16. Bourgain's projection theorem III

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Let us review the problem where we left off last time. Suppose that A is a (δ, s, C)-set. Lemma 14.8 tells us that there is a fixed polynomial Q so that $|Q(A)|_\delta \geq \delta^{-s\epsilon}$. We would like to iterate this lemma to prove a stronger lemma, which we now state.

Lemma 16.1. *For each $s > 0$ and each $\epsilon > 0$, there is a polynomial $P = P_{s,\epsilon}$ so that, if A is a (δ, s, C) set, then $|P(A)|_\delta \geq \delta^{-1+\epsilon}$.*

However, we cannot prove Lemma 16.1 just by iterating Lemma 14.8, because we don't yet know whether $Q(A)$ contains a $(\delta, s+\epsilon, C')$ set.

Using the uniformization lemma, we can reduce to the case that A is uniform. In this case, we know that A is a (ρ, s, C) set for all $\rho \geq \delta$. Now, by Lemma 14.8, we know that $|Q(A)|_\rho \geq \rho^{-s-\varepsilon}$ for all $\rho \geq \delta$. If we knew that $Q(A)$ was uniform, then it would follow that $Q(A)$ is a $(\delta, s+\epsilon, C')$ set with a reasonable C'. However, just because A is uniform, it does not tells us that $Q(A)$ is uniform.

The main enemy here is that $|Q(A)|_\rho$ may be large, and $|Q(A)|_\delta$ may be large, but it could still happen that there is a subset $B \subset Q(A)$ so that $|B|_\rho \ll |Q(A)|_\rho$ and yet $|Q(A) \setminus B|_\delta \ll |Q(A)|_\delta$. (It's a good exercise to draw a picture of this scenario.)

Recall that the map Q is a polynomial map from $\mathbb{R}^k$ to $\mathbb{R}$ for some k. And recall that $Q(A)$ is shorthand for $Q(A^k)$. In Lemma 14.8, we showed that the entire image $Q(A^k)$ is large: $|Q(A^k)|_\delta \geq \delta^{-s-\varepsilon}$. To deal with this technical problem, it is very helpful to have a more robust estimate.

Lemma 16.2. *There is a polynomial $Q : \mathbb{R}^k \to \mathbb{R}$ so that the following holds. If A is a (δ, s, C) set and $X \subset A^k$ with $|X|_\delta \gtrapprox |A^k|_\delta$, then $|Q(X)|_\delta \geq \delta^{-s-\varepsilon}$.*

In Subsection 16.1, we will sketch how the robust lemma, Lemma 16.2, implies Lemma 16.1. Then in Subsection 16.2, we will sketch the proof of Lemma 16.2.

In these sketches, we will deal with an important technical issue in the theory : formulating theorems in a robust way. We will see that more robust estimates are more useful – for instance because they work better in iteration. So having a more robust estimate is really useful. But on the other hand, we will see that the more robust estimate in Lemma 16.2 does not follow from simple tweaks to our previous Lemma 14.8. It requires a really new input – the Balog-Szemeredi-Gowers theorem. This part further develops the ideas from Lecture 12 where we introduced BSG.

16.1. **Why robust estimates are useful.** Let us begin on the positive side and discuss how to use Lemma 16.2. Suppose that A is uniform and A is (δ, s, C). We will use Lemma 16.2 to show that $Q(A)$ contains a $(\delta, s+\epsilon/2, C')$ set. Such a result can then be iterated to prove Lemma 16.1.

We are going to build a $(\delta, s+\epsilon/2, C)$ subset of $Q(A)$. Let us recall the definition of a (δ, s, C) set. A set S is (δ, s, C) if, for every ball $B(x,r)$ we have

$$|S \cap B(x,r)|_\delta \leq C r^s |S|_\delta.$$

We are going to build a set which is (ρ, s, C) for every $\rho \in [\delta, 1]$. So for every $\rho \in [\delta, 1]$, and every ball $B(x,r)$, out set will obey

$$|S \cap B(x,r)|_\rho \leq C r^s |S|_\rho. \tag{27}$$

Consider a sequence of scales $1 > \rho_1 > \rho_2 > ... > \rho_N = \delta$. Assume these scales are very close together.

First consider $|Q(A)|_{\rho_1}$. Since A is uniform, we know that A is (ρ_1, s, C) and so $|Q(A)|_{\rho_1} \geq \rho_1^{-s-\epsilon}$. Cover $Q(A)$ with disjoint intervals I_1 of length ρ_1. We will pick some of these intervals I_1 to include in B. Initially, we include all of them, but as we continue through the construction, we will remove bad intervals.

We pick a small parameter $\eta > 0$ with $\eta < \epsilon$.

Next we consider scale ρ_2. We know that A is (ρ_2, s, C) and so $|Q(A)|_{\rho_2} \geq \rho_2^{-s-\epsilon}$. Cover $Q(A)$ with disjoint intervals I_2 of length ρ_2. Now we notice how many intervals I_2 lie in each interval I_1. We say that an interval I_1 is bad if

$$|Q(A) \cap I_1|_{\rho_2} > \rho_1^{s+\epsilon-\eta} |Q(A)|_{\rho_2}.$$

(Notice that a bad interval I_1 is a ball $B(x,r)$ that violates (27) with $\rho = \rho_2$.)

The number of bad intervals I_1 is at most $\rho_1^{-(s+\epsilon-\eta)}$. Next define $X_{1,bad} \subset A^k$ by

$$X_{1,bad} = \{(a_1, ..., a_k) \in A^k : Q(a_1, ..., a_k) \text{ lies in a bad interval } I_1\}.$$

Our robust estimate Lemma 16.2 tells us that $|X_{1,bad}|_{\rho_1} \ll |A^k|_{\rho_1}$. Since A is uniform, this also tells us that for every $\rho \leq \rho_1$,

$$|X_{1,bad}|_\rho \ll |A^k|_\rho.$$

Define $X_1 = A^k \setminus X_{1,bad}$.

Applying Lemma 16.2, we also see that

$$|Q(X_1)|_{\rho_1} \geq \rho_1^{-s-\epsilon} \tag{28}$$

$$|Q(X_1)|_{\rho_2} \geq \rho_2^{-s-\epsilon} \tag{29}$$

Typically, we have $|Q(X_1)|_{\rho_2} \approx |Q(A)|_{\rho_2}$. We will focus on that special case in this sketch. (If $|Q(X_1)|_{\rho_2} \ll |Q(A)|_{\rho_2}$, then we redefine bad intervals and repeat the argument above.)

We claim that $Q(X_1)$ obeys (27)with dimension $s = s + \epsilon - \eta$, in the special case where r and ρ are either 1 or ρ_1 or ρ_2. There are three cases here. If $r = 1$ and $\rho = \rho_1$, (27) boils down to (28. If $r = 1$ and $\rho = \rho_2$, then (27) boils down to (29). And if $r = \rho_1$ and $\rho = \rho_2$, then (27) boils down to the definition of a good interval:

$$|Q(X_1) \cap I_1|_{\rho_2} = |Q(A) \cap I_1|_{\rho_2} \leq \rho_1^{s+\epsilon-\eta} |Q(A)|_{\rho_2} \approx \rho_1^{s+\epsilon-\eta} |Q(X_1)|_{\rho_2}.$$

Now we continue by the same method working through all the scales ρ_j. In this way, we will find a subset $X = X_N \subset A^k$ so that $Q(X_N)$ obeys (27) at all the scales r, ρ of the form ρ_j. Since these cover essentially all scales, this finishes our proof sketch that $Q(A)$ contains a $(\delta, s + \epsilon/2, C')$ set.

16.2. **How to prove robust estimates.** In this Subsection, we will outline the proof of Lemma 16.2.

We first encountered the issue of robust estimates in the proof of the Bourgain-Katz-Tao projection theorem in Lecture 12. Recall that in the previous lecture, we had proven that if $A \subset \mathbb{F}_p$ with $|A| = p^{s_A}$ and $D \subset \mathbb{F}_p$ with $|D| = p^{s_D}$ with $0 < s_A, s_D < 1$, then there exists $t \in D$ so that $|\pi_t(A \times A)| \geq p^{s_A + \epsilon}$ for $\epsilon = \epsilon(s_A, s_D) > 0$. We wanted to replace the product set $A \times A$ by a general set $X \subset \mathbb{F}_p^2$ and to prove that there exists $t \in D$ so that $|\pi_t(X)| \geq p^{\epsilon} |X|^{1/2}$. By changing variables we could assume that our direction set D included horizontal and vertical projections, and then we could reduce to the case that $X \subset A_1 \times A_2$ with $|X| \geq p^{-2\epsilon} |A_1||A_2|$. So we only needed to make our previous estimates a little more robust, extending from the case when X is an honest product $A \times A$ to the case when X is a large subset of a product $A_1 \times A_2$.

But we found that this extension was not straightforward. It required a signficant new idea. The key idea to make this extension work is the Balog-Szemeredi-Gowers theorem. The BSG theorem can be used in a similar way in the proof of Lemma 16.2.

To prove the more robust estimate Lemma 16.2, we use the BSG theorem and follow some of the ideas from Lecture 12. We will ultimately prove Lemma 16.2 with $k = 3$ and with polynomial $\tilde{Q}(a_1, a_2, a_3) = a_1 + a_2 a_3$.

We sketch the steps of this argument. Each step is similar to proofs we have done in the last lectures. It is a good exercise to fill in the details of these arguments.

The first step is to prove that if A is a (δ, s, C) set, then there is an $a \in A$ so that

$$|A + aA|_{\delta} \geq \delta^{-s-\varepsilon} \tag{30}$$

By Lemma 14.8, we know that there is a polynomial Q so that $|Q(A)|_{\delta} \geq \delta^{-s-\varepsilon}$, and it's not hard to show that $Q(A)$ is a (δ, s, C) set. Using a careful double counting argument, we can then show that there exists $b \in Q(A)$ so that $|A + bA|_{\delta} \geq \delta^{-s-\epsilon}$.

And then using the contagious structure argument, based on Plunnecke-Ruzsa, we can find $a \in A$ so that $|A + aA|_\delta \geq \delta^{-s-\epsilon}$. This argument is similar to Lecture 11.

The second step is to upgrade this estimate by proving that there is some $a \in A$ so that if $X \subset A \times A$ is a large subset, then $|\pi_a(X)|_\delta \gtrapprox \delta^{-s-\epsilon}$. More precisely, we would prove that there is some $\eta > 0$ so that if $|X|_\delta \geq \delta^\eta |A \times A|_\delta$, then $|\pi_a(X)|_\delta \gtrapprox \delta^{-s-\epsilon}$. This upgrade is based on Balog-Szemeredi-Gowers and a symmetry argument, as in Lecture 12.

With just a little more work, we can prove that almost all $a \in A$ have the good property in the second step. To prove this upgrade, we set $A_{good} \subset A$ to be the set of $a \in A$ with the good property in the second step, and we set $A_{bad} = A \setminus A_{good}$. If A_{bad} is a large subset of A, then we can get a contradiction by applying our previous results to A_{bad}.

All together, we see that if $\tilde{Q}(a_1, a_2, a_3) = a_1 + a_3 a_2$ and $\tilde{X} \subset A \times A \times A$ is a large subset, then $|\tilde{Q}(\tilde{X})|_\delta \geq \delta^{-s-\epsilon}$. This finishes our proof sketch for Lemma 16.2.