Known-to-New Structure

18.200 Spring 2024

Recitations

Feb 7 Generating proofs

Feb 14 Checking their correctness

Today **How can we make a proof easy to follow?**

Strategies **Guiding text**

Flow

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Guiding text:

Ensure readers know *WHAT* will happen and *WHY.*

Proof. Let $S := \{s \in \mathbb{R}: 0 \le s, s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathbb{R}$, and we let $x := \sup S$. Note that $x > 1$.

We will prove that $x^2 = 2$ by ruling out the other two possibilities: $x^2 < 2$ and $x^2 > 2$. First assume that $x^2 < 2$. We will show that this assumption contradicts the fact that $x = \sup S$ by finding an $n \in \mathbb{N}$ such that $x + 1/n \in S$, thus implying that x is not an upper bound for S. To see how to choose n, note that $1/n^2 \le 1/n$ so that

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \le x^2 + \frac{1}{n}(2x + 1).$$

Then how to make details easy to follow?

Bartle & Herbert's Introduction to Real Analysis, 3rd Edition, p. 41.

Read the following sentence. Which sentence would you expect to follow it?

Conditional probabilities can behave non-intuitively.

A striking paradox in probability called Simpson's paradox illustrates this non-intuitive behavior.	4 respondents	13 %
This non-intuitive behavior is illustrated by a striking paradox in probability called Simpson's paradox.	26 respondents	87 %

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Readers (of English) expect each sentence to ***start*** with **known** information

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Readers (of English) expect each sentence to *start* with **known** information, which introduces the sentence's important **new** information.

Conditional probabilities can behave ***non-intuitively.***

This non-intuitive behavior *is illustrated by a paradox in probability called* ***Simpson's paradox****.*

This is known-to-new sentence structure.

Known-to-new sentence structure has a nice side effect:

To describe a broom, we recall the wave packet decomposition of Ef introduced by Bourgain [1]. The wave packet decomposition says that inside a large ball of radius R, we can decompose Ef into a sum over wave packets $Ef_{\theta,v}$. Each wave packet $Ef_{\theta,v}$ is essentially supported in a tube $T_{\theta,v}$ of length R and radius $R^{1/2+\delta}$ for some small $\delta > 0$. The axis of $T_{\theta,v}$ points in a direction depending only on θ, and the location of $T_{\theta,v}$ is described by v. The absolute value $|Ef_{\theta,v}|$ of a wave packet is approximately a constant function on $T_{\theta,v}$.

Text that flows well can help readers follow the flow of the logic.

This looks easy, but drafts often don't do it…

When drafting, it's normal to put "**new**" information first, and then connect to "**known**."

*Conditional probabilities can behave **non-intuitively**.*
A paradox in probability called Simpson's paradox
*illustrates **this non-intuitive behavior**.*

When revising, change **new**←**known** ordering to **known** →**new**.

*Conditional probabilities can behave **non-intuitively**.*
***This non-intuitive behavior** is illustrated by*
***a paradox in probability called Simpson's paradox**.*

When drafting it’s also normal to not recognize gaps in flow.

What is the effect of gaps?

Plugging our initial values into (2) we see the expression simplifies to 2+2. Four is less than…

Plugging our initial values into (2) we see the expression simplifies to 2+2. Perfect squares have the property…

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Activity by M. Effron

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Which gaps slow reading depends on the audience.
Look for and fill gaps in flow that may slow (or prohibit!) comprehension.

Activity by M. Effron

YOU TRY IT

Read the following sentences:

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By definition, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(B)}$. Events A and B are independent iff $\mathbb{P}(A \wedge B) = \mathbb{P}(A)\mathbb{P}(B)$. Thus $\mathbb{P}(A|B) = \mathbb{P}(A)$ iff events A and B are independent.

By definition, $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(B)}$. Furthermore, $\mathbb{P}(A \wedge B) = \mathbb{P}(A)\mathbb{P}(B)$ iff events A and B are independent. Thus $\mathbb{P}(A|B) = \mathbb{P}(A)$ iff events A and B are independent.

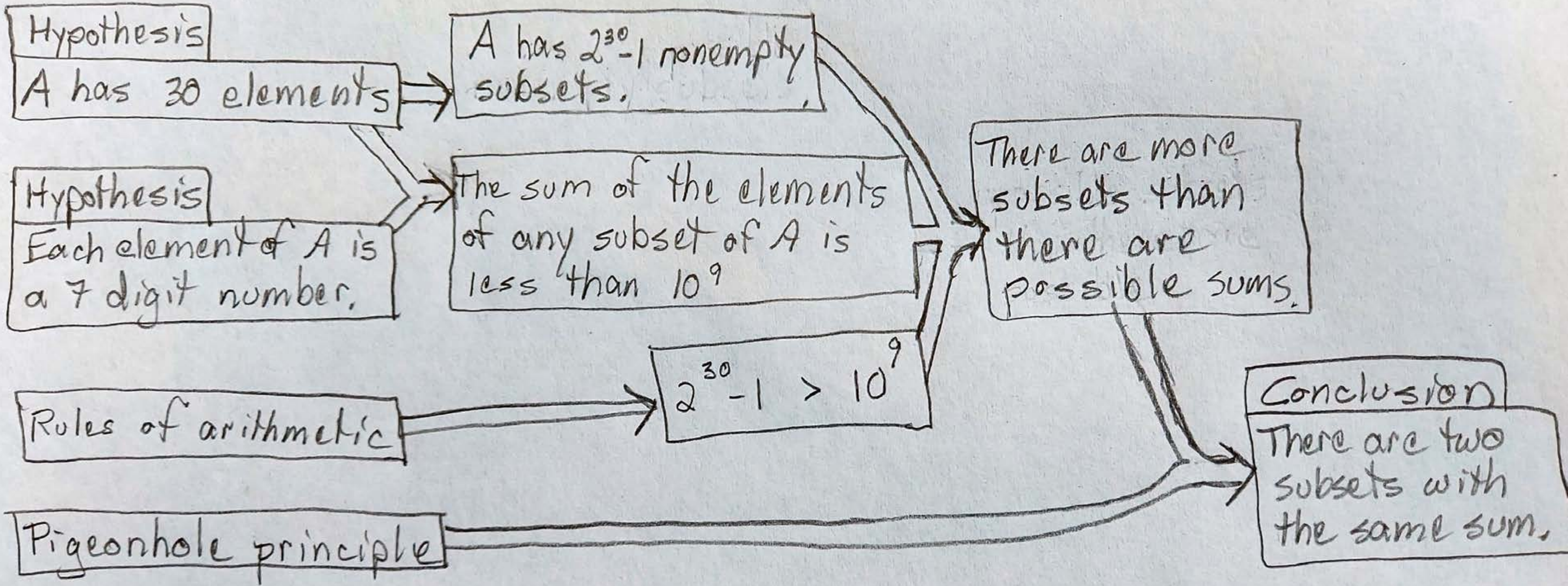
With known-to-new structure, adjacency helps readers see logical connections.

To describe a broom, we recall the wave packet decomposition of Ef introduced by Bourgain [1]. The wave packet decomposition says that inside a large ball of radius R, we can decompose Ef into a sum over wave packets $Ef_{\theta,v}$. Each wave packet $Ef_{\theta,v}$ is essentially supported in a tube $T_{\theta,v}$ of length R and radius $R^{1/2+\delta}$ for some small $\delta > 0$. The axis of $T_{\theta,v}$ points in a direction depending only on θ, and the location of $T_{\theta,v}$ is described by v. The absolute value $|Ef_{\theta,v}|$ of a wave packet is approximately a constant function on $T_{\theta,v}$.

Text is inherently one dimensional: one long thread of text. But logic is not!

Theorem If A is a set of 30 7-digit numbers, then A has two disjoint subsets whose elements have the same sum.

Proof



What if logic is more complicated?

We will investigate two different kinds of up-down walks: hill walks and cliff walks.

Hill walks are up-down walks in which the three allowed steps are all of width one and change the height by -1, 0, or 1. In vector notation, these steps are $(1,-1)$, $(1,0)$, and $(1,1)$. See Figure 1 for an example of a hill walk.

Cliff walks are up-down walks in which the three allowed steps also change the height by -1, 0, or 1, but only the flat steps have width 1; the other two steps have width $1/2$. In vector notation, these steps are $\left(\frac{1}{2},-1\right)$, $\left(\frac{1}{2},1\right)$, and $(1,0)$. See Figure 2 for an example of a cliff walk.

In this paper, we will analyze both types of up-down walks by using the general fact that every walk has at most one peak, or local maximum value, since the first down step must come

Use **guiding text** and **repeat key terms** exactly to build **connections across distance.**

YOU TRY IT Craft the flow of the text to reveal the flow of the logic.

Begin with the proof's second sentence. What can you do to improve flow?

- turn sentences around so known-to-new?
- add missing information?
- repeat key terms exactly, to build connectivity across distance?
- restructure the proof?
- add guiding text to help readers see how pieces fit together?

For Overleaf, copy tex from http://tinyurl.com/200S23K2N

In a graph, a *cycle* is a set of nodes $v_1, v_2, \ldots, v_k$ such that there is an edge between v_i and v_{i+1}, and also between v_k and v_1. If $v_i \neq v_j$ for all $i \neq j$, then the cycle is a *simple cycle.*

Lemma 2. *A graph with N vertices and minimum degree 3 must have a simple cycle of length at most $2\log_2 N$.*

Proof. We will construct a simple cycle of length at most $2\log_2 N$ by taking the union of two paths of length $\log_2 N$ that go from a fixed vertex v to another vertex $v\prime$. We'll use the pigeonhole principle to show that two such paths exist. The paths from v that have length $\log_2 N$ will be the pigeons: because each vertex has degree at least 3, there are two ways to proceed after the path from v reaches each vertex. If any of these paths contain a cycle, then we are done, so assume they are all simple paths. There are at least $2^{\log_2 N} = N$ such paths. Now consider the number of possible ending vertices. There are $N - 1$ pigeonholes. Thus at least 2 paths must end at the same vertex; call it $v\prime$. We are done. $\square$

Using known-to-new structure can craft “flow.”
But beware “superficial flow”

The market-determined price of a bond with fixed, known cash flows determines the bond’s internal rate of return, or yield. Different yields are typically approximately equal. Approximations can be provided by Taylor series. The Taylor series is due to James Gregory of Scotland. Scotland has 790 islands, including the Northern Isles and the Hebrides, according to Wikipedia. Wikipedia occasionally asks for donations.

“FLOW” SHOULD HELP READERS FOLLOW THE FLOW OF THE LOGIC