24. Sharp Projection Theorems III: Combining different scales

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Last lecture, we discussed some of the ideas in the proof of the AD regular case of the Furstenberg conjecture by Orponen-Shmerkin.

Building on their work, Ren and Wang proved the full Furstenberg conjecture. They used the AD regular case as a black box. The rest of the proof depends on two ideas, which we will explore in this lecture.

- Using a Fourier method in the well-spaced case.
- Combining different scales.

24.1. **Well spaced case.** For the well-spaced case, we want to have some sort of Geometric Measure Theory version of the SzemerediTrotter (theorem **??**). Let us remind ourselves what the classic theorem looks like in our setting.

Theorem 24.1. *If $E \subset \mathbb{R}^2$ is a set of N points and $L_R(E)$ the set of R-rich lines, then*

$$|L_R(E)| \lesssim \frac{N^2}{R^3} + \frac{N}{R}$$

Guth-Solomon-Wang proved an analogue of this theorem in the well spaced case.

Theorem 24.2 (GSW)**.** *Let $E \subset \mathbb{R}^2$ be a set of N δ-balls with $E \subset B_1$ which is well-spaced, in the sense that $|E \cap B_{N^{-1/2}}|_\delta \lesssim 1$.*
Let $\mathbb{T}_R(E)$ be a set of δ-tubes which are essentially distinct with $|T \cap E|_\delta \geq R$.
Assume also that $R > \delta^{-\varepsilon}\delta|E|_\delta$. Then

$$|\mathbb{T}_R(E)| \lessapprox \frac{N^2}{R^3}.$$

When we compare the two theorems two things stand out to us.

- First we no longer have a $\frac{N}{R}$ term. In Szemeredi-Trotter, the N/R term dominates only when $R > \sqrt{N}$ which isn't possible in the well-spaced case since each line intersects roughly $\sqrt{N}$ squares.
- The second difference is that we do need to assume some lower bound on R. To see why this is necessary, let us consider a random δ-tube T, then the expected number of balls on the line is

  $$\mathbb{E}[|T \cap E|_\delta] \sim \delta|E|_\delta.$$

  If R is equal to $\delta|E|_\delta$, then an average tube will be R-rich, and so $|\mathbb{T}_R(E)|$ can be comparable to the total number of essentially distinct δ-tubes (about δ^{-2}). In this regime, the theorem is not true. But if we increase R slightly, then we get the sharp bound in the theorem. It is quite remarkable that there is such a sharp phase transition once we increase R past the richness of a random tube.

Proof sketch. We will now sketch the proof of this theorem, the tools we will need are the Fourier Method, Double Counting, and Playing with different scales.

Using the Fourier method as in Lecture 4, you can prove that under the hypothesis of the theorem, we get that

$$|\mathbb{T}_R(E)| \lessapprox \delta^{-1}|E|_\delta R^{-2} = \delta^{-1} N R^{-2}.$$

(This is a good exercise on the techniques we have studied in the class.)

Now in the special case where $R = \delta^{-\varepsilon}\delta|E|_\delta$ then $\delta^{-1} \approx \frac{N}{R}$ so

$$\delta^{-1} N R^{-2} = \frac{N^2}{R^3},$$

which exactly matches the theorem. This special case is when R takes the smallest value allowed by our hypotheses. Unfortunately, this breaks down when we increase R. However, this bound gets better as we increase δ, that is if we increase the width of our tubes.

Recall that $|E|_\delta = N$. We know $\delta^{-\varepsilon}\delta|E|_\delta = \delta^{-\epsilon}\delta N \leq R \leq N^{1/2}$. We set the scale parameter ρ such that $\rho \cdot N \sim R$. This way

$$\delta < \rho = \frac{R}{N} < N^{-\frac{1}{2}}$$

We are going to study E_ρ, the ρ-neighborhood of E. Now we want to understand

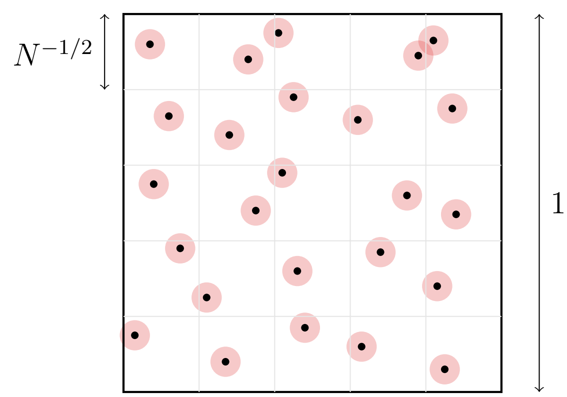


FIGURE 22. An example of a well spaced set with N points, along with its E_ρ neighborhood in red

δ-tubes that hit a lot of balls, but now that we have thickened our set, it makes sense to study thickened tubes intersecting our set. We define

$$\mathbb{T}_{\tilde{R}}(E_\rho) = \{\rho\text{-tubes } T_\rho : |T_\rho \cap E_\rho|_\rho \geq \tilde{R}\}.$$

Now we can again apply the Fourier method, and we get the bounds

$$|\mathbb{T}_{\tilde{R}}(E_\rho)| \lesssim \rho^{-1}|E|_\delta \tilde{R}^{-2} = \frac{N^2}{R \cdot \tilde{R}^2}.$$

where we importantly used the fact that $|E|_\rho = |E|_\delta = N$ because our set is well-spaced. In particular, if we pick $\tilde{R} = R$ then

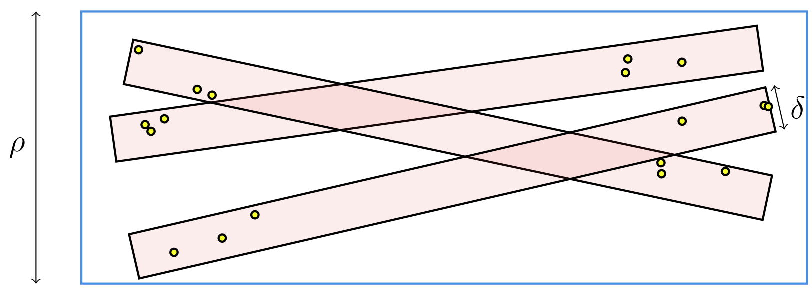
$$|\mathbb{T}_R(E_\rho)| \lesssim \frac{N^2}{R^3}.$$

Now one might think that we are now done, but this isn't quite the case. Recall that originally we want to count thin δ-tubes, where as this rescaling result gives us a bound for thick ρ-tubes. While each δ-tube can be expanded to give a single ρ-tube, each ρ-tube can contain many δ-tubes and so we are not quite done yet. So we need to estimate the number of R-rich δ-tubes contained in a $\tilde{R}$-rich ρ-tube.

For a given tube ρ-tube T_ρ, we define

$$\mathbb{T}_R(E, T_\rho) = \{\delta - \text{tubes} T : |T \cap E|_\delta \geq R, T \subset T_\rho\}.$$

By using an inductive argument, we can reduce to the case that for each δ-tube $T \in \mathbb{T}$ the δ-balls in $E \cap T$ are not concentrated on one side. The tubes in the picture below obey this two ends condition.



Here is the rough idea of the inductive argument. If the balls in a typical T concentrate in a much shorter tube $T_{short} \subset T$, then we study those shorter tubes and use an induction on scale.

Using the two ends condition, we can bound the number of thin tubes in each fat tube as follows.

Lemma 24.3. *Suppose that E is a well spaced set in $B^1 \subset \mathbb{R}^2$ in the sense that $|E| \sim N$ and $|E \cap B(x, N^{-1/2}| \lesssim 1$. Suppose that $\delta \leq \rho \leq N^{-1/2}$. Suppose T_ρ is a ρ-tube with $|T_\rho \cap E|_\rho \sim \tilde{R}$, and suppose that each δ-tube $T \in \mathbb{T}_R(E, T_\rho)$ obeys the two ends condition. Then*

$$|\mathbb{T}_R(E, T_\rho)| \lesssim \frac{\tilde{R}^2}{R^2}.$$

Proof. We apply double counting to the set

$$\{(T, x_1, x_2) \in \mathbb{T}_R(E, T_\rho) \times E \times E : x_1, x_2 \in T \text{ near opposite ends }\}$$

For each $T \in \mathbb{T}_R(E, T_\rho)$ we have $\gtrsim R^2$ choices of x_1, x_2, so the cardinality is at least $|\mathbb{T}_R(E, T_\rho)|R^2$. On the other hand, given x_1, x_2 there is $\lesssim 1$ choice of T, and so the cardinality is $\lesssim \tilde{R}^2$. □

Now to solve our original problem, we can dyadically sum over $\tilde{R}$ and apply Lemma 24.3. This gives us

$$\begin{aligned} |\mathbb{T}_R(E_\delta)| &\leq \sum_{\tilde{R}>R, \text{ dyadic}} |\mathbb{T}_{\tilde{R}}(E_\rho)| \cdot |\mathbb{T}_R(E, T_\rho)| \\ &\lessapprox \sum_{\tilde{R}>R, \text{ dyadic}} \frac{N^2}{R \cdot \tilde{R}^2} \cdot \frac{\tilde{R}^2}{R^2} \\ &\lessapprox \frac{N^2}{R^3} \end{aligned}$$

□

It can be instructive to check where we used each hypothesis of the result.

- The well-spaced hypothesis was only used to control the rescaled size $|E|_\rho$ of E, and a slightly weaker version was used for the Fourier analysis.
- The lower bound on R was necessary for the Fourier analysis part. It was necessary to assume because otherwise the lower frequencies of the characteristic functions of the tubes dominate and we get a bad bound.

Another thing that is interesting is that it seems oddly coincidental that the lower bound given by simple examples matches the upper bound given by this argument. There are several proofs of Szemeredi-Trotter, but in each case it feels like something of a coincidence that the upper bounds match examples and are therefore sharp. There are many cousin problems to Szemeredi-Trotter where lines are replaced by circles or parabolas or other curves, and in most of those problems the upper and lower bounds are far from matching.

24.2. **Combining scales.** So far, we have discussed proofs for two special cases of the Furstenberg conjecture: the AD regular case and the well spaced case. Ren and Wang realized that the general conjecture can be proven by dividing the range of scales $[\delta, 1]$ into pieces, and using one of these two techniques on each piece. This multiscale argument is short and elegant and it may have other applications. It builds on multiscale arguments developed by Keleti-Shmerkin and Orponen-Shmerkin.

Before we describe it, let's recall the main theorem.

Theorem 24.4 (OSRW)**.** *Let E be a (δ, t) set in $B_1 \subset \mathbb{R}^2$ and $|E| = \delta^{-t}$.*
For every $x \in E$ let $\mathbb{T}_x$ be a (δ, s) set of tubes passing through x with $|\mathbb{T}_x| = \delta^{-s}$. Set $\mathbb{T} = \bigcup_{x\in E} \mathbb{T}_x$.
Let $R = |E \cap T|_\delta$ be the size of a typical intersection between the tubes and E.
Then $R \lessapprox \max(\underbrace{1}_{A}, \underbrace{\delta^{-s}\delta^{-\frac{t}{2}}}_{B}, \underbrace{\delta^{1-t}}_{C})$.

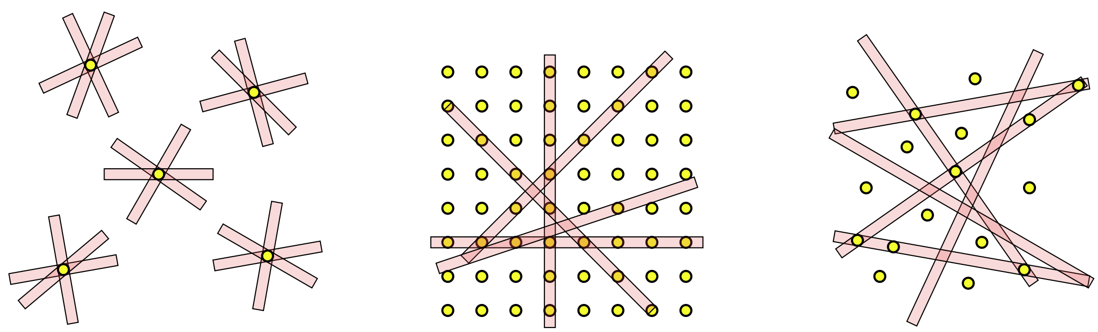


FIGURE 23. The 3 regimes of the OSRW theorem, A - unrelated balls with many tubes through each, B - an integer grid of balls, C - randomly picked tubes

[The picture C isn't quite what would be perfect. There should be many δ-balls in the picture, so many that every tube hits many δ-balls.]

We have already used many tools and techniques to prove this theorem for specific cases and regimes. Let us quickly document these.

(1) In the case where A dominates, i.e. $s \geq t$, this is true by D.C.
(2) In the case where C dominates, i.e. $s + t \geq 2$, this is true by the Fourier method.
(3) In the case where B dominates we have $s < t < 2 - s$. In this case we do not yet know if the theorem holds. However, we proved it for two special cases:
   - If E is AD-regular, we proved this last class (theorem 23.3).
   - If E is well-spaced, which we just showed.

The last idea of the proof which comes from Ren and Wang, comes in two steps. First they relax the well-spaced condition in the result we proved to a semi-well-spaced set, which we will define in a moment. Secondly they put together the known regimes by using a multiscale approach and a variant of the submultiplicative lemma we used for the AD-regular case.

Let ρ be a scale parameter with $\delta < \rho < 1$. Let us assume now that E is a general uniform set. So E_ρ is a collection of ρ-balls, and E contains about the same number of δ-balls in each ρ-ball of E_ρ.

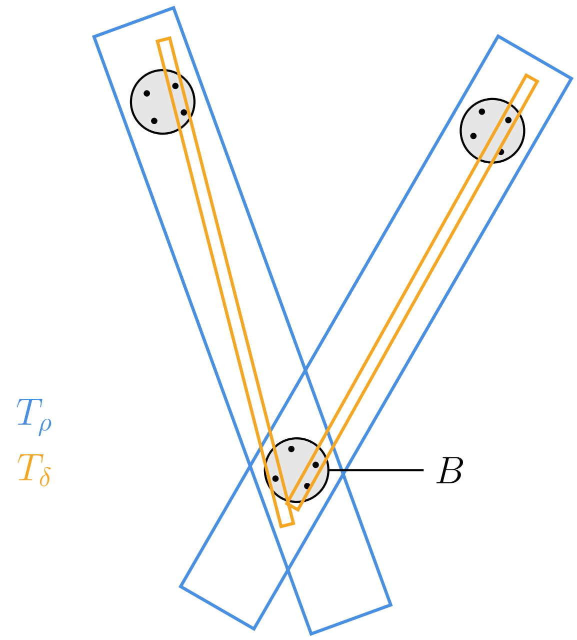


FIGURE 24. Multiscale argument

Now $R(E_\delta, \mathbb{T}_\delta)$ is the number of δ-balls of E_δ that are in T_δ. From the diagram we can compute this by first calculating the number of ρ-balls contained in T_ρ, which we will denote $R(E_\rho, \mathbb{T}_\rho)$. Then if we call one of these balls B, then for each such ball we take all the short segments of δ-tubes and see how many δ-balls each of them hits, we will call this $R(E_B, \mathbb{T}_B)$. We thus have

$$R(E_\delta, \mathbb{T}_\delta) \lessapprox R(E_\rho, \mathbb{T}_\rho) \cdot R(E_B, \mathbb{T}_B)$$

But now we can rescale B to B_1, so we will assume from now on that E_B is a set of $\frac{\delta}{\rho}$-balls, and $\mathbb{T}_B$ is a set of $\frac{\delta}{\rho}$-tubes.

We started with one scale δ, and using this multi-scale argument we broke it up into two similar problems with scales ρ and $\frac{\delta}{\rho}$. We can choose ρ freely. And we can then keep doing this splitting, breaking the problem into many subproblems. We hope to arrange that we can solve each of these subproblems with the tools we have. At that point we will also hope that we can multiply the bounds together to get a sharp bound for the original problem.

Now to discuss these scaling argument we will use the language of the branching function of a uniform set. Because we are concerned with scaling we will reparametrize the function by setting $f : \log_\delta(\rho) \mapsto \log_{\frac{1}{\delta}} |E_\rho|$, with domain $[0,1]$. What do we know about f?

(1) f is trivially increasing, since $|E| = \delta^{-t}$ we have that $f(0) = 0$ and $f(1) = t$.
(2) Because E is a (δ, t) set we know that $f(x) \geq t \cdot x$ for all $x \in [0,1]$.
(3) Because we are in 2 dimensional Euclidean space, we can always cover a $C\rho$ ball with C^2 smaller ρ balls and so our function satisfies $f(x + \Delta x) \leq f(x) + 2\Delta x$, i.e. is 2-Lipschitz.

All of these properties give us a range of 'admissable' branching functions, which we can represent in the following graph.

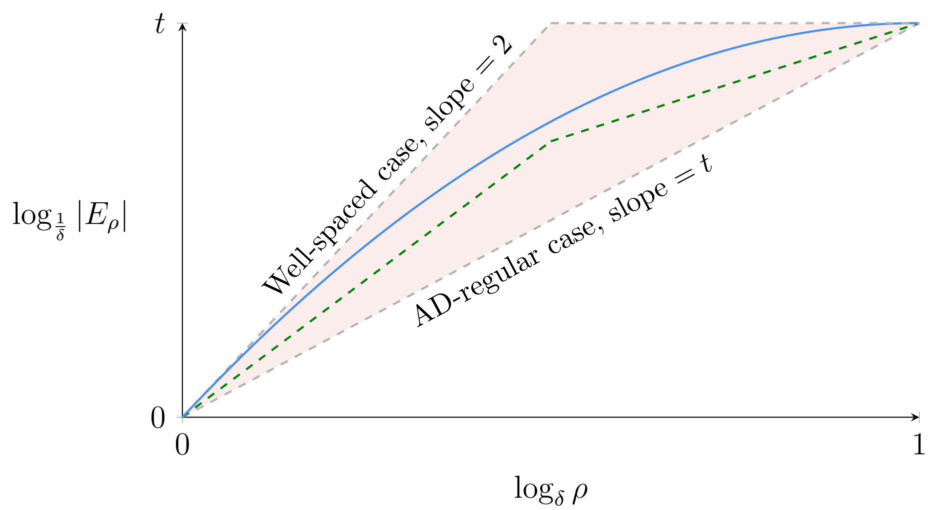
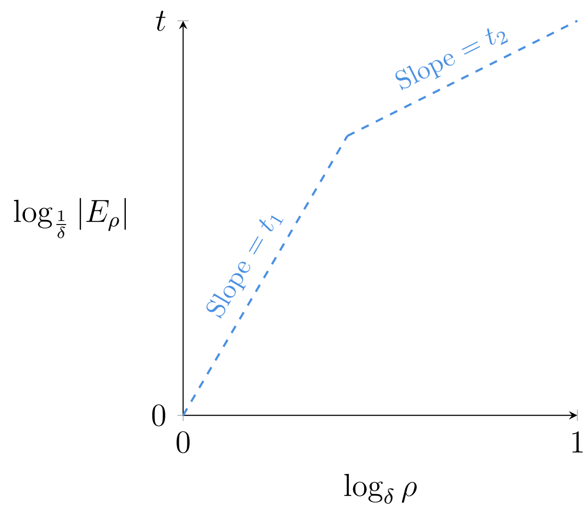


FIGURE 25. An example of a branching function (blue). The two known cases of an AD-regular set and a Well-spaced set bound an admissible region in which the function can lie (red). The semi-well-spaced case corresponds to any function lying above the green dashed line.

Using this language we can define what a semi-well-spaced set is. A well spaced set is formed on the graph with two lines of slope 2 and 0 which meet in the middle. We then slightly weaken this to have two lines of slope $2 - s$ and s. Any branching function above this new graph corresponds to a semi-well-spaced set. Ren and Wang adapted the Fourier method to prove the Furstenberg conjecture in the semi-well-spaced case.

Now how does our multiscale argument interact with this branching function? The branching function of E_ρ corresponds to the branching function of E on scales $[0, \log_\delta(\rho)]$. Similarly, if B is a ball of radius ρ, then the branching function of E_B corresponds to the branching function of E restricted to $[\log_\delta(\rho), 1]$. In essence the multiscale argument splits our branching function into two pieces which we can analyze separately.

In our graph this looks like splitting the graph into a left and a right part. The left part corresponds to the branching function of E_ρ and the right describes the branching function of E_B. Because the branching function of E_δ can be recovered from the two pieces by placing them side by side, we will call this the Concatenation method. Let us work out an explicit example.



Consider a branching function as above, by splitting at ρ corresponding to where the two lines meet, we get

$$\delta^{-t} = |E_\delta| = |E_\rho||E_B| = \rho^{-t_1} \cdot \left(\frac{\delta}{\rho}\right)^{-t_2}$$

Now let us try to estimate $R(E_\delta, \mathbb{T}_\delta)$ using this splitting. We already know that

$$R(E_\delta, \mathbb{T}_\delta) \leq R(E_\rho, \mathbb{T}_\rho) R(E_B, \mathbb{T}_B),$$

Now we have two scenarios that can happen depending on the values t_1, t_2.

- $s < t_1, t_2 < 2 - s$. In this case we can estimate both $R(E_\rho, \mathbb{T}_\rho)$ and $R(E_B, \mathbb{T}_B)$ by the B bound in the theorem. This gives us

$$R(E_\rho, \mathbb{T}_\rho) R(E_B, \mathbb{T}_B) \leq \rho^{\frac{s}{2}} \rho^{-\frac{t_1}{2}} \left(\frac{\delta}{\rho}\right)^{\frac{s}{2}} \left(\frac{\delta}{\rho}\right)^{-\frac{t_2}{2}} = \delta^{\frac{s}{2}} \delta^{-\frac{t}{2}},$$

where we used the equation for δ^{-t} we had above. This is exactly the bound we want when t is in the B regime.

- $2 - s < t_1$ and $t_2 < s$. Now when we bound $R(E_\rho, \mathbb{T}_\rho)$ we get the C bound of the theorem, and when we bound $R(E_B, \mathbb{T}_B)$ we get the A bound of the theorem. This gives us

$$R(E_\rho, \mathbb{T}_\rho) R(E_B, \mathbb{T}_B) \leq \rho^{1-t_1} \cdot 1 \gg \delta^{\frac{s}{2}} \delta^{-\frac{t}{2}}.$$

Unfortunately, in this regime, we do not get the desired bound.

What can we learn from this? We can assume from the start that $s < t < 2 - s$, so that we are in scenario B. When we split our branching function in pieces, we want each piece to be in scenario B, and we want to be able to analyze each piece. So we want each piece to be in scenario B, and we want each piece to be either AD regular or semi-well-spaced.

The last argument of the theorem is then to show that such a decomposition is always possible.

Lemma 24.5. *If $f : [0,1] \to \mathbb{R}$ is 2-Lip, increasing with $f(1) = t$, $f(x) \geq t \cdot x$ and $s < t < 2 - s$.*
Then there is a decomposition $[0,1] = \bigsqcup I$ (plus some tiny leftovers) where on each interval I either

- *f is almost linear with slope t_I, $s < t_I < 2 - s$.*
- *f is semi-well-spaced.*

We do not show the full proof here, but an interesting tool used here is the Radamacher theorem. Because our function is 2-Lipschitz our function must be differentiable almost everywhere. Thus as we split into smaller and smaller pieces, our pieces will look more and more like constant slope functions, i.e. the AD-regular case. We then use the semi-well-spaced case to get rid of the slopes that are outside our range.

This lemma was the last tool in our outline and finishes our sketch of the proof of the Furstenberg conjecture.

18.156 Projection Theory
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