[SQUEAKING]

[RUSTLING]

[CLICKING]

PETER SHOR: So, if you have a-- so, how would you go about choosing k things from n objects? What's the easy way to do it? Choose them one at a time.

So, n objects. n ways to choose first thing. n minus 1 ways to choose the second thing because once you've chosen the first thing, you can't choose again. n minus 2 ways to choose [INAUDIBLE] third thing. And then with n minus k [INAUDIBLE]

So this gives us n, n minus 1, n minus 2, through n minus k plus 1. So I guess this should be n minus k plus 1 [INAUDIBLE]. So, is this correct? No. Why not? Yeah?

STUDENT: You want to divide their order.

PETER SHOR: Yeah. It's basically because we've overcounted. I mean, if you had a set of 1, 2, 3, 4, 5, you could choose 3 and 4, but we can also choose 4 and 3, and these two ways will give us the same set.

So what we need to do is we need to divide by the number of ways of getting each of these objects, and that's k factorial because we could have gotten them in any order. So we divide by n by k factorial. So this is n choose k is equal to--

Now, there's another expression for this, which is actually maybe a little simpler [INAUDIBLE], we can multiply the top by n minus k factorial and the bottom by n minus k factorial. And we see that this is equal to n factorial over k factorial n minus k.

Now, one of the really nice techniques in proving, I guess, theorems about binomial coefficients and other kind of things is you count things in two different ways. And if you count the same set of things in two different ways, these two expressions have to be the same.

So what can we do? Well, let's count ways to choose k things. Well, either the last thing is in the set or it's not in the set. If it's in the set, we have-- in the set, we have n minus 1 things here. And the last thing is in this set, and also, I guess, k minus 1 things. And the first n minus 1 things in the set.

And if the last thing is not in the set, then we'll color it white rather than blue, and you get this here. So what's the number of ways of choosing k minus 1 things out of n minus 1 things? Yeah?

STUDENT: N minus 1 choose k works.

PETER SHOR: Yeah, exactly. So n minus 1 choose k minus 1 ways up here, and we get n minus 1 choose k ways of choosing these things here. So that shows us that n choose k is equal to n plus 1 choose k minus 1 plus n minus 1 choose k.

So, let's see. So we've got this expression by counting the number of ways of choosing k things out of n in two different ways. First, we colored one of them, and then did the-- used induction. And the other way is we just prove the binomial coefficient n choose k. So counting things in two different ways can give you a theorem. And, in fact, it can give you very deep theorems which-- as well as relatively easy theorems.

And there's this beautiful proof of Cayley's tree theorem in the notes, which depends on counting things in different ways, and I will eventually get to it, probably not until next class. So what do I want to say now? Yeah. This is-- this formula gives us Pascal's triangle here. I'm just saying 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1.

And if this row is n, so n equals 1, n equals 0, n equals 2, et cetera. And this is k equals 0, k equals 1, k equals 2. Well, you get this number from adding the number above it, and that follows from this identity.

And from Pascal's triangle, you can see that n choose k equals n choose n minus k because Pascal's triangle is symmetric. The way of getting this number is exactly the same-- the ways of computing this number is exactly the same way computing the-- way we compute this number because each row is a palindrome. But you can also see this by the formula n factorial over k factorial n minus k factorial.

OK, so the multinomial coefficients. If a plus b plus c equals n, n choose a, b, c is equal to n factorial over a factorial, b factorial, c factorial, and this is the number of ways-- this is the number of ways of choosing a-- sets of size a, b, c from n things. And these sets are labeled, but the things within the sets are-- I mean things within the sets are interchangeable.

So, and how do you prove this? Well, there's two ways of proving this. First, you can use this to-- and use a, b, c to n minus 1 choose a minus 1, b, c, n minus 1 choose a, b minus 1, c, plus n minus 1 choose a, b, c minus 1. So it's one big glick.

Because you just asked, which set was the last thing in? And then the n minus 1 do many things. You know what the size of the sets are, or you can prove it with n factorial-- I'm sorry-- n choose a, b, c is 1, n choose a times n minus a, b, c.

So here, you first choose the first set, which has size a. And now you choose the second set for the remaining n minus 1 thing-- a minus things. And that has size b. And this is equal to our formula for the binomial coefficient, n factorial, a factorial, n minus a factorial times i s.

n minus a is b plus c, so b plus c, factorial, over b times the real, c times the real. And now remember that b plus c equals n minus a. So these two terms cancel. And you get n factorial of a factorial, b factorial, c factorial. And of course, there's no reason that you would need to stop these three sets.

So, we get n choose a1, a2, choose ak is equal to n factorial over the product i jth 1 through k, a sub bar, with a1 plus k2 plus ak equals n. So this is multinomial coefficients. Oops.

So now I want to talk about bijection, which is another way of counting things. And common, of course, really like finding bijective proofs of n and a's, although these are not necessarily the easiest piece, but often they are, so bijection.

The function is a bijection f is projective, is injective and surjective. I guess, a to b, or a and b, are sets of things. Injective means f of a1 is not equal to f of a2 if a1 is not equal to a2. So you don't get mapping to things onto the same object.

And surjective is if b is in d, then there exists an a such that f of a is b. So surjective means that everything in b is in the range of f. Also, f is a bijection if and only if f has f, an inverse of fg. Now for inverse, we need f of g of a whose a and g of f of bc.

And again, we'll prove one of these directions. If f has an inverse, it is a bijection. So what we need to prove is that it's injective and then surjective.

If f of a1 equals f of A true, then g of f a1 equals g of f of k2, just because g is a function. But this has to equal a2, and this has to be a1. So this implies a1. So f is injective.

And now if b is in the range of f, f of g of b equals phi. So there's an element that maps onto b. So it's subjective. And we've proven that if f has an inverse, that is a bijection.

And the nice thing about bijections is that f is a bijection. The size of a is equal to the size of b. And so if we counted the things in a, then we will also have counted the things in b. And we will see several examples of this today.

And the first one I want to show you is another thing with binomial coefficients. Let's see. How many different ways of coloring are OK, n balls with j colors. And I should tell you what I mean by different here.

So let's do an example. n equals 4 and j is 3. Well, we could color all the same color. I want to claim that there's only one way of doing that. For each color, there's only one way of doing that. So there are three ways total.

We could color one of them one color, and the other three another color. And I want to claim there are six ways of doing this, because you choose one color for the first one and three colors for the other one. So there are three ways to choose the first color and two ways to choose the second color. So there are six ways.

So note that we're not distinguishing the coloring red, blue, blue, blue from the color of blue, red, blue, blue. So these, we're just asking for the total number of balls of each color. And you could also color two of them one color and two of them another color. And I want to claim there's three ways of doing this, because there are three colors you don't use. And once you've decided which color you're not going to use, the rest of them are given. So that's 12 so far.

And is there another way I've missed? Yeah?

STUDENT: It's like 2, 1, 1.

PETER SHOR: Yeah, 2, 1, 1. OK, so we could have red, blue, green, green. So how many ways are there of doing this? Yeah?

STUDENT: There are three ways. So just choose which color will be two.

PETER SHOR: Yeah, there are three ways. Because once you've chosen which color you're going to make two balls, then the rest is set. So there are three ways.

And if we add this up, we get 15. And 15 just happens to be equal to 5 choose 2. And just from this one example, it might be a coincidence, but you can try a whole bunch of other examples. And it always works out to be a binomial coefficient. So it turns out not to be a coincidence. It's a theorem that we'd like to prove. And 5 choose 2 is a number of ways of choosing two things out of five things.

So how can we take these colorings of balls and turn it into finding a subset of two things out of five things? And this is actually a fairly clever bijection, so I'm not going to ask you to figure it out. Oh. You'll notice I'm going to do this [INAUDIBLE].

So let's pick one of these. Maybe we should pick two of them. Let's pick red, blue, green, green. So first step, step bijection-- order balls with some fixed ordering. And in this case, can be red, blue, green.

OK, second step is on certain dividers, so we're going to put a divider between-- actually, you know what, nobody caught my mistake. 15 is 6 choose 2, and not 5 choose 2. So, because it's 6 times 5 divided by 2, that's 15. 5 choose 2 is 10. So I should warn you that I cannot do elementary arithmetic when I'm lecturing.

So the second step is to insert dividers. So what I do is we're going to take this and put dividers between the colors.

OK, now I want to claim the third step is in these colors. So now we have two dividers and six things, because we had four balls and two dividers.

And I want to claim that this is a bijection. Why? Because we can reverse it. If we know where the divisors are, what we do is we color everything before the first divider red, everything between the first and second divider blue, and everything after the second divider green. And if we don't have any red balls, we just put the first divider at the beginning.

So this can be reversed and this can be reversed. So that means this function that maps from colored balls to balls and dividers is invertible. So that means that it's a bijection. OK.

So that means that the number of-- OK, maybe I should-- ways of coloring of n balls with j colors. So, how many dividers did we use? So we have j colors, how many dividers did we use? Yeah?

STUDENT: j minus 1.

PETER SHOR: j minus 1, because we put a dividers between all the pairs of colors. And we have n plus j minus 1 things. These are the total dividers and colors. And we'll choosing j minus 1 dividers out of them.

So this shows proofs of bijection. And this is not a completely trivial one. And I want to claim that there are other ways you could have done this.

So now I want to shift gears and talk about Dyck walks, which are in Catalan numbers, which are really very nice, a very nice example. And we will see them again when we do generate functions. OK.

First, let's define. So a Dyck path is a path from coordinate 0, 0 to 2n, 0. And I guess I should call that a path of length 2n. And each step is 1, comma, plus/minus 1. And it's always above x-axis.

And further, the number of Dyck paths length 2n is c sub n, the nth Catalan number. So we get a better feel for Dyck-- yes?

STUDENT: So the first step is not legal to have, like, 1 minus 1?

PETER SHOR: The first step always goes up. Yes. And the last step always goes down, because it has to come back to 0. And if you have a Dyck path, the reverse of it is also a Dyck path.

But what I'm going to do is I'm going to give you some examples. c1 is equal to-- well, how many paths are there of length 2? Well, the first step has to go up, and the second step has to go down. So c1 is what?

Now let's figure out Dyck paths of length 2. Well, you could either go up, down, up, down, or you could go up, up, down, down. So c2 equals 2, c3 equals 5.

So we have that one. We could use this one. So we could go up twice, down twice, and one more. And we could do the same thing. And we could also go up three and down three.

And there's one I'm missing. Anyone want to tell me what it is? Yeah?

STUDENT: Up, up, down, up, down, down.

PETER SHOR: Yeah, very good. Up, down, up, down, down. So there are five Dyck paths [INAUDIBLE] three-- or six, I should say. OK, I did not write these.

And let's, this Catalan number's down here. Catalan numbers-- noted 1, 2, 5, 14, 42, 132. And we do not have to memorize this, although maybe you should memorize. I mean, we're going to be doing a lot of stuff with Catalan numbers. So maybe you should memorize the first three or four in the sequence.

So Richard Stanley was a professor at MIT before he retired. He was a combinatorist. And he came up with something like 200 things that are counted by the Catalan numbers. So there are really a lot of-- I mean, they really appear all over mathematics, and--

STUDENT: Is there a pattern relating them to the n?

PETER SHOR: Yes, there is. And we will get to that eventually. But the first thing I'm going to do is tell you two more things that are counted by Catalan numbers. OK.

So there are plane trees, and n equals 1. I'm going to say with roots. n equals 1, we have a tree. And there's only one way to draw a tree, really. n equals 2, well, there are two ways you can draw tree with two nodes. So that's 1, 2, just like that.

n equals 3, so let's see we can draw a tree. Yes. We can draw a tree that looks like this. We can draw a tree that looks like this. We can draw a tree that looks like this. And there's one there.

And you'll notice that these left and right count when you're talking about plane trees. So this is the mirror image of this, but they count as two different trees. So which leaf, which vertex something is attached to matters inflections. And the next one has 14, but we're not going to draw that. And let's move out here.

I know what I'm saying, last thing is binary trees where every node has two children. So n equals 1, there's only one binary tree on, I guess, three vertices, two leaves. n equals 2, well, we can either add 2 to either develop this node, or add, develop for this node. n equals 3, OK, let's see if I can do this without taking my nodes. OK, I have to put them on the other side.

And that is supposed to be all of them, although it's not immediately clear. But you can only get one of these by taking one of these and adding a pair of leaves to a vertex-- or a pair of leaves to a leaf. And there are only-- well, you could take care of these two.

Anything here occur, we use any of the three vertices here. But when you do it to this vertex and this vertex, you get the same tree. So this is fine. Yes?

STUDENT: Is n counting the pair of the leaves?

PETER SHOR: n is counting-- n is the number of non-leaf nodes, or 1 less than the number of leaf nodes. And you can prove that every tree has one more leaf node than interior node, because note this is true for n equals 1.

And n equals 2, you replace the leaf with an interior node and two more leaves. So you preserve the number of leaf nodes minus non-leaf nodes. So for binary trees, every node, the number of non-leaf node is one fewer than the number of leaf nodes. OK.

STUDENT: Well, wasn't n in the plane tree case the number of edges or--

PETER SHOR: n is the number of edges, yeah.

STUDENT: OK.

PETER SHOR: n plus 1 equals the number of leaves, or no-- number of vertices. And you can similarly prove that a tree always has one more vertex than edge.

OK, so we have three things. We counted five Dyck paths. And I think what I'm going to do is, first, I'm going to give you the formula for the Catalan number. And this is a really nice-- well, this is a really nice proof of the formula for Catalan number, although it's a little bit complicated.

And we will-- this is one of those proofs. It's not clear how anyone came up with that. So, c sub n is equal to 1 over n plus 1, 2n choose n. So that's the formula for the number of Catalan numbers.

So let's try it for n equals 3. n equals 3-- no, let's try for n equals 2. Sorry, for n equals 2. So 6 choose 3 as 1 equals 4. 6 minus 5 minus 4, divided by 3 times 2 times 1. And actually, there's another 4 here.

And when we cancel these out, so 4's cancel, 3's cancels the 2, and we get 5. So it works, at least for one case. And that will prove it in general. So how are we going to prove it? OK. Let p sub n be 0, be the number of paths of paths from 0, 0 to 2n, 0. So this is all paths from 0, 0 to 2n, 0.

OK. Let P sub n be a set of Dyck walks from 0, 0 to 2n. And we'll let Dn bar be P sub n minus Dn. So everything that's not a Dyck walk from 2n, 0 to 0. And what we're going to do, we will count the number of things in Dn bar.

And now to get the number of Dyck walks, we just take the number of paths and subtract Dn bar, and we get Dn. So can someone tell me what the number of paths is from 0, 0 to 2n, 0? Yeah?

STUDENT: 2n choose n.

PETER SHOR: 2n choose n, because we need n steps up and n steps down. And we don't care whether it goes below the x-axis or not. So what is Dn [INAUDIBLE]?

Well, let P minus 2n be paths from 0, 0 to 2n minus 2. And we don't care which way they go. So a path from 0, 0 to 2n minus 2 is going to have to have two more down-steps than up-steps. And so the number of Pn is equal to 2n choose n minus 1.

And I want to claim that the number of non-Dyck paths is equal to the number of paths drawn-- 0, 0 to 2n minus 2. And that will give us the Catalan number when we subtract.

And why did I show you that first? Because that's pretty great.

So a path from Pn minus 2 is equal to 2n choose n minus 1, P sub n, 0 equals 2n choose n. Pn 0 minus Pn minus 2 is equal to-- what is 2n factorial over n factorial, n factorial, minus 2n factorial over n plus 1, factorial, n minus 1 is equal to-- well, this is just 2n choose n.

And this, if we take 2n choose n, and multiply in the denominator by n plus 1, and the numerator by n, we get this thing. So that's minus n over n plus 1, 2n.

And you see, when you multiply by n plus 1 n in the n factorial, the denominator is n minus 1 factorial. When we multiply by n plus 1, the denominator turns an n factorial to an n plus 1 factorial. So to get from here to here, we multiply by n over plus 1. And this is just equal to 2n choose n, divided by n plus 1, which is the Catalan number. So what we have to do now is show that every non-Dyck path show a bijection between non-Dyck paths and paths from 2n to n from 0, 0 to 2n minus 2.

And this, I think the best way is by drawing a picture, because I'm not going to be able to describe it in words. So we have a path here. And it starts at 0 and it goes up, down, down, up, down, maybe. At some point, it has to go below the x-axis, because it has to end up at minus 2, and which is why it's not a Dyck path. This, I believe, is minus 2.

So this is in D bar. I guess D bar is fixed, if it makes any difference. What are we going to do? What we're going to do is we're going to take the line x equals minus 1-- so this is x equals 2-- and we're going to reflect this path, then, around x equals minus 1 from every point after the first time it hits minus 1 from, well, let's say, x minus 1 to the end, which would be, I guess, 2n minus 2 around x equals minus 1 axis.

OK, so let's do this set. Well, when we flipped the last thing in the path, it's going to end up on 0. So now, I want to claim this is a non-Dyck path. So why is it a non-Dyck path? Well, I mean, we have a point x on it that is x equals minus 1. So it has to be a non-Dyck path, because it goes below the x-axis.

And it is a path from 0 to 2n with every step up, either up or down. So it is in Pn plus 0. And I'm going to claim this is a bijection between non-Dyck paths and paths from 0 to 2n minus 2. P 2n.

Now why is that? Well, I mean, if you have a path from Pn, well, you have a path from 0 to 2n minus 2, you take this point, which is the first point that hits [INAUDIBLE] x equals minus 1, so we flip this around this axis and you get a unique path.

On the other hand, if you have a path from x equals 0 to x equals 2n that goes below the x-axis, you can take the first time it hits the x equals minus 1 line, and you end up this function. So it's a function, f, which has an inverse, g. So it's a bijection.

And the number of paths from x equals 0 to x equals minus 2 is indeed 2n factorial of n plus 1 factorial, n minus 1 factorial. So this shows that the number of non-Dyck paths is the right number. So that shows that the number of Dyck paths is the Catalan number. So that's our bijection-- or that's our proof that Dyck paths are counted by Catalan numbers.

So I want to say, how would you ever find this proof? And I really don't know. However, if you wait a couple of weeks, we will be doing generating functions. And generating functions is a completely straightforward way of figuring out the formula for the Catalan number, although it's actually not as nice and easy as this. It involves some calculations.

And now the next thing I want to show is a bijection between binary trees and Dyck paths. And that will show you that binary trees are also counted by Catalan numbers. Oh. OK. So how does this work? Well, let's do an example. So let's draw a tree. OK, so this is a binary tree.

And what we're going to do is we're going to traverse the nodes in that first order. Traverse nodes in that order. And what does that mean? Well, what it means, in this case, is we're going to just run around the outside of the nodes.

And we get the first node we do this one, we do a second node we visit, this is the third node we visit. I'm going to go back and go down and go up. And this is the fourth we visit. So we're only counting the first time we visit a node. This is y 6. And now we go down, back down and up here. This is the seventh node, eighth node, ninth node, tenth node up.

OK. So what do we do with these things? Well, what we're going to do is we're going to associate each of the nodes-- interior node, we'll call it plus. And a leaf, we'll call it a minus 1.

So what is this? This is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. And 1 and 2 are both pluses, because they're interior nodes. 3 is a minus, 4 is a plus, 5 is a minus, and 6 is a minus. 7 is a plus, 8 is a minus, 9 is a plus, 10 and 11 are minuses. And what we do is we use this for our Dyck path. We go up when it's plus and down when it's a minus.

So what does our Dyck path look like? Well, it starts out going up twice. And now we go down and up and down and down. And then we go up-- 1 plus, minus, minus, plus, minus, plus, minus, minus. So remember, it is always one more leaf than interior node. So this always ends up at 2n plus 1 and minus 1. And this starts at 0, 0.

But the last step has to be a leaf. So the last step has to be down. So we just ignore this last step. And the Dyck path is here. So we have to prove this is the Dyck path. And we also have to prove that every Dyck path corresponds to a tree.

So why is this a Dyck path? Well, how many up steps does it have? And how many down steps does it have? So there are 2n plus 1 leaves. I'll write it n plus 1 leaves and n internal nodes. But we're ignoring the last step. So there are n steps up and n steps down.

So there are n steps up, n steps down. And why does it never go below the x-axis? So suppose it went below the x-axis.

Look at the tree and just nodes-- well, just the nodes up to the point where it goes to minus 1. I get 2k plus 1, and minus 1. Well, this subtree must have, well, one more leaf than interior node. So it has one more leaf than interior node. And it's a complete-- [INAUDIBLE] complete because it's a full binary tree.

Because basically, if you take a tree and you remove some part of it before-- let's stop when it gets to this thing-- remove some part of it, then I want to claim this part always has at least as many leaves as interpreted. Why? Well, I mean, if you remove some part of it, there's always node degree 1, because if every node has degree 2, it's a complete binary tree and you can't add anything more to it.

So that means that if you traverse the whole tree, so there's a node of degree 1 somewhere. And that-- yeah, I mean, well, there's either an interior node of tree. Either have-- but if you stop at this node and you remove these two leaves, you either have this guy or you could stop here and notch through this leaf, then you have this guy, which just has degree 1. So if you stop early, you will always have more interior nodes than leaves. And that means you haven't actually gotten to the end yet.

So if you take a tree, then it's a Dyck path. If you take a binary tree and you stop early, you do not get to the end of this top, which has limit 2n plus 1 minus n. So that was a little bit hand-wavy, but you can go through that if you want.

And every path will give you a tree, because you just go through this and you put a plus 1-- if it's a plus 1, you put 2. Goes on, if it's a minus 1, you put leaves on. And eventually, you will get a tree put up.

I mean, because you have labeled this one, this one, and this one, well, I mean, this is a plus 1, in case you put the leaves on. Or it's a minus 1 in case [INAUDIBLE] and you end. And there's always something to do. I mean, as long as this tree does not have more interior nodes-- does not have one more than interior node, you can always add to it. And as soon as you get to the end of the path and it has one more leaf than interior node, then it's a full binary tree, and we're done.

So that is more or less what I wanted to tell you today. I also wanted to tell you what Stirling's formula was. But, well, I'll just give it to you. And we can talk about him more next time. OK, I have time.

So n factorial is equal to approximately root 2 pi n, n over e theta. And from this, you can make 2n choose n. It's approximately root 4 pi n over root 2 pi n squared. 2n over e to the 2n, over n over e to the n, and squared. Because we're just taking n factorial. We're taking this approximation for 2n factorial and dividing it by the approximation for n factorial squared.

And you can see that most of these things cancel out. This e cancels this e. This n cancels this n. And more of this cancels. And we get 2n choose n is approximately 4 to the n over square root of pi. And that tells us that c sub n is going to 1 over n plus 1, 2n choose n, is 4 to the n over root pi n to 3/2. OK. So we have an approximation. We know roughly how fast c sub n goes.

So next time I will do more examples of counting things. Any questions?

We're not going to give a proof of Stirling's formula. But I wanted to say that the only hard part about proving Stirling's formula is this square root of pi. Everything else-- I guess the square root of 2 pi-- everything else can be done with elementary calculus or elementary probability. But the root pi, but knowing that this constant, this root pi, is tricky.

OK, thanks.