18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Last lecture, we were discussing binary trees: we found that when our trees are unlabeled, the number of binary trees on n vertices are the same as the number of full binary trees on $2n+1$ vertices, and both are equal to the Catalan number C_n.

Here's the weighted version of that:

Theorem 1

We have lots of results here!

- The number of labeled binary trees on n vertices - that is, the binary trees where the vertices are marked with the numbers 1 through n - is $n!C_n$.
- The number of increasing binary trees - that is, those whose entries are larger as we move away from the root - is $n!$.
- The number of left-increasing binary trees - that is, those who always have left children larger than parent - is $(n+1)^{n-1}$.
- The number of increasing full binary trees (those whose vertices either are leaves or have both children) on $2n+1$ vertices is A_{2n+1}, the number of alternating permutations on $2n+1$ numbers.

We can come up with simple bijections for some of these.

Proof. We can construct a bijection between increasing binary trees on n vertices and permutations in S_n: this will prove the second and fourth point. Basically, we just stick 1 in our permutation, put everything on its left before it, and put everything on its right after it: this is inductively "unfolding" the tree!

It's easy to go backwards: just find 1 and make it the root of our large tree, and for each of the left and right parts, find the smallest number in that subsequence and make those the roots of the smaller trees.

So this is a bijection, and since there are $n!$ permutations, there are $n!$ increasing binary trees. But if our tree is full, we will always get a **down-up permutation** of size $2n+1$ (which are just mirrors of the number of alternating permutations)!

As an exercise, we should find the other bijections. $\square$

It's time for us to start talking about **non-crossing paths**, which might help with the remaining problems on the problem set.

Problem 2

Let's say we have two vertices A and B in a square grid, where B is a steps up and b steps to the right: we want to find the number of lattice paths from A to B that go only up and right.

This is just the binomial coefficient $\binom{a+b}{a}$.

Problem 3

Now let's say we have two initial vertices A_1, A_2 and two targets B_1, B_2 in the square grid, and we want to connect A_1 to B_1 and A_2 to B_2. How many ways can we pick two paths that are non-crossing (in other words, they share no vertices)?

This is actually possible, and it's known as the Lindstrom or Gessel-Viennot lemma!

Let G be an acyclic digraph (no directed loops), and let's say we have selected vertices $A_1, \cdots, A_k, B_1, \cdots, B_k$. Define $N(A_1, \cdots, A_k, B_1, \cdots, B_k)$ to be the number of ways to connect each A_i with its corresponding B_is using k pairwise non-crossing paths, such that the paths do not share any vertices.

Suppose that $N(A_1, \cdots, A_k, B_{w(1)}, \cdots, B_{w(k)}) = 0$ unless w is the identity (in other words, the only way to connect them is in the original configuration). In particular, this means our lemma holds for the case where both the A_is and the B_is are arranged southeast of each other, but not for the case where the A_is are arranged going northeast and the B_is are arranged going southeast!

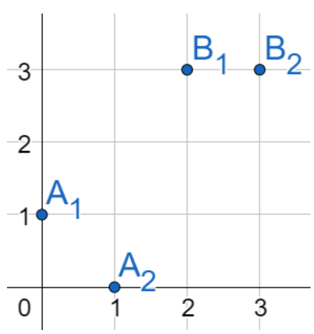
Theorem 4 (Lindstrom)

Then $N(A_1, \ldots, A_k, B_1, \cdots, B_k)$ is the determinant of the matrix C, where

$$C_{ij} = N(A_i, B_j).$$

Example 5

Let $A_1 = (0,1)$, $A_2 = (1,0)$, $B_1 = (2,3)$ and $B_2 = (3,3)$. How many noncrossing paths are there from A_1 to B_1 and A_2 to B_2?



Then by Lindstrom, we can just find the determinant

$$\det \begin{pmatrix} \binom{4}{2} & \binom{5}{2} \\ \binom{4}{1} & \binom{5}{2} \end{pmatrix} = 20.$$

Theorem 6

If we remove the bolded condition above, then we get a slightly more complicated formula: the determinant of C is equal to

$$\sum_{w \in S_n} (-1)^{\ell(w)} N(A_1, \cdots, A_k, B_{w(1)}, \cdots, B_{w(k)}).$$

It turns out the theorem and proof are both very nice in this case!

Proof. We use the involution principle. Let $P(A_1, \cdots, A_k, B_1, \cdots, B_k)$ be the number of ways to connect the A_is with B_is with any paths: this is just

$$P(A_1, B_1) \cdot P(A_2, B_2) \cdots$$

where each $P(A_i, B_i)$ is just the binomial coefficient representing the number of ways to get from start to finish with no restrictions. Now by definition of the determinant,

$$\det C = \sum_{w \in S_k} (-1)^{\ell(w)} C_{1,w(1)} C_{2,w(2)} \cdots c_{k,w(k)},$$

and now we can represent this as

$$\sum_{w \in S_k} P(A_1, \cdots, A_k, B_{w(1)}, \cdots, B_{w(k)}),$$

because we construct our paths independently. **Essentially, think of this as constructing all possible paths from the A_is to the B_is, not caring about whether or not our paths cross.** Our goal is to cancel out all families of paths with at least one crossing: we'll do this with a sign-reversing involution!

Basically, find the "first crossing" of two paths: let's say they intersect at point X. Now just take everything after point X and swap it: the rest of path P_i goes to P_j, and the rest of path P_j goes to P_i. (Keep all the other paths the same.) So now our permutation has introduced the transposition (ij), so the sign reverses.

What does "first crossing" mean? We want to make sure it's consistent for both the forward and reverse direction. We have to be careful: if we try the lexicographic minimum (i, j), that doesn't quite work! Swapping might create a different lexicographic minimum.

So instead, we **find the minimal i for which the path intersects anything**. Now, find the first intersection point on that path: find the minimal j that goes through that path, and take (i, j)! $\square$

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