18.200 HW ?
? PM RECITATION

YOUR NAME
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Exercise 1.

Theorem 1. *If F_i is the i^{th} Fibonacci number, defined by $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$ and $F_0 = F_1 = 1$, then for any $k \geq 2$,*

$$2F_k = F_{k+1} + F_{k-2}. \tag{1}$$

Note that the assumption $k \geq 2$ is needed; otherwise Equation (1) would not make sense.

Proof. We use strong induction.

We start with the inductive step. Assume that (1) holds for all k less than n. We must show that (1) holds also for $k = n$. In particular, assume that (1) is true for $k = n - 1$ and $k = n - 2$ where $n \geq 4$. We thus have the following two equations:

$$2F_{n-1} = F_n + F_{n-3}$$
$$2F_{n-2} = F_{n-1} + F_{n-4}.$$

Adding the respective sides of these equations yields

$$(2F_{n-1} + 2F_{n-2}) = (F_n + F_{n-1}) + (F_{n-3} + F_{n-4}).$$

Since the Fibonacci numbers are defined by $F_i = F_{i-1} + F_{i-2}$, we can simplify within each set of parentheses:

$$2F_n = F_{n+1} + F_{n-2}.$$

This completes the inductive step of the proof.

The base case is when $k = 2$ or $k = 3$. We verify that

$$2F_2 = 2 * 2 = 3 + 1 = F_3 + F_0$$

and

$$2F_3 = 6 = 5 + 1 = F_4 + F_1.$$

This completes this proof by induction. □

Exercise 1. Add guiding text to the following proof. Guiding text is text that guides the reader through the content by communicating the structure and/or purpose of the text to the reader. The purpose is to ensure readers understand what you're doing, why you're doing it, and why it works. Whenever possible, the relevance of steps and claims should be clear as they arise–avoid requiring readers to wait until later to see how everything fits together. See the writing resources on Canvas for examples.

As a reminder from recitation: The probabilistic method is a powerful technique whereby we prove the existence of certain objects or structures. For example, instead of constructing objects directly, if we show that a randomly chosen object has the desired property with a non-zero probability, we may conclude that it exists.[1]

Theorem 2. *Let $G = (V, E)$ be a graph with n vertices and let $d_1, \dots, d_n$ be the degrees of the vertices of G. There is an independent set of G with at least*

$$\sum_{v=1}^{n} \frac{1}{d_v + 1}$$

vertices.

Lemma 3. *Let σ be a uniformly random permutation on $[n] := \{1, \cdots, n\}$, and let $A \subseteq [n]$. Then, for any $v \in A$,*

$$\mathbb{P}[\sigma(v) = \min_{u \in A} \sigma(u)] = \frac{1}{|A|}.$$

Proof of Lemma.

The statement is trivial if $|A| = 1$.

For $|A| > 1$, let $q_v = \mathbb{P}[\sigma(v) = \min_{u \in A} \sigma(u)]$, and let $v \neq v'$ with $v, v' \in A$. Then the function f on the set of permutations which exchanges $\sigma(v)$ and $\sigma(v')$ is a bijection, so $q_v = q_{v'}$. Since $\sum_{v \in A} q_v = 1$, we derive that $q_v = \frac{1}{|A|}$. ☐

Proof of Theorem.

Let σ be a uniformly random permutation on the set V of vertices of the graph.

Let

$$S = \{v \in V | \sigma(v) < \sigma(w) \text{ for all } w \text{ with } \{v, w\} \in E\}.$$

We claim that S is an independent set in the graph G. Therefore,

$$\mathbb{E}[|S|] = \mathbb{E}\left[\sum_{v \in V} 1_{v \in S}\right] = \sum_{v \in V} \mathbb{P}[v \in S] = \sum_{v \in V} \frac{1}{d_v + 1}.$$

[1] This explanation of probabilistic method was drafted with Chat GPT3.5 and then revised for clarity, correctness, and relevance.

This implies the existence of an independent set S with $|S| \geq \sum_{v \in V} \frac{1}{d_v+1}$. □

Exercise 2.

Theorem 4. *For each n, the set $\mathcal{T}_n$ of plane trees with n edges is the same size as the set $\mathcal{D}_n$ of Dyck paths with $2n$ steps:*

$$|\mathcal{T}_n| = |\mathcal{D}_n| \text{ for all } n.$$

Proof. We define a map Φ from plane trees to Dyck paths as follows: given any tree $T \in \mathcal{T}_n$, perform a *depth-first search* of the tree T (as illustrated in Figure 1) and define $\Phi(T)$ as the sequence of up and down steps performed during the search. A Dyck path $D \in \mathcal{D}_n$ is obtained from T because $\Phi(T)$ has n up steps and n down steps (one step in each direction for each edge of T), starts and ends at level 0, and remains non-negative. Because Φ is a bijection between $\mathcal{T}_n$ and $\mathcal{D}_n$, we conclude $|\mathcal{T}_n| = |\mathcal{D}_n|$. □

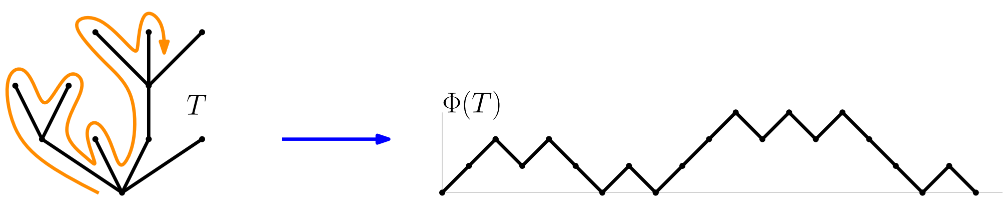

FIGURE 1. A plane tree T and the associated Dyck path $\Phi(T)$. The depth-first search of the tree is represented graphically by a tour around the tree (drawn in orange): first visit the leftmost subtree entirely, then the next subtree, etc.

Here's an example of some more complicated LaTeX formatting in case you ever want to do something like this. Feel free to adapt this code for your own use, if you like.

$$\overset{1}{\begin{pmatrix}1\\1\end{pmatrix}} \; \overset{4}{\begin{pmatrix}2\\1\end{pmatrix}} \; \overset{3}{\begin{pmatrix}2\\2\end{pmatrix}} \; \overset{7}{\begin{pmatrix}3\\1\end{pmatrix}} \; \overset{2}{\begin{pmatrix}2\\3\end{pmatrix}} \; \overset{6}{\begin{pmatrix}3\\2\end{pmatrix}} \; \overset{5}{\begin{pmatrix}3\\3\end{pmatrix}} .$$

APPENDIX A. GENERATIVE AI

Important: Before using generative AI, read the syllabus.
Which AI did you use?
Share here the prompt and the AI's response: