23. Sharp Projection Theorems II: AD Regular Case

May 8

A set E is called AD regular if the spacing of the set E behaves similarly at all scales. AD regular sets include classical fractals such as the Cantor set. Orponen and Shmerkin proved the AD regular case of the Furstenberg set conjecture. We discuss their proof and how the self similar spacing comes into play.

Recall that we want to prove the following:

Theorem 23.1 (OSRW)**.** *If $E \subset \mathbb{R}^2$ is a (δ, t, C)-set and for all $x \in E$, $\mathbb{T}_x$ is a set of δ-tubes going through X, $\mathrm{Dir}(\mathbb{T}_X)$ is a (δ, s, C)-set, with $\mathbb{T}_X$ uniform, $|\mathbb{T}_x| \sim \delta^{-s}$, and $s > 0$, then*

$$|\mathbb{T}| \geq c_\epsilon \delta^\epsilon C^{-O(1)} \min\left(\delta^{-s-t}, \delta^{-\frac{t}{2}-\frac{3s}{2}}, \delta^{-1-s}\right).$$

When δ^{-s-t} is the minimum, call this case A. When $\delta^{-\frac{t}{2}-\frac{3s}{2}}$ is the minimum, call this case B. And if δ^{-1-s} is the minimum, call this case C.

In case A, $s \geq t$ and the result follows by double counting. In case C, $s + t \geq 2$, and we can deduce the theorem using the Fourier method. This leaves case B, which is the essentially new content of this theorem.

It will be a little easier to think about things in terms of

$$R(E, \mathbb{T}) := \text{“typical number of } \delta\text{-balls of } E \text{ on a typical tube of } \mathbb{T}\text{”}.$$

More precisely,

$$R(E, \mathbb{T}) = \frac{|E|\delta^{-s}}{|\mathbb{T}|}.$$

We will be interested in the AD-regular case. Suppose E is uniform. Let $\delta = \Delta^m$ (m large). Then

$$|E \cap Q_{\Delta^j}|_{\Delta^{j+1}} \sim B_j,$$

where B_j is the branching number, for all dyadic cubes Q_{Δ^j} intersection E.

Definition 23.2. *E is (δ, t, C)-**AD-regular** if*

$$\frac{1}{C}(\Delta^J)^{-t} \leq \left|\prod_{j=1}^{J} B_j\right| \leq C(\Delta^J)^{-t}.$$

Let

$$R_{AD}(s, t, \delta, C) = \max_{E,\ \mathbb{T} \text{ obey hypotheses of theorem, } E \text{ is } (\delta,t,C)\text{-AD-regular}} R(E, \mathbb{T}).$$

We won't worry about C, so we'll just set $C = 1$. The argument works if $C \lessapprox 1$. And we'll abbreviate the above to $R_{AD}(\delta)$. Then in terms of these quantities, the theorem in the AD-regular case is

Theorem 23.3. *[OS]*

$$R_{AD}(s,t,\delta) \lessapprox \max\left(1, \delta^{-\frac{t}{2}}\delta^{\frac{s}{2}}, \delta^{1-t}\right).$$

The AD regular case seems like a very special case, but as we'll see, this is a very important case that is crucial to proving the theorem. Breaking into cases was an important step to proving the general theorem.

The AD regular case is special because it interacts in a very nice way with multiscale arguments. This gives us special tools for studying the AD regular case. If E is an AD regular set, of dimension t then if we take $E \cap B(x,\rho)$ and rescale it to diameter 1, we get an AD regular set of dimension t. In contrast, if E is just a (δ, s, C) set, and if we take $E \cap B(x,\rho)$ and rescale it to diameter 1, then we can say much less about it. This feature of AD regular sets leads to the following key lemma.

Lemma 23.4 (Submultiplicative Lemma)**.** *If $\delta = \delta_1\delta_2$, $\delta_1, \delta_2 < 1$, then*

$$R_{AD}(\delta) \lessapprox R_{AD}(\delta_1)R_{AD}(\delta_2).$$

Proof Sketch. The idea is to take a set E of δ-balls and $\mathbb{T}$ of δ-tubes and thicken it to set E_1 of δ_1-balls and a set $\mathbb{T}_1$ of δ_1-tubes. We can also restrict E and $\mathbb{T}$ to a δ_1-ball and magnify it. Then we'll get a set E_2 of δ_2-balls and a set $\mathbb{T}_2$ of δ_2-tubes. Then $(E_1, \mathbb{T}_1)$ and $(E_2, \mathbb{T}_2)$ satisfy the hypotheses, and

$$\begin{aligned} R_{AD}(\delta) &\le (\text{number of } \delta_1\text{-balls in a } \delta_1\text{-tube}) \\ &\quad \cdot (\text{number of } \delta\text{-balls in a } \delta\text{-tube within one } \delta_1\text{-ball}) \\ &\le R_{AD}(\delta_1)R_{AD}(\delta_2). \end{aligned}$$

□

(1) If E and $\mathbb{T}_x$ are uniform and E is (δ, t), then E_1 is (δ_1, t). If $\mathbb{T}_x$ is (δ, s), then $\mathbb{T}_{1,x}$ is (δ_1, s). If E is AD-regular then so is E_1.
(2) Because E is AD-regular, $E \cap B_{\delta_1}$ magnifies to a set that is (δ_2, t) and AD-regular.

This is why we need to work with AD-regular sets.

Remark. This lemma is analogous to a submultiplicative lemma from decoupling theory in Fourier analysis. In both cases, multiscale analysis turns out to be very powerful. Beyond that, it's not clear to me whether the two theories are parallel.

Next we give several applications of the submultiplicative lemma and then discuss some of the ideas in the proof of Theorem 23.3.

23.1. **Brute force proof.** One can give a brute force proof of the AD-regular OS theorem. For some specific δ_0, check by brute force

$$R_{AD}(s,t,\delta_0) \le \max(1, \delta_0^{-\frac{t}{2}}\delta_0^{\frac{s}{2}}, \delta_0^{1-t})\delta_0^{-\epsilon}$$

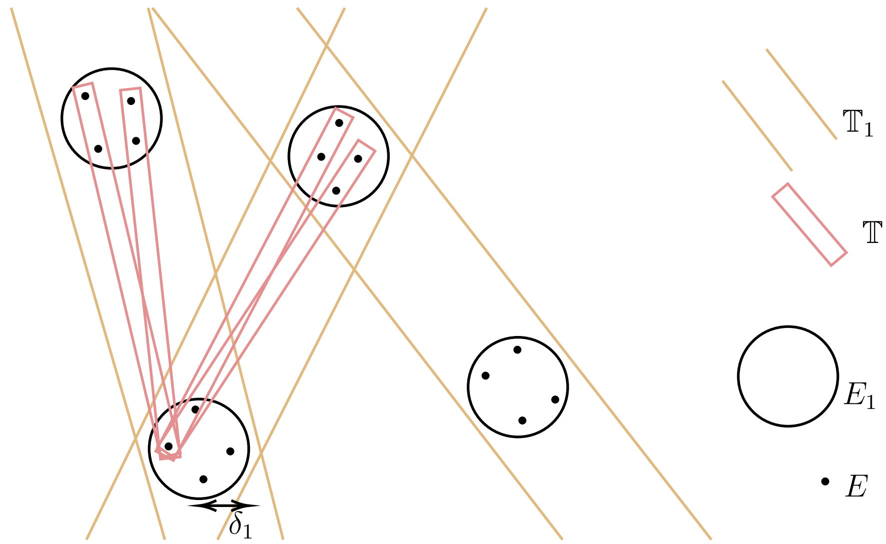

FIGURE 20. Submultiplicative lemma

for some $\epsilon > 0$. There are essentially only finitely (but a very very large number!) many cases for a fixed δ_0, so this can theoretically be check by brute force. Then we can use the submultiplicative lemma many times to get

$$R_{AD}(s, t, \delta_0^2) \leq \max(1, (\delta_0^2)^{-\frac{t}{2}}(\delta_0^2)^{\frac{s}{2}}, (\delta_0^2)^{1-t})(\delta_0^2)^{-\epsilon}$$

and so on.

On the one hand, the brute force part is completely unmanageable, and so this is not a realistic of proof. Nevertheless, it is interesting to note that in principle one can prove a nearly sharp Furstenberg estimate in the AD regular case just by using the simple submultiplicative lemma and brute force. Most deep questions in math cannot be easily reduced to a (hopelessly large) brute force computation. I think this argument, while it is impractical, still suggests that the AD regular case of Furstenberg may be especially approachable.

23.2. **General AD vs Projective AD.** Theorem 23.3 is related to projection theory but it is more general.

Definition 23.5. *We say* $(E, \mathbb{T})$ *is* ***projective*** *if* $\mathrm{Dir}(\mathbb{T}_{x_1}) = \mathrm{Dir}(\mathbb{T}_{x_2})$ *for any* $x_1, x_2 \in E$.

Let

$$R_{AD,\mathrm{proj}}(\delta) = \max_{(E,\mathbb{T}) \text{ satisfy hypotheses, } E \text{ is AD-reg, } (E,\mathbb{T}) \text{ projective}} R(E, \mathbb{T}).$$

Then clearly $R_{AD,\mathrm{proj}}(\delta) \leq R_{AD}(\delta)$.

Notice that from the proof of the submultiplicative lemma, if we let $\delta_1 = \delta_2 = \sqrt{\delta}$, then the small ball problems are all projective: We can only distinguish the angles of tubes for the small ball up to $\sim \sqrt{\delta}$, so they are the angles of the larger $\sqrt{\delta}$-tubes. So

$$\begin{aligned} R_{AD}(\delta) &\lesssim R_{AD}(\delta^{1/2}) R_{AD,\mathrm{proj}}(\delta^{1/2}) \\ &\lesssim R_{AD}(\delta^{1/4}) R_{AD,\mathrm{proj}}(\delta^{1/4}) R_{AD,\mathrm{proj}}(\delta^{1/2}) \\ &\lesssim \cdots \end{aligned}$$

So to prove the theorem, it suffices to check the projective case.

We also note that the proof of the submultiplicative lemma applies to the projective case giving

Lemma 23.6 (Submultiplicative Lemma, projective version)**.** *If $\delta = \delta_1\delta_2$, $\delta_1, \delta_2 < 1$, then*

$$R_{AD,proj}(\delta) \lessapprox R_{AD,proj}(\delta_1) R_{AD,proj}(\delta_2).$$

23.3. **Sketch of the proof for the AD regular case.** When Pablo Shmerkin was visiting me, he described to me the philosophy of the proof in a way that has stuck with me. He said, “The goal of the proof is get an ϵ-improvement to the submultiplicative lemma.”

Let us state this in a precise way. Let us write $RHS(\delta)$ for the right-hand side of Theorem 23.3, so $RHS(\delta) = \max\left(1, \delta^{-\frac{t}{2}}\delta^{\frac{s}{2}}, \delta^{1-t}\right)$.

Lemma 23.7 (ϵ-improvement to submultiplicative lemma)**.** *Fix s, t. For every $\alpha > 0$ there is some $\epsilon > 0$ so that either*

$$R_{AD,proj}(\delta^{1/2}) \lesssim \delta^{-\alpha} RHS,$$

or

$$R_{AD,proj}(\delta) \lesssim \delta^{\epsilon} R_{AD,proj}(\delta^{1/2})^2.$$

Given this lemma, a simple iteration argument shows that $R_{AD,proj}(\delta) \lessapprox RHS$.

To prove the lemma, we have to examine the situation when the submultiplicative lemma is almost sharp in the sense that

$$R_{AD,proj}(\delta) \gtrsim \delta^{\epsilon} R_{AD,proj}(\delta^{1/2})^2.$$

So what does it mean for the submultiplicative lemma to be (almost) sharp? Let's recall a little bit of the setup of the submultiplication lemma. We have E a set of δ balls and $\mathbb{T}$ a set of δ-tubes, and we want to estimate $R(E, \mathbb{T})$, they typical number of δ-balls of E in a δ-tube $T \in \mathbb{T}$. We let $\mathbb{T}_1$ be the set of $\delta^{1/2}$-tubes formed by

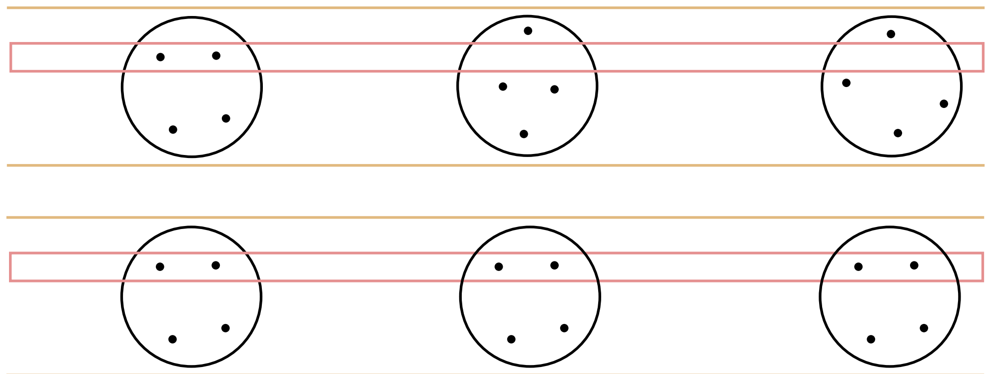

FIGURE 21. Top: Submultiplicative Lemma is Not Sharp, Bottom: Submultiplicative Lemma is sharp

thickening tubes of $\mathbb{T}$, and we let E_1 be the set of $\delta^{1/2}$-balls formed by thickening balls of E. Given the spacing conditions of E and of $\mathbb{T}_x$, we see that each tube of T intersects $\lessapprox R_{AD,proj}(\delta^{1/2})$ thick balls of E_1. And we see that the restriction of T to a ball of radius $\delta^{1/2}$ intersects at most $R_{AD,proj}(\delta^{1/2})$ δ-balls of E. This gives the submultiplicative bound $R_{AD,proj}(\delta) \lessapprox R_{AD,proj}(\delta^{1/2})^2$. If the argument is tight, then each step must be tight. In particular, for a typical tube T that intersects a typical ball $\tilde{B} \in E_1$, we must have $|T \cap \tilde{B} \cap E|_\delta \sim R_{AD,proj}(\delta^{1/2}$.

So in the two pictures below, E must resemble the bottom picture in the following figure.

In this picture, you may see a hint of a product structure. We're going to make this precise. Let $T_1 \in \mathbb{T}_1$ be a $\delta^{1/2}$ tube. We are going to study $E \cap T_1$. Choose coordinates so that T_1 is described by $0 < x_2 < \delta^{1/2}$, $0 < x_1 < 1$. Let A be the projection of $E \cap T_1$ on the x_2 axis and let B be the projection of $E_1 \cap T_1$ on the x_1 axis. Now we see that $E \cap T_1 \subset A \times B$.

The set $A \times B$ is a union of horizontal rectangles of dimensions $\delta^{1/2} \times \delta$. When the submultiplicative lemma is sharp, then a fraction $\gtrapprox 1$ of these rectangles contain $\approx R_{AD,proj}(\delta^{1/2})$ δ-balls of E. Let $X \subset A \times B$ be the union of rectangles that do contain $\approx R_{AD,proj}(\delta^{1/2})$ δ-balls of E.

Now we study the projection of $E \cap T_1$ onto almost vertical lines. Suppose that $|c| \leq \delta^{1/2}$, and let ℓ_c be the line at angle c from the x_2 axis. Let $\pi_c : \mathbb{R}^2 \to \ell_c$ be orthogonal projection. Notice that since $|c| \leq \delta^{1/2}$, we have

$$\pi_c(E) \cap B_{\delta^{1/2}} = \pi_c(E \cap T_1) = \pi_c(X).$$

We are studying the projective case of the Furstenberg set problem. So let $D \subset S^1$ be the set of directions in which we are projecting. Let $C \subset D$ be the subset of D corresponding to projections onto lines ℓ_c with $|c| \le \delta^{1/2}$ as above.

When we choose the tube T_1, we can arrange that $A = \pi_0(X)$ has typical size, and therefore we get

$$|\pi_c(X)|_\delta \lessapprox |A|_\delta \text{ for all } c \in C.$$

Because we are assuming that the submultiplicative lemma, the set X is almost a product set. Using a cousin of the Balog-Szemeredi-Gowers theorem called the asymmetric BSG theorem, it is possible to reduce to the case that X is a product set, $X = A \times B$. Now we have

$$|A + cB|_\delta \lessapprox |A|_\delta \text{ for all } c \in C.$$

At this point, we can use Plunnecke-Ruzsa to get stronger inequalities of the form

$$|A + c_1 B + c_2 B + c_3 B|_\delta \lessapprox |A|_\delta \text{ for all } c \in C.$$

The full details of this argument are somewhat complicated, and we do not give them here. First one needs to determine the spacing properties of A, B, C. To discuss this, it is convenient to first change coordinates. The set A is a set of δ-intervals inside of $B(\delta^{1/2})$. It is natural to rescale A to a set of $\delta^{1/2}$ intervals inside $[0, 1]$. Similarly, we can rescale C to a set of $\delta^{1/2}$-intervals inside $[0, 1]$. Let us set $\rho = \delta^{1/2}$. After rescaling, we have that $|A + cB|_\rho \lessapprox |A|_\rho$ for all $c \in C$.

The spacing properties of A, B, C fall into different cases. The most interesting case is when

- A is a (ρ, a)-set with $|A| \sim \rho^{-a}$.
- B is a (ρ, b)-set with $|B| \sim \rho^{-b}$.
- C is a (ρ, c)-set with $|C| \sim \rho^{-c}$.
- For any $c \in C$, $|A + cB|_\rho \lessapprox |A|_\rho$.

Orponen-Shmerkin formulated and proved a projection estimate called the ABC sum product estimate.

Theorem 23.8. *(ABC sum product theorem, Orponen-Shmerkin)*
Under the hypotheses in the bullet points above, $a \ge b + c$.

This theorem is sharp: if $a = b + c$ there is a natural example that satisfies the hypotheses above, given by

$$A = [0, 1] \cap \delta^a \mathbb{Z},$$

$$B = [0, 1] \cap \delta^b \mathbb{Z},$$

$$C = [0, 1] \cap \delta^c \mathbb{Z}.$$

Using the ABC sum product theorem and some computation, Orponen-Shmerkin check that $E, \mathbb{T}$ must obey the conclusion of the Furstenberg set conjecture.

We will not prove the ABC sum product theorem here, but we make a few comments about it.

The proof of the ABC sum product theorem is based on two key inputs. One key input is the continuum Beck theorem from the last lecture. The ABC sum product theorem would be false over $\mathbb{C}$. Orponen-Shmerkin reduce it to continuum Beck theorem, our first example of a sharp projection theorem distinguishing $\mathbb{R}$ from $\mathbb{C}$. The second key input is from additive combinatorics. The setup of the ABC sum product theorem involves sum sets, and so Plunnecke-Ruzsa and other tools from additive combinatorics naturally come into play, as we hinted above. These tools give us a lot of leverage, and they allow the reduction from ABC sum product to continuum Beck.

The ABC sum product theorem can be considered as a special case of the Furstenberg set conjecture. (The Furstenberg set conjecture directly implies the ABC sum product theorem.) But it is a special case with extra structure, especially the product structure, which makes it more accessible to tools from additive combinatorics. The ABC sum product theorem has an analogue over prime fields, and the finite field analogue has a short proof using additive combinatorics, even though the analogue of the Furstenberg set conjecture over prime fields remains open.

To finish, let us summarize the ideas we have discussed about the AD regular case.

- In the AD regular case, we have the submultiplicative lemma.
- The submultiplicative lemma allows us to reduce to the AD regular projection case.
- In a worst case example, the submultiplicative lemma must be sharp, and this forces E to have some product structure.
- This product structure lets us use tools from additive combinatorics like Plunnecke-Ruzsa.
- With these tools, Orponen-Shmerkin reduce the problem to the continuum Beck theorem.
- As we discussed in the last lecture, the continuum Beck theorem reduces to the Orponen-Shmerkin projection theorem, an ϵ-improvement on a simple double counting argument. And this theorem in turn reduces to the Bourgain projection theorem.

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