18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

April 8, 2019

Problem sets were returned - we'll be going over some problems today.

Problem 1

Show that the number of noncrossing partitions of $\{1, 2, \cdots, n\}$ is the Catalan number C_n.

Solution by Vanshika. We construct a bijection between a Dyck path of length $2n$ and a noncrossing partition of $[n]$. As an example, consider the partition where 1 and 3 are connected, $4, 7, 8$ are connected, and $5, 6$ are connected. The idea is that every element should correspond to 2 letters in our Dyck path, so for each partition, draw an extra line from the first to last number in the partition (so now we have closed cycles). If there's two arcs to the right, we write "UU," if there's two arcs to the left, we write "DD;" if there's no arc, we write "UD," and if there's an arc going to and coming out of the number, we write "DU."

This is a Dyck path because we can't end an arc before we start it! To reverse it, take each D from left to right and connect it to the nearest U before it, removing duplicates. $\square$

Solution by Sanzeed. Mark all of the "up" steps in order from 1 to n in a Dyck path of length n, and now match the down steps by finding the corresponding "levels." Then every maximal continual sequence of down steps is in its own partition! To reverse this, arrange partitions in descending order, and insert a partition $\{a_1 > a_2 > \cdots\}$ right after the a_1th up step. $\square$

Problem 2

Find a closed formula for the number of saturated chains from the minimal element $\hat{0} = (1|2|\cdots|n)$ to the maximal element $\hat{1} = (1 \cdots n)$ in the partition lattice Π_n.

Proof by Congyue. A saturated chain corresponds to merging elements in some order: for example, we can go from $(1|2|3|4|5)$ to $(12|3|4|5)$ to $(124|3|5)$ and so on. At the beginning, we have n blocks, so we need $n-1$ steps to finish the process.

On the kth step, there are $\binom{n-k+1}{2}$ ways to merge blocks, since we have $n-k+1$ blocks. Thus, our answer is

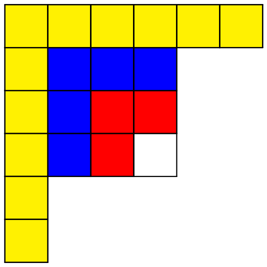
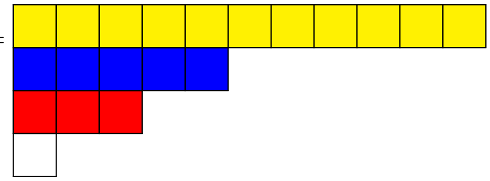
$$\prod_{k=1}^{n-1} \binom{n-k+1}{2} = \frac{(n-1)!n!}{2^{n-1}}.$$

$\square$

Problem 3

Find a bijection between partitions with odd distinct parts and self-conjugate partitions.

Solution by Sophia. Consider the Young diagram representation of a partition. Bend each odd/distinct partition at the middle and join them together along the diagonal! For example,

$(6, 4, 4, 4, 1, 1) =$  $\Longleftrightarrow$ $(11, 5, 3, 1) =$ 

This is reversible, so we do have a bijection. $\square$

Problem 4

Find the number of paths in Young's lattice that take $2n$ steps from $\hat{0}$ back to itself.

Solution by Yogeshwar. Consider the words of length $2n$ with n U's and n D's: for each one, we need to sum up

$$\sum_{W \text{ word}} W\hat{0}.$$

Recall that for a sequence of U's and D's, such as

$$DDUDUU\hat{0},$$

each "up" needs to be matched with a "down" to its left: if we have any word, that just means we can match each U with any D as we want. So if we have the numbers $1, 2, \cdots, 2n$, there's $2n-1$ ways to match 1, then $2n-3$ ways to match the next number in the list that hasn't been matched, then $2n-5$, and so on, which yields a final answer of $(2n-1)!!$. $\square$

Problem 5

Show that the Bell number is given by

$$B(n) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

Solution by Wanlin. From class, we have that

$$x^n = \sum_{m=0}^{n} S(n, m)(x)_m$$

where $(x)_m$ denotes the falling power of x. Replacing x by k,

$$k^n = \sum_{m=0}^{n} S(n, m)\binom{k}{m} m!.$$

This is also equal to (due to constraints on S and the binomial coefficient)

$$= \sum_{m=0}^{k} S(n, m)\binom{k}{m} m!$$

and plugging this in, we're trying to show that

$$B(n) = \frac{1}{e}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{S(n,m)\binom{k}{m}m!}{k!}$$

and now a lot of things cancel:

$$= \frac{1}{e}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{S(n,m)}{(k-m)!}$$

Switching the order of summation,

$$= \frac{1}{e}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{S(n,m)}{(k-m)!} = \sum_{m=0}^{\infty}S(n,m) = \sum_{m=0}^{n}S(n,m),$$

which is exactly the Bell number. □

Problem 6

Find a bijection between distinct partitions and odd partitions.

Solution by Sarah. Let's say we have a partition with all distinct parts, $\lambda = \lambda_1 + \cdots + \lambda_n$, where $\lambda_i = \lambda_i^{\text{odd}} 2^{m_i}$. Then we turn this into λ_1^{odd} with multiplicity 2^{m_1}, and so on.

To do the reverse, start with an odd partition $n = \lambda_1^{a_1}\lambda_2^{a_2}\cdots$ (where the exponents are multiplicity). For each a, write it in base 2 and expand it as a sum of powers of 2: $\lambda_1^{a_1}$ then becomes $\sum \lambda_1 \cdot 2^m$, where $\sum 2^m = a_1$. □

Problem 7

Construct a non-recursive description of the Fibonacci lattice.

Solution by Chiu Yu-Cheng. First of all, we show that the number of compositions of n with all parts equal to 1 or 2 is the Fibonacci number F_{n+1}: this is because we either start a composition with 1 or 2 and then fill in the rest in F_{n-1} or F_n ways by induction.

Now, consider a graph where every vertex is a composition of n with 1s and 2s. Two vertices v_1, v_2 are connected if v_2 is obtained by v_1 by either (1) inserting a 1 anywhere to the left of the leftmost 1 in v_1 or (2) changing the leftmost 1 to a 2. Similarly, v_1 is obtained from v_1 by either (1*) removing the leftmost 1 or (2*) chanigng a 2 to a 1, when 2 is on the left of the leftmost 1.

We claim this graph is isomorphic to the Fibonacci lattice. Two vertices are only connected if the sum of the composition parts differ by 1, so we can define the rank to be that sum. To show this the isomorphism, we need to show that for any $v_1 \neq v_2$ with some rank n, the number of common upper neighbors is equal to the number of common lower neighbors, and for every v_1, it has one more upper neighbor than lower neighbor. This was how we defined the Fibonacci lattice!

If v_1 and v_2 are different, a common neighbor must be applied by having one apply 1* and the other apply 2* to get a lower neighbor, or having one apply 1 and the other apply 2 to get an upper neighbor. So some vertex v' is obtained by 1* on v_1 and 2* on v_2, we can also apply 2 to v_1 and 1 to v_2 to get a v'', so there's a bijection between upper and lower neighbors.

On the other hand, if v_1 has i copies of 2 to the left of its leftmost 1, there will be i lower neighbors and $i+1$ upper neighbors if there is no 1, and there will be $i+1$ lower neighbors and $i+2$ upper neighbors if there is a 1, which finishes the construction.

Finally, we want to show that this is actually a lattice. Note that $a \leq b$ if and only if there is a walk with only upper steps from a to b: it's not hard to check that this satisfies the poset axioms. We define "join" as follows: if either of x

and y start with 1, there is a unique lower neighborhood, so $x' \wedge y = x \wedge y$. Meanwhile, if they both start with a 2, the subgraph starting with 2 is isomorphic to the initial graph, so we can just ignore the first term of the composition and do the rest by induction. Similarly, to find meet, every element of rank at most $2n$ with n pieces can work up to $2, 2, 2, \cdots$, so this is a common upper neighbor, and then we can work down from there. $\square$