Spring 2024

18.200 Homework 1

Instructions: Indicate your recitation and collaborators. We encourage collaboration (you could write for example: 'John Doe, my recitation groupmates, and Peter's office hours'), or state that you worked only on your own. In any case, you must write up your own proofs.

For all problems marked as writing problems, use the homework template provided on this website; you may use this template for all problems if you like. The LaTeX template contains writing guidance, to see it, you will need to download the file and view the tex code. Your grade on the writing problems will be based on both the math and the quality of the writing, so be sure to read and follow the writing guidance provided in the template and in recitation.

1. (10 points each) Consider **two** of the following statements (at your choice) from the recitation and, for each, write the statement as a theorem and prove it. Your proofs should be easily understandable by another 18.200 student.
   (a) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.
   (b) Prove that any set of 41 integers between 1 and 80 has the property that one of them divides another one.
   (c) Let a and b be integers. Show that $a^2 + b^2$ cannot have a remainder of 3 when divided by 4.
   (d) Show that for a random subset of $\{1, 2, \cdots, n\}$ the sum of the elements of this subset is odd with probability 1/2. Here we choose the set uniformly at random so that each element i is included independently with probability 1/2.
   (e) Show that for any irrational x and positive integer n, there is a rational number p/q with $1 \leq q \leq n$ so that $|x - p/q| \leq 1/nq$.
   Hint: Try using the pigeonhole principle, and considering values $qx - \lfloor qx \rfloor$, where $\lfloor \alpha \rfloor$ is the *floor* function, i.e., the greatest integer less than or equal to α.

2. **Writing Problem** (20 points; 10 points for math, 10 points for writing)

   Let π_1, π_2, π_3 be permutations of the numbers $[n] := \{1, 2, 3, \ldots, n\}$. A *common subword* between π_i, π_j, is a subsequence that appears in the same order for both of them. For example, if $\pi_1 = (12345678)$, $\pi_2 = (87654321)$, and $\pi_3 = (54832716)$, then 58 is a common subword of π_1 and π_3, and 821 is a common subword of π_2 and π_3.

   Let $n = m^3 + 1$ for some positive integer m. Prove that for any permutations π_1, π_2, π_3 of $[n]$, we can find two permutations π_i and π_j which have a common subword of length at least $m + 1$.

LaTeX Resources:

- Overleaf (`www.overleaf.com`) is a user-friendly interface that offer autocomplete coding suggestions and explanations of errors. You can get a free account from MIT: `https://www.overleaf.com/edu/mit`
- Resources are provided in the LaTeX module on Canvas.
- Detexify allows you to draw the symbol you want to use and will output the tex code you need (`detexify.kirelabs.org`).
- LaTeX Stack Exchange (`https://tex.stackexchange.com/`) has many answers to questions about LaTeX.
- Googling can often lead you to the answers, e.g. "latex left curly quotes"
- Consult your classmates or instructors.

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