20. Homogenenous Dynamics I

April 29

There has been recent striking work applying projection theory to homogeneous dynamics. We will try to give a friendly introduction to the field of homogeneous dynamics and how projection theory can help understand it.

In this lecture we introduce homogeneous dynamics and then explain in a simple example how projection theory connects to dynamics. In the next lecture, we flesh out this simple example. After that, we give a brief survey of the recent work connecting homogeneous dynamics and projection theory.

First we introduce homogeneous dynamics. Let G be a Lie group and Γ a discrete subgroup. The space $X = G/\Gamma$ is called a homogeneous space, because the group G acts on G/Γ, and for each $x \in X$, the orbit $Gx = X$. If $H \subset G$ is a subgroup, then we can study the orbits Hx inside of X. We focus on the case that Γ has finite covolume, meaning that X has finite volume. One important example is when $G = SL_n(\mathbb{R})$ and $\Gamma = SL_n(\mathbb{Z})$. In this case, the space $X = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ parametrizes the lattices in $\mathbb{R}^n$ with unit covolume. Here we could choose H to be a lower-dimensional subgroup, such as the diagonal matrices or the upper triangular matrices. Since H has infinite volume and X has finite volume, Hx "wraps around and around inside of X". There are examples where Hx is dense. There are other examples where Hx is contained in a lower dimensional submanifold inside of X. How might Hx look in general?

In this discussion, we have to be careful about left actions and right actions. An element of G/Γ is a coset of the form $h\Gamma$ where $h \in G$. The group G acts on the left on G/Γ, so an element $g \in G$ maps the coset $h\Gamma$ to the coset $g^{-1}h\Gamma$. (The inverse here makes it a left action and is traditional, but it's not that important in our discussion.)

The simplest example is $G = SL_2(\mathbb{R})$, and $\Gamma = SL_2(\mathbb{Z})$. Let m be a right invariant metric on G, which induces a metric on G/Γ. The left action of G on G/Γ distorts the metric but it preserves the volume. Define $U = \{\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}_{t\in\mathbb{R}}\}$ and $u_t = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$ A typical problem of homogeneous dynamics is to study the orbit $U \cdot x$ in G/Γ for $x \in G/\Gamma$.

Theorem 20.1. *(Hedlund 30's)*
$U \cdot x$ is either periodic or dense.

These questions are interesting in their own right and they also have applications to other areas of math. We describe one application to number theory.

Let $Q(x_1, \cdots, x_n)$ be a quadratic form.

Question: How is $Q(\mathbb{Z}^n)$ distributed?

Conjecture 20.2. *(Oppenheim) If $n \geq 3$, the signature of Q is mixed, and the coeffiencts of Q are not contained in $\mathbb{Z}\alpha$ for any α, then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

This conjecture was proven by Margulis in the 1980s. Raghunathan observed that the Oppenheim conjecture is related to homogeneous dynamics, and the proof uses this connection. Suppose that $n = 3$, which is the hardest case. Since the signature of Q is mixed, we can assume that it has signature $(2, 1)$. Then there is a linear change of variables that converts Q to a standard quadratic form of signature $(2, 1)$, such as $Q_1(x) = x_1^2 + x_2^2 - x_3^2$. This linear change of variables converts $\mathbb{Z}^3$ to some lattice Λ, and so we have $Q(\mathbb{Z}^3) = Q_1(\Lambda)$.

The key point is that the quadratic form Q_1 has many symmetries. In particular, $SO(2; 1) \subset SL(3; \mathbb{R})$ preserves the quadratic form. Therefore, for any $h \in SO(2, 1)$, we have

$$Q(\mathbb{Z}^3) = Q_1(\Lambda) = Q_1(h\Lambda).$$

Thus we are led to study the $SO(2, 1)$-orbit of Λ in the space of lattices. The space of lattices in $\mathbb{R}^n$ is $X_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. If $SO(2, 1)\Lambda$ is dense in X_3, then $Q(\mathbb{Z}^3)$ is dense in $\cup_{\Lambda \in X_3}(Q_1(\Lambda)) = \mathbb{R}$.

Margulis showed that $SO(2, 1)\Lambda$ is dense in X_3 except for some very special lattices Λ. When $SO(2, 1)\Lambda$ is not dense in X_3, Margulis showed that it must be a lower-dimensional homogeneous space contained in X_3. In terms of the original problem, this scenario implies that the quadratic form Q has coefficients in $\mathbb{Z}\alpha$ for some $\alpha \in \mathbb{R}$.

The Lie group $SO(2, 1)$ is a 3-dimensional Lie group. It contains a 1-dimensional unipotent subgroup $U \subset SO(2, 1)$. Most of the work in the proof is to show that $U\Lambda$ is either dense or is contained in a lower-dimensional homogeneous subspace of X_3. This can be viewed as a higher dimensional generalization of Hedlund's theorem, although the proof is much more difficult and involves new ideas. Ratner extended this work to a very general theorem that applies to all G/Γ and all unipotent orbits.

In these notes, we will sketch how projection theory leads to bounds related to the geometry of the orbits $U \cdot x$ in $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. While this will not lead to a full proof of Hedlund's theorem, it will give some interesting information. Then we will discuss why it is more difficult to understand unipotent orbits in higher dimensional homogeneous spaces like $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$. Finally, we will discuss some recent work applying projection theory to help understand unipotent orbits in higher dimensions.

It's important to note that Hedlund's thoerem is special for the unipotent group U. For the subgroup D of diagonal matrices, an orbit Dx may be neither periodic nor dense. For instance, the Hausdorff dimension of the closure of Dx may be strictly between 1 and 3. It is important to understand what is special about the unipotent group. In our discussion, the special feature will be the way the unipotent group interacts with the diagonal group. We need a little notation to state this interaction.

Define $a_r = \begin{bmatrix} e^r & \\ & e^{-r} \end{bmatrix}$. After some calculation, we see that

$$a_r u_t a_r^{-1} = u e^{2r} t.$$

Denote $U_{[0,T]}x = \{u_t x\}_{t\in[0,T]}$. Note that

$$U_{[0,T]}x = a_R U_{[0,1]} a_R^{-1} x$$

where $e^{2R} = T$. Also note that if $R = Jr$, $a_R = a_r^J$.

Goal: Understand how a_r acts on unipotent orbits.

We first spend some time visualizing how a_r acts on X. Then we will use this geometric information to prove bounds about how a_r acts on unipotent orbits. For this geometric discussion, it may be useful to look at the class video on the OCW page.

We write L_g for the left action of g on G or on G/Γ. So $L_g(h) = g^{-1}h$ and $L_g(h\Gamma) = g^{-1}h\Gamma$. (The inverse is traditional and makes it a left group action, but is not too important for us.) We write R_g for the right action of g on G. So $R_g(h) = hg$.

Since the metric m is right invariant, the map $R_g : G \to G$ preserves m. However, $L_g : G \to G$ does not preserve m. The mapping $L_{a_r^{-1}}$ does not preserve m. For any $h \in G$, $L_{a_r^{-1}}$ maps T_hG to $T_{a_r^{-1}h}G$. This mapping always has singular values $e^{2r}, 1$, and e^{-2r}. The singular vectors are $v_{exp}, v_0, v_{comp} \in T_hG$. Here v_{comp} is the singular vector with singular value e^{-2r} and we call it the compressing direction.

We shall consider a tube in the fundamental domain for G/Γ. By choosing coordinates on the tube, we can identify it with $D^2 \times [0,1]$ and put coordinates x, t with $x \in D^2$ and $t \in [0,1]$. We choose the coordinates so that each vertical line $x \times [0,1]$ is a piece of a U orbit, and so that $u_t(x, t_1) = (x, t + t_1)$.

When we apply $L_{a_r^{-1}}$ to the this tube, some directions get stretched and some directions get compressed. The tangent direction to the U orbits is stretched – the tangent direction is exactly v_{exp}. So the compressing direction v_{comp} is perpendicular to the orbits. Now the key geometric point is that the compressing direction is twisting relative to the unipotent orbits. The following picture illustrates how $L_{a_r^{-1}}$ acts on slices of the tube at various heights t.

If we slice the tube at a given height t, we get a disk. The map $L_{a_r^{-1}}$ approximately smooshes this disk to an ellipse. The direction v_{comp} is the direction of the original tube which is smooshed in this process. In the picture, at $t = 1$, the direction v_{comp} is vertical and at $t = 0$ the direction is horizontal. As t goes from 0 to 1, the direction v_{comp} twists gradually.

In the picture, there are three unipotent orbits. The three dots in each disk represent where the unipotent orbit intersects that disk. So we see that at height $t = 0$, two of the orbits get smooshed close together. On the other hand, at height $t = 1$, the action of $L_{a_r^{-1}}$ does not smoosh the orbits close together. The key point is

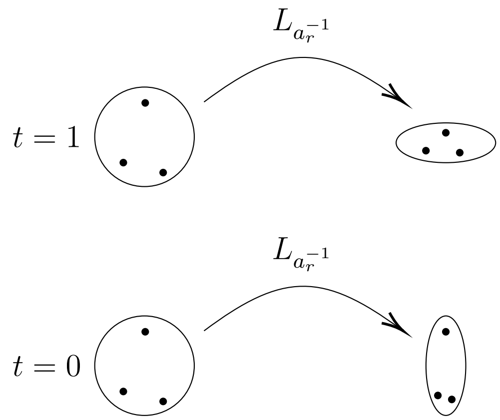

FIGURE 16. Action of $L_{a_r^{-1}}$ on slices of the tube at various heights t.

that at most heights t, the action of $L_{a_r^{-1}}$ does not smoosh the orbits together very much.

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