18.200 Homework 2

Instructions: Indicate your recitation and collaborators. We encourage collaboration (you could write for example: 'John Doe, my recitation groupmates, and Peter's office hours'), or state that you worked only on your own. In any case, you must write up your own proofs.

For all problems marked as writing problems, use the homework template provided on this website; you may use this template for all problems if you like. The LaTeX template contains writing guidance, to see it, you will need to download the file and view the tex code. Your grade on the writing problems will be based on both the math and the quality of the writing, so be sure to read and follow the writing guidance provided in the template and in recitation.

1. **Writing Problem** (20 points)

   This is a multi-part writing assignment. It is meant not to assess your writing, but rather to help you and the rest of the class to begin learning how to use generative AI effectively, both now and as it continues to evolve. Responses will be aggregated and reported back to the entire class, so some of your responses may be shared. Your grade will be based on the mathematical correctness of your own writing and on whether you engaged thoughtfully in critiquing the logic of the AI.

   (a) Let X be a uniformly random subset of $\{1, 2, \cdots, n\}$ (there are 2^n possible subsets, and each is chosen with probability $1/2^n$). Let Y be another independently chosen random subset. Determine:

   - the probability that $X \subseteq Y$. (*Hint:* The size of the sets X and Y should not be included in the final answer.)
   - $P(X \subseteq Y$ and X has k elements$)$, for a given integer k between 1 and n.

   Write your answers as theorems, and include proofs, writing them clearly for an audience of classmates who don't know how to do the proof. Do **not** use generative AI to help with this part. Ensure you have a correct proof before going on to the next part, since you'll need to understand this proof in order to assess the correctness of a proof generated by AI.

   (b) Ask your preferred generative AI to answer the same questions and provide proofs–optionally submitting follow-up prompts to try to get it to generate a good response. Are there any logical flaws in its proof? Describe them explicitly. Are there any parts of the proof that help you to improve your own proof? What are they?
   When you submit this homework, in the appendix, record which AI you used, your prompt(s), and the AI's response(s).

2. (10 points) Let's look at the following variation of the Monty Hall problem. This time, the game show host shows you five doors. Behind one of the doors is a big prize (maybe a car), and behind two of the other doors is a small prize (maybe a bicycle). You choose one of the

doors. He then opens a door and shows you a small prize behind it. If you want, you can switch to one of the three doors that isn't open yet.

(a) Suppose you want to maximize the probability that you win the big prize. Should you switch?

(b) Suppose you want to maximize the probability that you win any prize. Should you switch?

(c) Suppose the small prize has value A and the large prize has value B. What is the expected value of your prize if you switch? If you don't switch? What is the variance if you switch? If you don't switch? Suppose you want to maximize the expected value; when should you switch?

3. (10 points) Suppose that a baby is equally likely to be a boy or a girl, and that it is equally likely to be born on any day of the week. Furthermore, we assume that the gender of a baby and the day he/she was born are independent. And whenever we talk about 2 babies, their genders and days of birth are independent (no twins, etc.).

   (a) What is the probability of event A, that any given baby is a boy born on a Monday? (Yes, this is easy.)

   (b) Consider a family with two siblings. Given that at least one of them is a boy, what is the probability that they are both boys?

   (c) Consider a family with two siblings. Given that at least one of them is a boy born on a Monday, what is the probability that the other is also a boy (born on any day of the week)? (Hint: this should convince you that intuition is dangerous when dealing with probabilities...)

4. (10 points) Consider 11 independent tosses of a fair coin. Let X be the number of occurrences (among the 11 tosses) of 5 consecutive tosses resulting in HHTHH.

   (a) What is $\Pr(X \geq 1)$, i.e. what is the probability that there are 5 consecutive tosses resulting in HHTHH?

   (b) What is $\mathbb{E}(X)$?

5. (10 points) Somebody proposes that you play the following dice game. They have three dice, a red one with sides labeled $\{1, 1, 5, 5, 5, 5\}$, a blue one with sides labeled $\{2, 2, 2, 2, 6, 6\}$, and a green one with sides labeled $\{3, 3, 3, 4, 4, 4\}$. You each roll a die, and whoever rolls a higher number wins. They generously (?) offer to let you pick your die first, so that you can choose the best one.

   Which die should you choose, and if you choose it, what is the probability that you win the game? Assume that the dice are equally likely to land on any of their six faces.