18.200 Homework 5

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Instructions: Indicate your recitation and collaborators or state that you worked only on your own. In any case, you must write up your own proofs.

1. Add guiding text to the following proof. Guiding text is text that guides the reader through the content by communicating the structure and/or purpose of the text to the reader. The purpose is to ensure readers understand what you're doing, why you're doing it, and why it works. Whenever possible, the relevance of steps and claims should be clear as they arise–avoid requiring readers to wait until later to see how everything fits together. See the writing resources for examples.

   The tex code for this problem is available in the homework template.

   As a reminder from recitation: The probabilistic method is a powerful technique whereby we prove the existence of certain objects or structures. For example, instead of constructing objects directly, if we show that a randomly chosen object has the desired property with a non-zero probability, we may conclude that it exists.[1]

Theorem 1. *Let $G = (V, E)$ be a graph with n vertices and let $d_1, \ldots, d_n$ be the degrees of the vertices of G. There is an independent set of G with at least*

$$\sum_{v=1}^{n} \frac{1}{d_v + 1}$$

vertices.

Lemma 2. *Let σ be a uniformly random permutation on $[n] := \{1, \cdots, n\}$, and let $A \subseteq [n]$. Then, for any $v \in A$,*

$$\mathbb{P}[\sigma(v) = \min_{u \in A} \sigma(u)] = \frac{1}{|A|}.$$

Proof of Lemma.

The statement is trivial if $|A| = 1$.

[1] This explanation of probabilistic method was drafted with Chat GPT3.5 and then revised for clarity, correctness, and relevance.

For $|A| > 1$, let $q_v = \mathbb{P}[\sigma(v) = \min_{u \in A} \sigma(u)]$, and let $v \neq v'$ with $v, v' \in A$. Then the function f on the set of permutations which exchanges $\sigma(v)$ and $\sigma(v')$ is a bijection, so $q_v = q_{v'}$. Since $\sum_{v \in A} q_v = 1$, we derive that $q_v = \frac{1}{|A|}$. $\square$

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Proof of Theorem.

Let σ be a uniformly random permutation on the set V of vertices of the graph. Let

$$S = \{v \in V | \sigma(v) < \sigma(w) \text{ for all } w \text{ with } \{v, w\} \in E\}.$$

We claim that S is an independent set in the graph G. Therefore,

$$\mathbb{E}[|S|] = \mathbb{E}\left[\sum_{v \in V} 1_{v \in S}\right] = \sum_{v \in V} \mathbb{P}[v \in S] = \sum_{v \in V} \frac{1}{d_v + 1}.$$

This implies the existence of an independent set S with $|S| \geq \sum_{v \in V} \frac{1}{d_v+1}$. $\square$

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