Spring 2024

18.200 Homework 3

Instructions: Indicate your recitation and collaborators or state that you worked only on your own. In any case, you must write up your own proofs. Your solutions to the writing exercise must be typeset using LaTeX; you can adapt the provided template for your submission. Submit just your .pdf file to Gradescope. The other exercises could be handwritten (legibly) and scanned, although we encourage you to use LaTeX for them as well.

1. **Writing Exercise**: (10 points) Read the theorem and proof below.

   (a) Starting with the second sentence of the proof, identify the "known" information and the important "new" information in each sentence.

   (b) Identify at least one instance of poor information order AND at least one gap in connectivity. (Both could be caused by the same text, but explain).

   (c) Revise to improve the information order and the connectivity, without revising the theorem, the first sentence of the proof, or the figure caption. Feel free to add or remove text as needed.

   A *Dyck path* with $2n$ steps is a path from $(0,0)$ to $(2n,0)$, where each step goes to the right by 1 unit, either goes up or down 1 unit, and never goes below the x-axis.

   Theorem 1. *For each n, the set $\mathcal{T}_n$ of plane trees with n edges is the same size as the set $\mathcal{D}_n$ of Dyck paths with $2n$ steps:*

   $$|\mathcal{T}_n| = |\mathcal{D}_n| \textit{ for all } n.$$

   Proof. We define a map Φ from plane trees to Dyck paths as follows: given any tree $T \in \mathcal{T}_n$, perform a *depth-first search* of the tree T (as illustrated in Figure 1) and define $\Phi(T)$ as the sequence of up and down steps performed during the search. A Dyck path $D \in \mathcal{D}_n$ is obtained from T because $\Phi(T)$ has n up steps and n down steps (one step in each direction for each edge of T), starts and ends at level 0, and remains non-negative. Because Φ is a bijection between $\mathcal{T}_n$ and $\mathcal{D}_n$, we conclude $|\mathcal{T}_n| = |\mathcal{D}_n|$. □

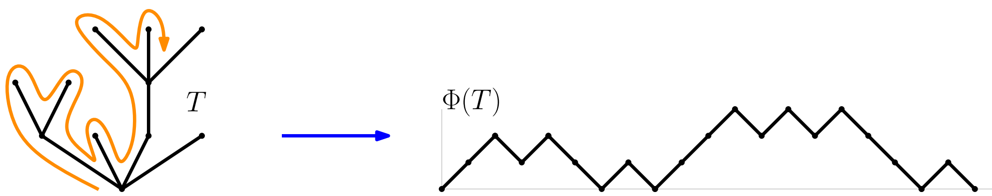


Figure 1: A plane tree T and the associated Dyck path $\Phi(T)$. The depth-first search of the tree is represented graphically by a tour around the tree (drawn in orange): first visit the leftmost subtree entirely, then the next subtree, etc.

2. **Writing exercise.** (8 points writing, 10 points math)

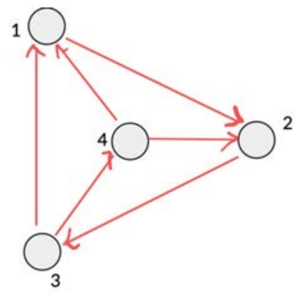
   (a) Revise your writing of Exercise 2 of Problem Set 1, and resubmit it: the audience is people comparable to 18.200 classmates (e.g., who are taking a class similar to 18.200 elsewhere) who don't know how to do the proof. Take into account the feedback you've received and the writing guidance provided in the LaTeX template and in recitation, including using known-to-new structure and ensuring readers know what you're doing and why you're doing it. The feedback you received may include comment codes; if so, they refer to the Common Comments document in the module of General Writing Resources on Canvas. **Do not use generative AI to help with this part;** you may do so in part 2c below.

   If you have questions about any of the feedback or writing guidance, please ask either the communication instructor for your recitation or the one who provided the feedback. Very simple questions may be asked via email or Piazza, but we encourage you to sign up for a time to meet: sign-up links are in the syllabus on Canvas.

   (b) Briefly summarize the revisions you made in part 2a and why you made them. As you do so, note at least one place where you improved information order or connectivity, or one place where these were already handled well.

   (c) **Optional for extra credit** (2 points) Do not do this part until you've completed part (a) to the best of your ability, since the experience of carefully revising your proof on your own will enable you to give a more informed critique of the AI's response. Use AI to redo any aspect of part 2a or to further improve your revision. Submit your prompts and the AI's response, and critique the pros and cons of using AI to revise mathematics. For example, What does it do well? What does it do poorly? How can you help it to revise mathematics well? Remember that the goal of the writing is to help the target audience to understand the proof.

3. (20 points) A *tournament* on n vertices is an orientation of a complete graph on n vertices, i.e. for any two vertices u and v, exactly one of the directed edges (u, v) or (v, u) is present. A Hamiltonian path in a tournament is a directed path passing through all vertices exactly once. For example, in the example below, $1 \to 2 \to 3 \to 4$ is a Hamiltonian path since all edges $(1, 2), (2, 3)$ and $(3, 4)$ are directed in the direction of traversal and it visits every vertex once, while for example $1 \to 2 \to 4 \to 3$ is not a Hamiltonian path since the second and third edges are directed in the wrong direction. The tournament in the figure has five Hamiltonian paths: $1 \to 2 \to 3 \to 4$; $2 \to 3 \to 4 \to 1$; $3 \to 4 \to 1 \to 2$; $4 \to 1 \to 2 \to 3$; $4 \to 2 \to 3 \to 1$.



(a) (6 points) Show that for every $n \geq 2$, there is a tournament on n vertices with only one Hamiltonian path.

(b) (7 points) Show that every tournament on $n \geq 2$ vertices has at least one Hamiltonian path.
Hint: use induction.

(c) (7 points) For every $n \geq 2$, show that there are tournaments with at least $n!/2^{n-1}$ Hamiltonian paths. For example for $n = 4$, this shows that there exists a tournament on 4 vertices with at least 3 Hamiltonian paths.
Hint: show that the expected number of Hamiltonian paths in a random tournament, where each edge is oriented randomly, is $n!/2^{n-1}$.

4. (10 points) Suppose we have two random variables, X and Y. The *covariance* of two random variables is

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

So if X and Y are independent, $\mathrm{Cov}(X, Y) = 0$.

Show that

$$\mathrm{Var}\Big(\sum_{i=1}^{k} X_i\Big) = \sum_{i=1}^{k} \mathrm{Var}(X_i) + 2 \sum_{1 \leq i < j \leq k} \mathrm{Cov}(X_i, Y_i)$$

5. (10 points)

(a) Suppose you flip a coin 26 times. Let X be the number of heads you toss. What is $\mathbb{E}(X)$? What is $\mathrm{Var}(X)$?

(b) Suppose you have a standard deck of cards, with 26 red cards and 26 black cards. You deal out 26 of them. Let X be the number of red cards among these 26. What is $\mathbb{E}(X)$? What is $\mathrm{Var}(X)$?
Hint: You may want to use the results of the previous problem.

6. (10 points) A math professor has 24 socks in his sock drawer, 8 brown, 6 white, 6 black, and 4 blue. When he wakes up, he grabs four socks at random from the drawer. What is the probability that these four socks contain a matching pair?

7. (10 points) Prove the following combinatorial identities, using the "counting two ways" technique, where m, n, k, and r are non-negative integers.

(a) $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$,

(b) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$,

(c) $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.