12. Proof of Bourgain-Katz-Tao projection theorem

Tues Apr 1

In this section, we introduce the Balog-Szemeredi-Gowers theorem and use it to finish the proof of the Bourgain-Katz-Tao projection theorem.

The Balog-Szemeredi-Gowers theorem is an important result from additive combinatorics which has many applications.

12.1. **Proof of BKT.**

Recall the Bourgain-Katz-Tao theorem for $\mathbb{F}_p$:

Theorem 12.1 (BKT)**.** *Let $X \subset \mathbb{F}_p^2$ be a subset with size $|X| = p^{s_X}$, $0 < s_X < 2$ and $D \subset \mathbb{F}_p$ be a set of direction with $|D| = p^{s_D}$, $s_D > 0$. Then*

$$\max_{t \in D} |\pi_t(X)| \gtrsim p^{\epsilon} |X|^{1/2}$$

for $\epsilon = \epsilon(s_X, s_D) > 0$.

The same statement fails for nonprime field $\mathbb{F}_q$, as one can see by taking (X, D) to be $(\mathbb{F}_p^2, \mathbb{F}_p)$ where $q = p^a$.

Our proof will be based on Theorem 12.1 in previous lectures, which we state here for reader's convenience.

Theorem 12.2. *Let A, D be subsets of $\mathbb{F}_p$ with $|A| = p^{s_A}$, $0 < s_A < 1$ and $|D| = p^{s_D}$, $s_D > 0$. Then*

$$\max_{D} |A + tA| \geq p^{\epsilon_1} |A|$$

for $\epsilon_1 = \epsilon_1(s_A, s_D)$.

It can be viewed as a special case of BKT where X takes the special form $A \times A$. Let's try to prove BKT by contradiction using this theorem. Assume that $S_D(X) \ll p^{\epsilon}|X|^{1/2}$ for $\epsilon > 0$ to be determined. Since the size of projections are invariant under projective transformations, we may assume $0, \infty \in D$ without loss of generality.

Let $A = \pi_0 X \cup \pi_\infty X$. Then $X \subset A \times A$. By Theorem 12.1, we have $\max_{t \in D} |\pi_t(A \times A)| \gtrsim p^{\epsilon_1} |A \times A| \geq p^{\epsilon_1} |X|^{1/2}$ with $\epsilon_1 = \epsilon_1(s_X/2, s_D)$. We will win if the size of projections of $A \times A$ does not differ too much from that of X. But this is not always the case.

Consider the following enemy scenario. Here the red plots are points of X and the blue plots are some random elements we added to form $A \times A$. One can easily see that even if the size of X is comparable to the size of $A \times A$, some of their projections may still be quite different.

To be more precise, let X be the grid example $X = [N] \times [N]$. Then it has small projections along rational directions. Now choose $A = [N] \cup \tilde{A}$ to be the projection of X plus some unstructured "garbage" $\tilde{A}$ with $|\tilde{A}| = N$. For a generic choice of $\tilde{A}$ it will cause large projections along most directions. To avoid this kind

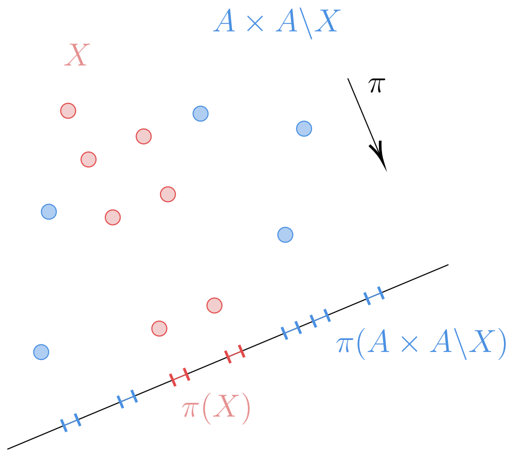

FIGURE 13. Enemy scenario

of difficulties, it is necessary to remove the "garbage" part of A. Fortunately, the BalogSzemerédiGowers theorem says this is always possible.

Theorem 12.3 (BSG). *Let X, A, B be subsets of an abelian group $(G, +)$. For $t \in G$, let $\pi_t : G \times G \to G$ denote the projection operator $(g_1, g_2) \mapsto g_1 + tg_2$. Assume that $|A|, |B| \leq N$, $X \subset A \times B$ and $|X| \geq K^{-1}N^2$, $|\pi_t X| \leq KN$ for some $t \in G$. Then exists $A' \subset A$, $B' \subset B$ such that $|X'| \gtrsim K^{-O(1)}N^2$, $\pi_t(A' \times B') \leq K^{O(1)}N$ where $X' = X \cap (A' \times B')$.*

Assumeing this theorem, it is easy to prove BKT:

Proof. (of BKT assuming BSG) Assume $S_D(X) \leq p^{\epsilon}|X|$ with $\epsilon > 0$ t.b.d., $\{0, \infty\} \subset D$. Let $A = \pi_0 X \cup \pi_\infty X$. By our previous discussion, $\max_{t \in D} |\pi_t(A \times A)| \gtrsim p^{\epsilon_1}|A \times A| \geq p^{\epsilon_1}|X|^{1/2}$ with ϵ_1 depending only on s_A, s_D. Write $|A| = N \geq |X|^{1/2}$. Then by assumption we have $N^2 \leq p^{2\epsilon}|X|$. Fix some $t_1 \in D \setminus \{0, \infty\}$ and apply BSG, we obtain $A', B' \subset A$, $X' = X \cap (A' \times B')$ such that $|X'| \geq p^{-O(\epsilon)}N^2$, $|A' + t_1 B'| \leq p^{O(\epsilon)}N$.

Since $X' \subset A' \times B'$ has small projection along one direction, we expect it to be highly structured and hence has small projections along many other directions. This is done by the following argument. Let $t \in G$. Consider the map

$$\pi_t\left(A' \times \frac{-1}{t_1}A'\right) \times X' \to (A' - A') \times (A' - t_1 B') \times \pi_t(Y)$$

given by

$$\left(a_1 - \frac{1}{t_1}a_2, (a, b)\right) \mapsto (a_1 - a, a_2 + t_1 b, a + tb).$$

This map is clearly injective. Hence

$$\left|\pi_t\left(A' \times \frac{-1}{t_1}A'\right)\right| \le \frac{|A'-A'||A'+t_1B'||\pi_t(X')|}{|X'|} \le p^{O(\epsilon)}\frac{N^2}{|X'|}|\pi_t(X')|,$$

where we used Plünnecke-Ruzsa to bound $|A'-A'|$. Therefore,

$$\max_{t\in D}|\pi_t(X')| \ge p^{-O(\epsilon)}\max_{t\in D}\frac{|X'|}{N^2}|\pi_{-t/t_1}(A'\times A')| \ge p^{\epsilon_1 - O(\epsilon)}|X|^{1/2}$$

by Theorem 12.1. A contradiction if ϵ is sufficiently small w.r.t. ϵ_1. □

12.2. **Additive energy and robust estimates.** Le A, B be finite subsets of an abelian group $(G,+)$. Define the energy

$$E(A,B) = |\{a_1, a_2 \in A, b_1, b_2 \in B : a_1 + b_1 = a_2 + b_2\}|.$$

There is a close relation between $E(A,B)$ and the size of the sumset $A+B$. Basically the energy counts the number of additive relations between A and B. Thus $E(A,B)$ must be large if $|A+B|$ is small. One may see this from the following proposition.

Proposition 12.4. *We have*

$$|A|^2|B|^2 \le |A+B|E(A,B)$$

Proof. For $z \in G$, write $r_{A,B}(z) = |\{(a,b) \in A\times B : a+b = z\}|$. Then $E(A,B) = \sum_{A+B} r_{A,B}(z)^2$. By Cauchy-Schwarz,

$$|A+B|E(A,B) \ge \left(\sum_{A+B} r_{A,B}(z)\right)^2 = (|A||B|)^2.$$

□

However, the converse if not true. Even if there are many additive relations between the elements of A, B, these subsets may still contain garbage with size comparable to the size of themselves which forces $|A+B|$ to be very large. One may exhibit this by taking $A = B \subset \mathbb{Z}$ to be $[N]\cup$ (some random subset of $\mathbb{Z}$ with size N). Instead, we have the BSG theorem for energy below.

Our previous results like Theorem 12.1 and BSG, BKT can also be formulated in terms of energy instead of cardinality. Let's record some results here without proof. The proofs use variations of the ideas we have presented. First we recall the BKT theorem that we have proven.

Theorem 12.5. *[BKT] Let A, D be subsets of $\mathbb{F}_p$ with $|A| = p^{s_A}$, $0 < s_A < 1$ and $|D| = p^{s_D}$, $s_D > 0$. Then there exists $t \in D$ so that*

$$|A + tA| = |\pi_t(A\times A)| \ge p_2^{\epsilon}|A|,$$

for $\epsilon_2 = \epsilon_2(s_A, s_D) > 0$.

Theorem 12.6 (BSG var)**.** *Let A, B be subsets of $(G, +)$ with $|A|, |B| = N$ and $E(A, B) \geq K^{-1}N^3$. There exists $A' \subset A$, $B' \subset B$ with $|A'|, |B'| \geq K^{-O(1)}N$ such that $|A' + B'| \leq K^{O(1)}N$.*

Theorem 12.7. *[BKT 2] With the same setting of BKT, there exists $t \in D$ such that for any subset $Y \subset X$ with $|Y| \geq p^{-\epsilon}|X|$, we have $|\pi_t(Y)| \geq p^{\epsilon}|X|^{1/2}$.*

Theorem 12.7 is a more robust version of Theorem 12.5. Proving more robust versions of this kind is important for applications in projection theory.

Let's remark that there are both advantages and disadvantages of working with energy. It makes the problem behaves better when passing to large subsets. But there is also a major drawback: Recall that P-R inequality yields the contagious structure of $|A + tA|$ (see Lemma 12.x with $1 \leq x \leq 3$). This is no longer the case for energy. Intuitively, to say that $E(A, B)$ is large is equivalent to say that a large part of A is "friendly" (has a lot of nontrivial additive relations) with a large part of B. Even if for each i there is a piece of A being friendly with t_iA, they are not necessarily the same for each i. Consider the example $A = [N] \cup t_1[N] \cup t_2[N]$. Then both $E(A, t_1A)$ and $E(A, t_2A)$ are large, but $E(A, (t_1 + t_2)A)$ doesn't need to be large in general.