18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Today we're going to discuss parking functions! Sometimes, it's hard to find a good parking spot.

Let's say we have a one-way road, and there are n cars on the road. All n cars are trying to park, and there are n parking spots along the road. However, the drivers prefer some spots to others: specifically, we have a function

$$f : [n] \to [n]$$

(from cars to spots), such that the ith car prefers to park in the $f(i)$th spot. It's possible that two cars prefer the same spot: what's the process? Each car tries to park in its favorite spot sequentially, and if the spot is already taken, the car keeps driving and will take the next available spot afterwards.

Example 1

For example, if $(f(1), f(2), f(3), f(4)) = (2, 3, 1, 2)$, the first car parks in spot 2, the second car parks in spot 3, the third parks in spot 1, and the fourth parks in spot 4.

Example 2

On the other hand, if $(f(1), f(2), f(3), f(4)) = (2, 3, 3, 2)$, the first car parks in spot 2, the second parks in spot 3, the third parks in spot 4, but the fourth car is out of luck!

Definition 3

Call such a function f a **parking function** if all n cars are able to park.

So the first example is a parking function, while the second is not. As an observation, all permutations are parking functions (since there are no conflicts), but there are others as well.

What are some examples of bad functions? In the bad example above, there are 4 cars that all want spots 2, 3, or 4, which is not satisfiable. It turns out this describes all parking functions!

Lemma 4

f is a parking function if and only if for any $1 \leq k \leq n$, the number of is for which $f(i) \geq n + 1 - k$ is at most k.

The idea is that permutations are good, and that we can also take a permutation and decrease some of the entries! For example, $(2, 3, 1, 2)$ is obtained by decreasing some entries of $(2, 3, 1, 4)$.

Lemma 5

f is a parking function if and only if there exists a permutation $w_(1), \cdots, w_(n)$ of 1 through n such that $f(i) \leq w(i)$ for all i.

These proofs will be left as an exercise! So let's try to find the number of parking functions in terms of n. One thing that makes our life easier: the last lemma tells us that the **order in how the cars come in does not matter**!

For $n = 2$, we can have $(1, 1), (1, 2)$, or $(2, 1)$, so there are 3 parking functions. For $n = 3$, there are 6 permutations, and then we can decrease some of the entries. There are 3 ways to get $(1, 2, 2)$, 3 ways to get $(1, 1, 3)$, 3 ways to get $(1, 1, 2)$, and 1 way to get $(1, 1, 1)$, for a total of 16 parking functions. This looks a lot like Cayley's formula!

Theorem 6 (Pyke, 1959, Konheim-Weiss 1966)

The number of parking functions $f : [n] \to [n]$ is $(n + 1)^{n-1}$.

The following will either be left as an exercise or covered later: can we find a bijection between spanning trees and parking functions from $[n]$ to $[n]$? Instead, we're going to come up with a very short proof that doesn't need them at all!

Proof. Assume we have $n + 1$ parking spots instead, and let's say we have a circular road, so cars drive counterclockwise. So now our function goes from $[n] \to [n + 1]$ - we still have the same preferences.

The important difference here is that all cars will always park (by the Pigeonhole principle, and also because we can loop around). In addition, one spot will stay empty!

Lemma 7

The total number of functions $f : [n] \to [n + 1]$ such that the ith spot stays empty is the same for all i.

This is because the problem is entirely symmetric: if we have a function that leaves the ith spot empty, just add 1 to all values of f, mod $(n + 1)$, and we get a function that leaves the $(i + 1)$th spot empty.

Lemma 8

Functions f that result in spot $n + 1$ staying empty are exactly the parking functions.

This is because $n + 1$ staying empty means no cars want spot $n + 1$, and no cars pass by the first n spots (because then they would take it)!

So because there are $(n + 1)^n$ total functions, and $\frac{1}{n+1}$ of them are parking functions, we have $(n + 1)^{n-1}$ functions in total, as desired. □

This doesn't explain why this is related to trees, but it proves the formula!

So let's go back to our original set-up, and let's pick a uniformly random function $[n] \to [n]$. The probability this is a parking function is

$$\frac{(n + 1)^{n-1}}{n^n} = \frac{1}{n}\left(1 + \frac{1}{n}\right)^{n-1}.$$

As $n \to \infty$, this approaches

$$\approx \frac{1}{n} \cdot e = \frac{e}{n},$$

so the probability is proportional to $\frac{1}{n}$ as n grows large. That explains why it's so hard to park in real life (or something)!

Now let's think a little bit about statistics on parking functions. First of all, can we say anything about the sum

$$f \to s(f) = \sum_{i=1}^{n} f(i)?$$

We know that f is always dominated by a permutation, so the sum is at most

$$s(f) \le \binom{n+1}{2}.$$

This is obtained by the $n!$ permutations. On the other hand, the minimum value n is only obtained by $(1, 1, \cdots, 1)$.

Problem 9

What's the number of parking functions with some given sum?

Alternatively, we can try to understand this statistic in terms of trees. It turns out there's a natural way to interpret it there!

Definition 10

Let T be a labeled tree on $(n+1)$ vertices, where we label the vertices $0, 1, \cdots, n$. (Think of 0 as the root.) A pair (i, j) for $1 \le i < j \le n$ is called an **inversion** if j belongs to the shortest path from vertex i to the root 0. Denote the number of inversions of T as $\text{inv}(T)$.

Theorem 11 (Kreweras, 1980)

Summing over all parking functions,

$$\sum_{\substack{f:[n]\to[n] \\ \text{parking function}}} x^{\binom{n+1}{2} - s(f)} = \sum_{\substack{T \text{ tree on} \\ 0,\cdots,n}} x^{\text{inv}(T)}.$$

This is called the **inversion polynomial for trees**, $I_n(x)$.

For example, $I_n(1) = (n+1)^{n-1}$, since we just count each term on the left side once. Similarly, $I_n(0) = n!$, since we want to the permutations with maximal sum of entries.

The $n!$ trees with no inversions are called **increasing trees**: we see now that there are $n!$ of them! This is because we add 1 to the root (since nothing can be between it and 0), and then add 2 in one of the two spots, then 3 in any of the three spots, and so on.

There are some other interesting values that we can plug in as well: if we plug in $x = -1$, we count the difference between the number of trees with an even versus odd number of inversions: it turns out to be plus or minus the number of **alternating permutations** of size n, which is defined to be those where $w_1 < w_2 > w_3 < \cdots$. These are related to the Euler and Bernoulli numbers!

The problem set will be posted soon, and there will be several questions around this theorem.