14. Bourgain's projection theorem over $\mathbb{R}$, part 1

Tues Apr 8

Over the next three lectures, we discuss Bourgain's projection theorem over $\mathbb{R}$. Bourgain's projection theorem is analogous to the BKT projection theorem which we studied in the last four lectures, but with balls in $\mathbb{R}^2$ in place of points in $\mathbb{F}_p^2$. The proof ideas are analogous but there are some new issues in $\mathbb{R}^2$. To motivate the statement of the theorem, we begin by recalling what we learned about the finite field case.

14.1. Finite field case.

Let us first recall the BKT projection theorem over $\mathbb{F}_p$.

Theorem 14.1 (Bourgain-Katz-Tao). *Let $0 < t < 2, 0 < s \leq 1$ and p be a prime. Then there exists some $\epsilon = \epsilon(s,t) > 0$ such that for all $X \subset \mathbb{F}_p^2$ with $|X| = p^t$ and all $D \subset \mathbb{F}_p$ with $|D| = p^s$, we have*

$$\max_{\theta \in D} |\pi_\theta X| \geq p^{t/2+\epsilon}$$

and

$$\max_{\theta \in D} \min_{Y \subset X, |Y| \geq p^{-\epsilon}|X|} |\pi_\theta Y| \geq p^{t/2+\epsilon}.$$

We proved the first part of this theorem in a previous lecture and made some comments about the second part.

Remark 14.2. *Note that if we instead consider $\epsilon = 0$ and $|D| \geq 2$, then the bound becomes trivial. Indeed, for any $\theta_1 \neq \theta_2$, we have an injective map $X \to \pi_{\theta_1} X \times \pi_{\theta_2} X$, which implies* $\max_{\theta \in D} |\pi_\theta X| \geq |X|^{1/2}$.

14.2. Real case.

Now, let us consider the analogous theorem for unit balls in $\mathbb{R}^2$. Let R be some positive real number and let $X \subset B_R$ be a (not necessarily disjoint) union of unit balls. Let $D \subset [0,1]$ be a $\frac{1}{R}$-separated set, and set $\pi_\theta(x_1, x_2) = x_1 + \theta x_2$ like in the $\mathbb{F}_p$ case.

Note that without any additional assumptions, the trivial bound in Remark 14.2 does not hold in the real case. So to state Bourgain's projection theorem we will need additional assumptions on X, D.

Example 14.3.

(1) *Consider when X is a $1 \times R$ rectangle packed with unit balls. Then if we set $D = [0, R^s]$ then we get* $\max |\pi_\theta X| \sim R^s$*, so if $s < \frac{t}{2}$ then we get* $\max |\pi_\theta X| < R^{t/2}$.

(2) *Let $X = B(0, R^{1/2})$. Then $|X| \sim R$ and $|\pi_\theta X| \sim R^{1/2}$ for all θ. So in this case we do not get* $\max |\pi_\theta X| \geq R^{t/2+\epsilon}$.

Theorem 14.4 (Bourgain). *Let $0 < t < 2, 0 < s \leq 1$. Then there exist $\epsilon, \eta > 0$, both functions of s, t, such that for all X with $|X| = R^t$, D with $|D| = R^s$, if for all $x \in B_R, r \leq R, \theta \in [0,1], \rho \in [0,1]$ we have*

$$|X \cap B(x,r)| \leq R^\eta \left(\frac{r}{R}\right)^t |X|, \quad |D \cap B(\theta, \rho)| \leq R^\eta \rho^s |D|,$$

then there exists some $\theta \in D$ such that

$$\inf_{Y \subset X, |Y| \geq R^{-\eta}|X|} |\pi_\theta Y| \geq R^{t/2+\epsilon}.$$

Note that this theorem does not hold over $\mathbb{C}$. Indeed, if we take $X = B_R \cap \mathbb{R}^2$ and D the set of real directions, then we get a similar counterexample to the $\mathbb{F}_{p^2}$ case.

We would like to adopt the various inequalities we used in the $\mathbb{F}_p$ case (Ruzsa triangle inequality, Plunnecke inequality, Balog-Szemeredi-Gowers) to the real case.

Carrying out this program, many of the steps work smoothly, but there are two particular steps that require new ideas. In these notes, we will identify these two steps and describe the new issue that arises and the idea to get around it.

First we introduce a new notion of size of a set.

Definition 14.5. *Let $X \subset \mathbb{R}^d$. Then for any $\delta > 0$, the δ-**covering number** $|X|_\delta$ is the smallest number of δ-balls needed to cover X.*

We make a few observations about delta covering numbers:

- If X is 2δ-separated, then $|X|_\delta = |X|$.
- If X is a union of δ-balls, then $|X|_\delta \sim_d \delta^{-d}|X|$.
- Let $\mathcal{D}_\delta = \{\delta k + [0,\delta)^d, k \in \mathbb{Z}^d\}$. Then $|X|_\delta \sim_d |\{Q \in \mathcal{D}_\delta, Q \cap X \neq \emptyset\}|$.

In lieu of this last observation, we define

$$X^{(\delta)} = \{k : (\delta k + [0,\delta)^d) \cap X \neq \emptyset\},$$

so we have $|X|_\delta \sim |X^{(\delta)}|$.

Now, the Ruzsa triangle inequality, Plunnecke inequality, and Balog-Szemeredi-Gowers all hold for δ-covering numbers. For example, for the Ruzsa triangle inequality the statement is now

$$|B|_\delta |A - C|_\delta \lesssim |A - B|_\delta |B - C|_\delta$$

for all $A, B, C \subset \mathbb{R}^d$.

Recall the key idea for expanding sets over $\mathbb{F}_p$:

Lemma 14.6. *There exists a polynomial Q such that given $s \in (0,1)$, there exists some $\epsilon(s) > 0$ such that for all $A \subset \mathbb{F}_p$ with $|A| = p^s$, we have $|Q(A)| \geq p^{s+\epsilon}$.*

Iterating this lemma, we could obtain all of $\mathbb{F}_p$ within some polynomial of A (that depends on s). In the proof of this lemma, the key idea was to consider the set

$B = \frac{A-A}{A-A}$. If $B = \mathbb{F}_p$, we could run an argument to imply the lemma, and if $B \neq \mathbb{F}_p$, then there would be some $x \in B$ such that $x + 1 \notin B$, and we could use this x to prove the lemma. We would like to extend these ideas to the real case.

However, there are some problems with the real case. This is the first set of issues in dealing with the real case. First, B can be unbounded, as the denominator $A - A$ could be very small. Also, if A is a segment, then $A + A, A \cdot A$ are segments with $|A + A|, |A \cdot A| \sim |A|$, so we have no real growth when we take a polynomial of A. It is also not immediately clear what the equivalent of adding 1 to get from $x \in B$ to $x + 1 \notin B$ is in the real case. Finally, $\mathbb{R}$ has subgroups of uncountable size, so we need to be able to "escape" such a subgroup.

Definition 14.7. *Let $X \subset B^d(0,1), \delta \in (0,1), s \in [0,d], C \geq 1$. Then X is a $(\delta, s, C)_d$-**set** if $|X \cap B(x,r)|_\delta \leq Cr^s|X|_\delta$ for all x and all $\delta \leq r \leq 1$.*

For Bourgain's projection theorem, we will take $C = \delta^{-\eta}$.

Lemma 14.8. *There exists a polynomial Q such that given $s \in (0,1)$, there exists some $\epsilon(s) > 0$ and $\eta(s) > 0$ such that for all $A \subset [0,1]$ with $|A|_\delta = \delta^{-s}$, if A a $(\delta, s, \delta^{-\eta})$-set, then $|Q(A)|_\delta \geq \delta^{-s-\epsilon}$.*

Proof idea. Pick some $\gamma \in (0,1)$. Then set

$$B = \{\frac{a_1 - a_2}{a_3 - a_4} : a_i \in A, |a_3 - a_4| > \delta^\gamma\} \cap [0,1].$$

This γ will have to be chosen carefully to make the rest of the proof work, but we omit the details here.

Lemma 14.9. *Let $B \subset [0,1]$ be closed with $0, 1 \in B$, and let ρ be the supremum of the lengths of the segments in $[0,1] \setminus B$. Then there exists a $b \in B$ such that either $d(\frac{b}{2}, B) \geq \frac{\rho}{5}$ or $d(\frac{b+1}{2}, B) \geq \frac{\rho}{5}$.*

Proof. Let $B' = \frac{B}{2} \cup \frac{B+1}{2} \subset [0,1]$. Then it suffices to show there is an element of B' that is a distance $\frac{\rho}{5}$ away from B. Note that $\frac{1}{2} \in B'$ since $0 \in B$, so the longest segment in $[0,1] \setminus B'$ has length at most $\frac{\rho}{2}$. Now, consider an interval of length $\rho - \epsilon$ in $[0,1] \setminus B$, and consider the middle $\frac{\rho}{2}$ interval inside it. By the above this middle interval contains some point in B'. But by construction this middle interval has distance at least $\frac{\rho}{5}$ from B, which completes the proof. $\square$

Now, for $\rho \in (0,1)$, we have two cases. First, if B is ρ-dense in $[0,1]$, then we have an argument similar to the $\frac{A-A}{A-A} = \mathbb{F}_p$ case in the finite field version of this lemma. Otherwise, by the above lemma there is some $b \in B$ such that either $\frac{b}{2}, \frac{b+1}{2}$ are far from B, in which case we can run an argument similar to the case in the $\mathbb{F}_p$ version where we have $x \in B, x + 1 \notin B$. $\square$