

An Example of a Proof with Guiding Text

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...The last example I want to present has to do with permutations. A permutation of n is an ordering of the numbers from 1 to n . For example, a permutation of 7 is

1 4 3 7 2 6 5.

We will prove the following theorem:

Theorem 1. Every permutation of n has either an increasing subsequence or a decreasing subsequence of length $\lceil \sqrt{n} \rceil$.

Here $\lceil \cdot \rceil$ ("pronounced ceiling") means to round up to the next integer. For a permutation of 7, the theorem guarantees an increasing or a decreasing sequence of length at least $\lceil \sqrt{7} \rceil = 3$. The permutation above has an increasing subsequence 1, 4, 7 and a decreasing subsequence 4, 3, 2.

Proof. We will associate to every number in the permutation an ordered pair of two integers. The first integer associated with a number k will be the length of the longest increasing subsequence ending with k , and the second will be the length of the longest decreasing subsequence ending with k . Thus, for the subsequence above, the ordered pairs will be

$$\begin{matrix} 1 & 4 & 3 & 7 & 2 & 6 & 5 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \end{pmatrix} & \begin{pmatrix} 3 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \end{pmatrix} \end{matrix}.$$

Here, the ordered pair associated with 2 is (2, 3) because the longest increasing subsequence ending with 2 is 1, 2, which has length 2, and the longest decreasing subsequence is 4, 3, 2, which has length 3.

We claim that all these ordered pairs are distinct. First, we use the pigeonhole principle to show how this claim implies our theorem, and then we will prove this claim.

Use LaTeX theorem and proof environments to emphasize the theorem statement and to clearly indicate the beginning of the proof. If it would be helpful to readers, you may include explanatory text between the theorem statement and the proof.

A theorem statement must be precise, and should be worded generally enough so it can be applied in other contexts. For example, "The permutation above has an increasing subsequence of length 3" is not suitable because it cannot be applied out of context.

"Guiding text" is text that ensures readers know what's happening and why it's happening, e.g., by communicating the structure or purpose of the text.

Explicitly identifying the pigeons and holes helps readers see how the principle is applied.

The pigeons will be the numbers $1, 2, \dots, n$, of the permutation, and the holes will be the ordered pairs. To find a bound on the number of holes, let ℓ_i be the length of the longest increasing subsequence of our permutation, and let ℓ_d be the length of the longest decreasing subsequence. It is clear that all the ordered pairs are less than or equal to (ℓ_i, ℓ_d) , so the number of pairs is at most $\ell_i \ell_d$.

If we assume the claim that each pigeon fits into a different hole, then we have that the number of pigeons is at most the number of holes. There are n pigeons and at most $\ell_i \ell_d$ holes. This gives us

$$n \leq \ell_i \ell_d,$$

showing that either $\ell_i \geq \sqrt{n}$ or $\ell_d \geq \sqrt{n}$. Since ℓ_i and ℓ_d are integers, we can strengthen this to $\ell_i \geq \lceil \sqrt{n} \rceil$ or $\ell_d \geq \lceil \sqrt{n} \rceil$.

This is another example of helpful guiding text.

It remains to show that all the ordered pairs are distinct. Suppose that we have two numbers in our permutation s and t , with s coming before t , and let (a, b) and (c, d) be the ordered pairs associated with them.

$$\begin{array}{ccccc} \dots & s & \dots & t & \dots \\ \dots & \left(\begin{array}{c} a \\ b \end{array} \right) & \dots & \left(\begin{array}{c} c \\ d \end{array} \right) & \dots \end{array}$$

There is an increasing sequence of length a that ends with s . If $s < t$, we can add t to this increasing sequence to obtain an increasing sequence of length $a + 1$ that ends with t , showing that $c \geq a + 1$. A similar argument shows that if $s > t$, then $d \geq b + 1$. Thus, the two ordered pairs associated with s and t are distinct. This proves the claim and completes the proof of the theorem. \square

It is a fairly easy exercise to find permutations of n that have no increasing or decreasing sequences longer than $\lceil \sqrt{n} \rceil$ for any n , showing that our theorem is the strongest possible result.

When you end the proof environment with `\end{proof}`, LaTeX inserts a Halmos box to signal the end of the proof to readers.

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