Recitation 2: More wrong proofs

Theorem 1. *Suppose we roll two fair 6-sided dice. Let X be the value on the first die, and let Y be the value on the second die. Then let Z be a variable equal to the parity of $X+Y$, with $Z=0$ when $X+Y$ is even and $Z=1$ when $X+Y$ is odd. Then the probability $P((X=a)\wedge(Y=b)\wedge(Z=c))$ is constant, regardless of the choice of $1 \le a, b \le 6$ and $c \in \{0,1\}$.*

Proof. We show that the three random variables X, Y, Z are independent. We know X and Y are independent since they are given by separate dice rolls. For X and Z, we have

$$P((X=a)\wedge(Z=c)) = P((X=a)\wedge(Y \equiv (a-c) \mod 2) = P(X=a)P(Y \equiv (a-c) \mod 2) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}.$$

Since we have shown that the probability is independent of our choice of a, and since $P(X=a)=\frac{1}{6}$ for any $1 \le a \le 6$, it follows that

$$P((Z=c)|(X=a)) = \frac{P((X=a)\wedge(Z=c))}{P(X=a)} = \frac{1}{2}$$

is independent of a, so X and Z are independent random variables. This also shows that $P(Z=c)=\frac{1}{2}$ for any choice of $c \in \{0,1\}$. Similarly, Y and Z are independent random variables. Thus, all three random variables are independent, and

$$P((X=a)\wedge(Y=b)\wedge(Z=c)) = P(X=a)P(Y=b)P(Z=c) = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{2} = \frac{1}{72}$$

is constant. □

Theorem 2. *Suppose we have a fair 2-sided coin. We decide on a three-letter sequence of 'H's and 'T's, with 'H' corresponding to heads and 'T' corresponding to tails. We then flip our coin repeatedly until the last three coin flips produce that sequence. For example, if our chosen sequence is "HTT", then we flip our coin until we consecutively get heads, followed by two tails. Then the number of coin flips before we stop is independent of the sequence we choose.*

Proof. Since our coin is fair and each coin flip is independent, the probability of three coin flips producing a specific sequence is always $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$, regardless of the sequence. As a result, for any finite sequence of coin flips, each three-letter combination has the same chance of being produced at the end, so the expected number of coin flips must be independent of the sequence. □