p^\epsilon$ a contradiction. □

The above theorem 11.4 is a special case of the following theorem when we put $X = A \times A$.

Theorem 11.8. *(BKT)*
If $X \subseteq \mathbb{F}_p^2$, $|X| = p^{s_X}$ *with* $0 < s_X < 2$, *and* $D \subseteq \mathbb{F}_p$ *with* $|D| = p^{s_D}$ *such that* $0 < s_D$.

$$\max_{t \in D} |\pi_t(X)| \geq p^{\epsilon}|X|$$

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