

## Generating Functions

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We are going to discuss one more powerful tool for enumeration or counting: generating functions. We will also see that they can be used for solving recurrences.

## 1 What is a generating function?

A generating function is a different, often compact way, of writing a sequence of numbers. Here we will be dealing mainly with sequences of numbers  $(a_n)$  which represent the number of objects of size  $n$  for an enumeration problem. The interest of this notation is that certain natural operations on generating functions lead to powerful methods for dealing with recurrences on  $a_n$ .

**Definition 1.** Let  $(a_n)_{n \geq 0}$  be a sequence of numbers. The *generating function* associated to this sequence is the series

$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Also if we consider a class  $\mathcal{A}$  of objects to be enumerated, we call *generating function* of this class the generating function

$$A(x) = \sum_{n \geq 0} a_n x^n,$$

where  $a_n$  is the number of objects of size  $n$  in the class.

Note that the variable  $x$  in generating functions doesn't stand for anything but serves as a placeholder for keeping track of the coefficients of  $x^n$ . We say that  $A(x)$  is a *formal power series*. (Note: A powerful way of using generating functions are methods that analyze the “poles” (i.e. singularities) of the generating function, but it extends beyond the scope of this class.)

**Example 1.** The generating function associated to the class of binary sequences (where the *size* of a sequence is its length) is  $A(x) = \sum_{n \geq 0} 2^n x^n$  since there are  $a_n = 2^n$  binary sequences of size  $n$ .

**Example 2.** Let  $p$  be a positive integer. The generating function associated to the sequence  $a_n = \binom{k}{n}$  for  $n \leq k$  and  $a_n = 0$  for  $n > k$  is actually a polynomial:

$$A(x) = \sum_{n \geq 0} \binom{k}{n} x^n = (1 + x)^k.$$

Here the second equality uses the binomial theorem. Thus  $A(x) = (1 + x)^k$  is the generating function of the subsets of  $\{1, 2, \dots, k\}$  (where the *size* of a subset is its number of elements).

We see on this second example that the generating function has a very simple form. In fact, simpler than the sequence itself. This is the first magic of generating functions: in many natural instances the generating function turns out to be very simple.

Let us now modify this example in connection with probabilities. Suppose we have a coin having probability  $p$  of landing on heads and a probability  $q = 1 - p$  of landing on tails. We toss it  $k$  times and denote by  $a_n$  the probability of getting exactly  $n$  heads. What is the generating function of the sequence  $(a_n)$ ? Well, it can be seen that  $a_n = \binom{k}{n} q^{k-n} p^n$  thus the generating function is

$$A(x) = \sum_{n \geq 0} \binom{k}{n} q^{k-n} p^n x^n = (q + px)^k,$$

using the binomial theorem again.

Now, observe that the generating function is

$$(q + px)(q + px)(q + px) \cdots (q + px),$$

which is just multiplying  $k$  times the generating function  $(q + px)$  corresponding to a single toss of the coin<sup>1</sup>. This is the second magic of generating functions: the generating function for complicated things can be obtained from the generating function for simple things. We will explain this in details later on.

## 2 Operations on classes and generating functions

We start with an easy observation. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are disjoint classes of objects, and  $\mathcal{C} = \mathcal{A} \uplus \mathcal{B}$  is their union (the symbol  $\uplus$  denotes disjoint union). For instance  $\mathcal{A}$  could be the set of permutations and  $\mathcal{B}$  could be the set of binary sequences. Can we express the generating function of  $\mathcal{C}$  in terms of the generating function  $A(x) = \sum_{n \geq 0} a_n x^n$  of  $\mathcal{A}$  and  $B(x) = \sum_{n \geq 0} b_n x^n$  of  $\mathcal{B}$ ? Well yes it is simply

$$C(x) = \sum_{n \geq 0} (a_n + b_n) x^n = \sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} b_n x^n = A(x) + B(x)$$

since the number  $c_n$  of objects of size  $n$  in  $\mathcal{C}$  is  $a_n + b_n$ .

We have just seen addition of generating functions, and we will now look at multiplication of generating functions. Consider the following problem. We have a die with six faces (numbered 1 to 6) and a die with eight faces (numbered 1 to 8). We roll the dice and we consider the sum of the dice. We want to know the number of ways  $c_n$  of getting each number  $n$ .

We claim that the generating function  $C(x) = \sum_{n \geq 0} c_n x^n$  is given by

$$C(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6) \times (x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8).$$

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<sup>1</sup>This is not a coincidence: if we were to expand out the product into a sum, it would be a sum of  $2^k$  terms, each of which takes either a  $q$  or a  $px$  from each of the  $k$  terms in the product. Hence each terms can be seen as a particular sequence of tails (represented by  $q$ ) and head tosses (represented by  $px$ ). In this calculation, the  $x$ 's were a device for keeping track of the number of heads.

Indeed the first part accounts for the possible outcomes of the first die and the second part accounts for the possible outcome of the second die. For instance getting the sum 5 by getting 2 from the first die and 3 from the second die is accounted by the multiplication of the monomial  $x^2$  from the first parenthesis with monomial  $x^3$  from the second parenthesis  $x^3$ , etc. Multiplying this out, we get

$$C(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 6x^8 + 6x^9 + 5x^{10} + 4x^{11} + 3x^{12} + 2x^{13} + x^{14}.$$

In the above problem we see that multiplying generating function is meaningful. Let us now try to generalize the above reasoning. Given two sets,  $\mathcal{A}$  and  $\mathcal{B}$  the *Cartesian product*  $\mathcal{A} \times \mathcal{B}$  is defined as the set of pairs  $(a, b)$  with  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . So if  $\mathcal{A}$  and  $\mathcal{B}$  are finite the cardinality of these sets are related by  $|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ . We also suppose that the *size* of a pair  $(a, b)$  is the size of  $a$  plus the size of  $b$ . The fact that we assume that the size is *additive* is crucial for what follows.

For instance, in the example above the class  $\mathcal{A}$  represents the possible numbers of the first die, so that  $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$  and the class  $\mathcal{B}$  represents the possible number of the second die, so that  $\mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . Now  $\mathcal{C} = \mathcal{A} \times \mathcal{B}$  represents the possible numbers of the two dice. The *size* of a number on the first die is just that number, so the generating function for  $\mathcal{A}$  is  $A(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$  while the generating function for  $\mathcal{B}$  is  $B(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$ . Now the *size* of a pair of number  $(a, b) \in \mathcal{C}$  is the sum of the numbers of the two dice. So we want to determine  $c_n$  which is the number of pairs  $(a, b)$ . We have claimed above that  $C(x) = A(x) \times B(x)$ . We now prove a generalization of the above relation between generating functions.

**Theorem 1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be classes of objects and let  $A(x)$  and  $B(x)$  be their generating functions. Then the class  $\mathcal{C} = \mathcal{A} \times \mathcal{B}$  has generating function  $C(x) = A(x)B(x)$ .*

*Proof.* Let  $c_n$  be the number of objects of size  $n$  in the Cartesian product  $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ . These objects  $c = (a, b)$  are obtained by picking an object  $a \in \mathcal{A}$  of size  $k \leq n$  ( $a_k$  choices) and an object  $b \in \mathcal{B}$  of size  $n - k$  ( $b_{n-k}$  choices). Thus

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Now let us consider product of generating functions

$$A(x)B(x) = \left( \sum_{k \geq 0} a_k x^k \right) \times \left( \sum_{k \geq 0} b_k x^k \right).$$

In order to get a monomial  $x^n$  in this product, one must multiply a monomial  $a_k x^k$  for  $k \leq n$  from the first sum with a monomial  $b_{n-k} x^{n-k}$  from the second sum. Thus one has

$$A(x)B(x) = \sum_{n \geq 0} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n.$$

This completes the proof. □

We denote by  $\mathcal{A}^k = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$  the set of  $k$ -tuples of elements in  $\mathcal{A}$ . Using Theorem 1 we see that this class has generating function  $A(x)^k = A(x) \times A(x) \times \cdots \times A(x)$  (where  $A(x)$  is the

generating function of  $\mathcal{A}$ ). For instance, the generating function for the sum of numbers obtained by rolling 4 dice with 6 faces is

$$C(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

Lastly we define

$$\text{Seq}(\mathcal{A}) = \cup_{k \geq 0} \mathcal{A}^k,$$

as the set of finite sequences of elements in  $\mathcal{A}$ . For instance if  $\mathcal{A} = \{0, 1\}$  then

$$\mathcal{A}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

and  $\text{Seq}(\mathcal{A})$  is the set of binary sequences. Because of Theorem 1 we see that the generating function of the class  $\mathcal{C} = \text{Seq}(\mathcal{A})$  is

$$C(x) = \sum_{k \geq 0} A(x)^k$$

where  $A(x)$  is the generating function of  $\mathcal{A}$ . Observe also that  $\sum_{k \geq 0} A(x)^k = \frac{1}{1 - A(x)}$  since

$$(1 - A(x)) \times \sum_{k \geq 0} A(x)^k = (1 - A(x)) \times (1 + A(x) + A(x)^2 + A(x)^3 + \dots) = 1.$$

For instance for the binary sequences,  $\mathcal{A} = \{0, 1\}$  has generating function  $A(x) = 2x$  ( $\mathcal{A}$  contains 2 binary sequences of length 1 and nothing else) so the class of binary sequences  $\mathcal{C} = \text{Seq}(\mathcal{A})$  has generating function

$$C(x) = \sum_{k \geq 0} A(x)^k = \sum_{k \geq 0} (2x)^k = \frac{1}{1 - 2x}.$$

We will now use these results to treat various problems.

### 3 Number of ways of giving change

Let us look at the following simple question. Suppose we have 6 pennies, 1 nickel, and 2 dimes. For what prices can we give exact change? and in how many different ways? Let  $g_n$  be the number of ways we can give the exact changes for  $n$  cents ( $g_n = 0$  if we cannot make the change), and let  $G(x) = \sum_{n \geq 0} g_n x^n$  be the generating function for this problem.

We claim that

$$G(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \times (1 + x^5) \times (1 + x^{10} + x^{20})$$

Indeed here a way of giving change is determined by a triple  $(a, b, c)$  where  $a$  is the number of pennies,  $b$  is the number of nickels,  $c$  is the number of dimes. Moreover the “size” is the total number of cents it represents. So by Theorem 1 the generating function  $G(x)$  is  $A(x)B(x)C(x)$  where  $A(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$  is the generating function of the change which can be made in pennies,  $B(x) = (1 + x^5)$  is the generating function of the change which can be made in nickels, and  $C(x) = (1 + x^{10} + x^{20})$  is the generating function of the change which can be made in dimes.

What happens when we have an arbitrary number of dimes, nickels and pennies? Well in this case the generating function for the change which can be made in pennies becomes  $A(x) = \sum_{k \geq 0} x^k = \frac{1}{1-x}$ , generating function for the change in nickels becomes  $B(x) = \sum_{k \geq 0} x^{5k} = \frac{1}{1-x^5}$ , and the generating function for the change in dimes becomes  $C(x) = \sum_{k \geq 0} x^{10k} = \frac{1}{1-x^{10}}$ . So the generating function for the number of ways of giving the change if one has infinitely many pennies, nickels, and dimes is

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})}.$$

## 4 Dots and Dashes

Now, let's think about another problem. Suppose that you are sending information using a sequence of two symbols, say dots and dashes, and suppose that sending a dash takes 2 units of times, while sending a dot take 1 unit of time. Here the *size* of a message will be defined as the number of unit times it takes. So we ask the question: how many different messages can you send in  $n$  time units? Let's call this number  $f_n$ . We'll figure out for the first few  $f_n$ . We have

n	1	2	3	4	5
$f_n$	1	2	3	5	8
messages	.	..	...	....	.....
		-	.-	..-	...-
			..-	...-	....-
				....-	.....
					.....
					.....
					.....

You may already recognize the pattern: these are the Fibonacci numbers. But let's see what we can learn about Fibonacci numbers by using generating functions. The recursion for the Fibonacci numbers is

$$f_n = f_{n-1} + f_{n-2}.$$

It's not difficult to see why this works. The first symbol must either be a dot or a dash. If the first symbol is a dash, removing it leaves a sequence two units short, and if the first symbol is a dot, removing it leaves a sequence one unit shorter. Adding up these two possibilities gives us the above recursion relation.

Now, how does this connect to generating functions? Let us define

$$F(x) = \sum_{j=0}^{\infty} f_j x^j.$$

What is  $f_0$ ? It has to be 1, in order to have  $f_2 = f_1 + f_0$ . This makes sense intuitively: there is one message, the empty message, using zero units of time. What does this recurrence say about  $F(x)$ ? Let's look at the following equations

$$\begin{array}{rcll} F(x) & = & 1 & + & x & + & 2x^2 & + & 3x^3 & + & 5x^4 & + & 8x^5 & \dots \\ xF(x) & = & & & x & + & x^2 & + & 2x^3 & + & 3x^4 & + & 5x^5 & \dots \\ x^2F(x) & = & & & & & x^2 & + & x^3 & + & 2x^4 & + & 3x^5 & \dots \end{array}$$

We can see that by multiplying by  $x$  and  $x^2$  we have shifted the terms, so that instead of  $f_k x^k$  we get  $f_{k-1} x^k$  and  $f_{k-2} x^k$ . We thus get that the equation  $f_k = f_{k-1} + f_{k-2}$  nearly corresponds to the equation  $F(x) = xF(x) + x^2 F(x)$ . This isn't quite right. All the terms  $x^k$  for  $k > 0$  do cancel, but the constant term doesn't. To make the constant term correct, we need to add 1 to the right side, obtaining the correct equation

$$F(x) = xF(x) + x^2 F(x) + 1.$$

Now, this can be rewritten as

$$F(x) = \frac{1}{1 - x - x^2}$$

At this point the perceptive reader will have observed that there was a much faster way to obtain the above result. Indeed one can observe that  $F(x)$  is the generating function of the set of sequences  $\text{Seq}(\mathcal{A})$  of the class  $\mathcal{A} = \{\text{dot}, \text{dash}\}$ . Moreover the class  $\mathcal{A}$  has generating function  $A(x) = x + x^2$  since it has one element of size 1 and one element of size 2. Therefore by the discussion in the previous section one gets  $F(x) = \frac{1}{1 - A(x)} = \frac{1}{1 - x - x^2}$ .

Now we want to use the expression of  $F(x)$  in order to obtain some information on the coefficients  $f_n$ .  $F(x)$  is a *rational function*, i.e. it is the ratio of two polynomials. When we have a rational function, we can factor the polynomial in the denominator and express the rational function as a sum of simpler rational functions. That is, we first factor the denominator

$$1 - x - x^2 = (1 - \phi_+ x)(1 - \phi_- x)$$

where  $\phi_+ = \frac{1+\sqrt{5}}{2}$  and  $\phi_- = \frac{1-\sqrt{5}}{2}$ . (Note that  $\phi_+$  and  $\phi_-$  are not the roots of  $1 - x - x^2$ , but the inverses of the roots.)

We now use the *method of partial fractions* to rewrite this as

$$F(x) = \frac{a}{1 - \phi_+ x} + \frac{b}{1 - \phi_- x}$$

for some  $a$  and  $b$ . Elementary algebra gives

$$a = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right),$$

$$b = -\frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right).$$

Now, we need to remember the Taylor series for  $1/(1 - \alpha x)$ . This is

$$\frac{1}{1 - \alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \dots$$

Even if you don't remember Taylor expansions, you should recognize this as the formula for summing a geometric series.

We thus have that, expanding each of the fractions in the expression for  $F(x)$  above in a Taylor series,

$$F(x) = a \left( 1 + \left( \frac{1+\sqrt{5}}{2} \right) x + \left( \frac{1+\sqrt{5}}{2} \right)^2 x^2 + \left( \frac{1+\sqrt{5}}{2} \right)^3 x^3 + \dots \right) \\ + b \left( 1 + \left( \frac{1-\sqrt{5}}{2} \right) x + \left( \frac{1-\sqrt{5}}{2} \right)^2 x^2 + \left( \frac{1-\sqrt{5}}{2} \right)^3 x^3 + \dots \right)$$

Substituting in the values we know for  $a$  and  $b$ , we get

$$F(x) = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right) + \left( \frac{1+\sqrt{5}}{2} \right)^2 x + \left( \frac{1+\sqrt{5}}{2} \right)^3 x^2 + \left( \frac{1+\sqrt{5}}{2} \right)^4 x^3 + \dots \right) \\ - \frac{1}{\sqrt{5}} \left( \left( \frac{1-\sqrt{5}}{2} \right) + \left( \frac{1-\sqrt{5}}{2} \right)^2 x + \left( \frac{1-\sqrt{5}}{2} \right)^3 x^2 + \left( \frac{1-\sqrt{5}}{2} \right)^4 x^3 + \dots \right)$$

Now, this gives us a nice expression for  $f_n$ , the  $n$ th Fibonacci number. We equate the coefficients of  $x^n$  on the left- and right-hand sides of this equation. Since the  $n$ th Fibonacci number  $f_n$  is the coefficient on  $x^n$  in  $F(x)$ , we get

$$f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^{n+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{n+1} \right).$$

Since  $\left| \frac{1-\sqrt{5}}{2} \right| < 1$ , we can see that the second term goes to 0 as  $n$  gets large, and  $f_n$  grows as

$$C \left( \frac{1+\sqrt{5}}{2} \right)^n$$

for  $C = \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}$ . This means that we have found the asymptotic growth rate for the number of messages that can be encoded by our dots and dashes, and that the number of bits sent per time unit is

$$\log_2 \frac{1+\sqrt{5}}{2} \sim 0.694.$$

## 5 Generalized Recurrence Equations

We have seen an example of a linear recurrence equation in the last section. Here is a general method for solving linear recurrence equations (also called linear difference equations). If you have taken 18.03, you'll notice that this method looks a lot like the method for solving linear differential equations.

Suppose we have a recurrence equation

$$f_n = \alpha f_{n-1} + \beta f_{n-2} + \gamma f_{n-3}.$$

We are only writing this equation down with three terms, but the generalization to  $k$  terms is obvious, and works exactly like you would expect. How do we solve this? What we do is to write down the generating function

$$F(x) = \sum_{j=0}^{\infty} f_j x^j.$$

Then, using the same reasoning as before, we get an equation for  $F(x)$  of the following form:

$$F(x) = \alpha x F(x) + \beta x^2 F(x) + \gamma x^3 F(x) + p(x)$$

where  $p(x)$  is a low-degree polynomial that makes this equation work for the first few elements of the sequence, where the recurrence equation doesn't necessarily work. For the Fibonacci number example above, we have  $p(x) = 1$ . Note that if we don't have a  $p(x)$  term, we get the solution  $F(x) = 0$  which, while its coefficients (all 0's) satisfy the linear recurrence equation, doesn't tell us anything useful. The maximum degree on  $p(x)$ , if the recurrence equation has three terms, is quadratic, and if the recurrence equation has  $k$  terms, is  $k - 1$ . You can see this by noticing that for the  $x^3$  term (or  $x^k$  term in the case of  $k$ -term recurrence) and later, the recurrence is guaranteed to work.

As before, we next obtain

$$F(x) = \frac{p(x)}{1 - \alpha x - \beta x^2 - \gamma x^3}.$$

Let's suppose we can factor the denominator as follows:

$$1 - \alpha x - \beta x^2 - \gamma x^3 = (1 - r_1 x)(1 - r_2 x)(1 - r_3 x).$$

What happens if you have a double or triple root is left as a homework problem. We then use the method of partial fractions (which you may remember from Calculus) to get

$$F(x) = \frac{a_1}{1 - r_1 x} + \frac{a_2}{1 - r_2 x} + \frac{a_3}{1 - r_3 x}$$

where  $a_1$ ,  $a_2$ , and  $a_3$  are constants. See below for the easiest way to get these constants  $a_1$ ,  $a_2$  and  $a_3$ .

We can then see, by taking a series expansion for this generating function, that a generic term of our sequence will be

$$f_n = a_1 r_1^n + a_2 r_2^n + a_3 r_3^n. \tag{1}$$

How did we get the roots  $r_1$ ,  $r_2$  and  $r_3$ ? They are the zeroes of the polynomial

$$y^3 - \alpha y^2 - \beta y - \gamma = 0.$$

We can see this by taking  $y = \frac{1}{x}$ , so

$$\begin{aligned} 1 - \alpha x - \beta x^2 - \gamma x^3 &= x^3(y^3 - \alpha y^2 - \beta y - \gamma) \\ &= x^3(y - r_1)(y - r_2)(y - r_3) \\ &= (1 - r_1 x)(1 - r_2 x)(1 - r_3 x). \end{aligned}$$

To get the constants  $a_1$ ,  $a_2$  and  $a_3$ , the easiest is to plug in  $n = 0$ ,  $n = 1$  and  $n = 2$  in (1) and use the values for  $f_0$ ,  $f_1$  and  $f_2$ , to get a system of linear equations with 3 equations and 3 unknowns.



In general, for a  $k$ -term recurrence relation, we derive that  $F(x) = \frac{p(x)}{q(x)}$  where  $q(x)$  has degree  $k$  and  $p(x)$  is of degree  $s \leq k - 1$ . If  $q(x)$  has  $s$  single roots  $r_1, \dots, r_k$ , the approach used here simply applies and we write  $F(x)$  as

$$F(x) = \sum_{i=1}^k \frac{a_i}{1 - r_i x}$$

and  $f_n$  as

$$f_n = \sum_{i=1}^k a_i r_i^n.$$

We can determine the  $a_i$ 's by using the values for  $f_n$  for  $n = 0, \dots, k - 1$ . If  $q(x)$  has multiple roots, the expansion of  $f_n$  will involve terms like  $x^j r_i^n$  if  $r_i$  is a root of multiplicity at least  $j + 1$ .

## 6 Catalan numbers via generating functions

Let's go back to Catalan numbers for a bit. Previously, we calculated using bijections that the  $n$ -th Catalan number is  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . There is a more calculational, but still interesting way to get this result. Let's count binary trees with  $n$  nodes.

We will construct a recurrence for  $C_n$ . Since the subtrees of a  $n$ -node binary tree are themselves binary trees, with between 0 and  $n - 1$  nodes, we have

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

which looks tougher than what we solved before because it encodes quadratic relationships between the variables. Nevertheless, we'll be able to handle it using generating functions.

The next question is how we evaluate it using generating functions. Let  $G(x)$  be the generating function corresponding to Catalan numbers, that is

$$G(x) = \sum_{i=0}^{\infty} C_i x^i = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \dots$$

What happens when we square  $G(x)$ . We get

$$\begin{aligned} G(x)^2 &= 1 + (1 + 1)x + (1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1)x^2 + (1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1)x^3 + \dots \\ &= \sum_{k=0}^{\infty} x^k \sum_{j=0}^k C_j C_{k-j} \end{aligned}$$

You can see that the  $x^k$  expression on the right is the right-hand-side of the recurrence relation we found about for  $C_{k+1}$  (with  $n = k + 1$ ), so we get

$$G(x)^2 = \sum_{k=0}^{\infty} C_{k+1} x^k.$$

Multiplying by  $x$  gives a sum with the  $x^j$  coefficient equal to  $C_j x^j$ . We now have an expression relating  $xG(x)^2$  and  $G(x)$ . We need to make sure we get the smallest terms right. We can check

the constant term is the only one that is wrong, and we can fix that by adding 1 to the right hand side, to get the equation

$$G(x) = 1 + xG(x)^2.$$

**Finding the coefficients.** We now solve this equation. This is a quadratic equation in  $G(x)$ , so we can use the quadratic formula to solve for  $G(x)$ , obtaining

$$G(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

We now have a choice. Which of the two roots of this equation should we use. We can figure this out by looking at the first term. We should have  $G(0) = 1$ . Depending on which root we choose, when we plug in 0 we either get  $G(0) = 2/0$ , or  $G(0) = 0/0$ . Clearly, the first option gives the wrong answer. Using a little calculus, in the second case,

$$\lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{(1 - \sqrt{1 - 4x})(1 + \sqrt{1 + 4x})}{2x(1 + \sqrt{1 + 4x})} = 1$$

we indeed have  $G(0) = 1$ , so we get

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

This was the fun part; we have an algebraic expression for  $G(x)$ . We now need to expand it in a power series to find the  $x^k$  term. In this case, unfortunately, this happens to be somewhat tedious. We will go through the steps carefully.

The first step is expanding  $(1 - y)^{1/2}$  in a power series. We use the binomial formula

$$(1 - y)^{1/2} = 1 - \binom{1/2}{1}y + \binom{1/2}{2}y^2 - \binom{1/2}{3}y^3 + \binom{1/2}{4}y^4 - \dots$$

This might look odd if you haven't seen it before, but one can define  $\binom{\alpha}{k}$  even when  $\alpha$  is not a positive integer: the formula is

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}.$$

Simplifying,

$$(1 - y)^{1/2} = 1 - \frac{1}{2}y + \frac{1}{2}(-\frac{1}{2})\frac{y^2}{2!} - \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\frac{y^3}{3!} + \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\frac{y^4}{4!} + \dots$$

Now, we need to substitute  $y = 4x$ , and plug the resulting expression into the formula we got for  $G(x)$ . We obtain

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{1}{2x} \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 7) \cdot (2k - 5) \cdot (2k - 3)}{2^k} \frac{(4x)^k}{k!}$$

where we have the product of all odd numbers between 1 and  $2k - 3$  in the numerator, and  $k!$  in the denominator. All the  $-$  signs cancel out, as they should: since we're counting things, we have to get a positive integer.

How do we simplify this expression? Recall  $G(x) = \sum C_k x^k$ , so equating coefficients, we get

$$C_k = \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-5) \cdot (2k-3) \cdot (2k-1)}{2^{k+1}} \cdot \frac{(4)^{k+1}}{(k+1)!}$$

where we have had to replace  $k$  by  $k+1$  in the above formula to make up for the  $\frac{1}{x}$  in front of it. We can cancel out the powers of 2 in the numerator and denominator to get

$$C_k = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-5) \cdot (2k-3) \cdot (2k-1)}{(k+1)!} 2^k.$$

Now, let's multiply the top and bottom of the above expression by  $k!$ , and write  $k!2^k = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k)$ . We get

$$C_k = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2k-1) \cdot (2k)}{(k+1)(k!)^2}.$$

Thus, we have

$$C_k = \frac{1}{k+1} \frac{(2k)!}{(k!)^2} = \frac{1}{k+1} \binom{2k}{k},$$

which is the definition of the  $k$ 'th Catalan number.

The Catalan numbers turn up in quite a few places (as we've already seen). Prof. Richard Stanley has a section on his webpage (which is also in his book) giving 66 combinatorial interpretations of the Catalan numbers.

## 7 Diagonals in Pascal's Triangle

In this section we use generating function of more than one variable in order to solve a neat problem. Recall that Pascal's triangle is formed by binomial coefficients:

					1					
						1				
					1		1			
				1		2		1		
			1		3		3		1	
		1		4		6		4		1
			1	5		10		5		1
	1		6		15		20		15	
		1	7		21		35		35	
			1	7		21		35		1
1		8		28		56		70		56
	1	8		28		56		70		56
		8		28		56		70		56
			8		28		56		70	
				8		28		56		70
					8		28		56	
						8		28		70
							8		28	
								8		70
									8	
										8

We will compute the sums of the diagonal elements in Pascal's triangle. For example, the sum of the boldface elements above is

$$34 = 1 + 10 + 15 + 7 + 1 = \binom{4}{0} + \binom{5}{2} + \binom{6}{4} + \binom{7}{6} + \binom{8}{8}.$$

How do we do this with generating functions? Let's use a generating function of two variables. Consider the two variable function

$$G(x, y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^a y^b.$$

Note that the coefficient of  $x^a y^b$  of  $g$  is precisely the value in row  $a+b$ , column  $b$  of Pascal's triangle (both indexed starting from 0). We already saw the sum of the terms in the  $n$ 'th row of Pascal's triangle

$$(x+y)^n = \sum_{a=0}^n \binom{n}{a} x^a y^{n-a}$$

Now, let's sum over all rows.

$$G(x, y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^a y^b = \sum_{n=0}^{\infty} (x+y)^n = \frac{1}{1-x-y}$$

where the last equality comes from the sum of a geometric series.

What are we looking for? We're looking for the sum

$$\sum_{j=0}^m \binom{m+j}{2j}$$

for all  $m$ . The generating function for that would be

$$H(z) := \sum_{m=0}^{\infty} \sum_{j=0}^m \binom{m+j}{2j} z^m.$$

We would like to relate  $H$  to  $G$  somehow (since we have an expression for  $G$ ). So we'd like to somehow turn  $\binom{a+b}{a}$  into  $\binom{m+j}{2j}$ . Can we do this? Looking at this more closely, it involves setting  $a = 2j$  and  $b = m - j$ . This would involve turning  $x^a y^b$  into  $x^{2j} y^{m-j}$ . So we'd like to get  $x^{2j} y^{m-j} = z^m$ . Note we can do this if we let  $x = z^{1/2}$  and  $y = z$ . Dealing with square roots is not so nice, so let's square everything and let  $z = x^2$ ,  $y = x^2$ . Our hope will be that  $H(x^2)$  is related to  $G(x, x^2)$ .

So consider  $G(x, x^2)$ :

$$G(x, x^2) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \binom{a+b}{a} x^{a+2b}.$$

Now let's compute the coefficient of  $x^{2m}$  in  $G(x, x^2)$ ; in other words, the sum of all terms where  $a + 2b = 2m$ , or  $a = 2(m - b)$ . We obtain

$$\begin{aligned} \text{coeff. of } x^{2m} \text{ in } G(x, x^2) &= \sum_{b=0}^m \binom{2(m-b)+b}{2(m-b)} \\ &= \sum_{j=0}^m \binom{m+j}{2j} \quad \text{with the substitution } j = m - b. \end{aligned}$$

Success! This is exactly  $h_m$  (or in other words, the coefficient of  $x^{2m}$  in  $H(x^2)$ ). Note that we don't have  $H(x^2) = G(x, x^2)$ ; rather, we have

$$G(x, x^2) = H(x^2) + Q(x),$$

where  $Q(x)$  consists of only odd powers of  $x$ .

At any rate, we can now determine  $h_n$ , since we know

$$G(x, x^2) = \frac{1}{1 - x - x^2} = \sum_{r=0}^{\infty} f_r x^r,$$

where  $f_m$  is the  $m$ 'th Fibonacci number. So  $h_m = f_{2m}$ .

It turns out that once you know that you know the answer, it's easy to prove it by induction. Generating functions have the advantage that we didn't have to guess the answer first for the technique to work. There are also some identities which have straightforward generating function proofs, but for which it is quite difficult to find induction proofs.

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