18.200 Homework 9

Instructions: Indicate your recitation and collaborators or state that you worked only on your own. In any case, you must write up your own proofs.

1. Let C be the linear code over $\mathbb{Z}_2$ given by the following generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

   (a) How many codewords does C have?

   (b) Show that all nonzero codewords have Hamming weight at least four.

   (c) Show that the code C can correct one error.

   (d) Is C a perfect code? Justify.

2. Consider the binary alphabet $\{a, b\}$, with probability distribution $p(a) = 0.75$ and $p(b) = 0.25$. The Huffman code for this distribution is rather trivial, and has expected length 1. What is the Shannon entropy for this distribution? Now, consider the product probability distribution on blocks of two letters: $\{aa, ab, ba, bb\}$. Find the optimum Huffman code for this distribution and its expected length. Compare it with the Shannon entropy for this distribution.

3. Consider the set of frequencies

$$\left\{ \frac{1}{34}, \frac{1}{34}, \frac{1}{34}, \frac{2}{34}, \frac{3}{34}, \frac{5}{34}, \frac{8}{34}, \frac{13}{34} \right\}.$$

   The numerators are just the Fibonacci numbers with an extra 1. There are multiple different binary Huffman trees that give an optimal code for this set of frequencies. Find a Huffman tree with maximum height and one with minimum height. (You do not have to prove that these are the maximum and minimum.) What is the expected length of these codes? What is the entropy of this probability distribution?