18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

April 24, 2019

Recall from last class that we thought about the weighted matrix tree theorem: we have variables x_{ij} for $1 \le i, j \le n$, where $x_{ii} = 0$ (since we have no loops). Then we defined the Laplacian matrix

$$(L^{\text{out}})_{ij} = \begin{cases} \sum_{k \ne i} x_{ik} & i = j \\ -x_{ij} & i \ne j. \end{cases}$$

We could then formulate a general statement:

Theorem 1

The cofactor $(L^{\text{out}})^{rr}$ is the sum over all spanning in-trees

$$\sum_{\substack{T \text{ in-tree} \\ \text{rooted at } r}} \text{weight}(T),$$

where the weight of a tree is the product of its edges.

We will always have n^{n-2} terms on the right hand side, because we can take any undirected tree on n vertices and then direct everything towards the root! Then take the product of all x_{ij}, where there is an edge in our tree from i to j.

Let's show an interesting method for the proof.

Proof. Without loss of generality, let $r = n$. By definition, the cofactor $(L^{\text{out}})^{nn}$ is just a sum over permutations

$$\sum_{w \in S_{n-1}} (-1)^{\ell(w)} \prod_{i=1}^{n-1} (L^{\text{out}})_{iw_i}.$$

where $\ell(w)$ is the sign of the permutation w. Permutations break into cycles, so we can instead write this as a product over the individual cycles:

$$\sum_{\substack{\text{subdivision of } [n] \\ \text{into disjoint cycles} \\ C_1 \cup C_2 \cdots C_m \cup \{\text{fixed points}\}}} \pm \left(\prod_{\substack{i \to j \text{ edge of} \\ \text{some } C_j}} x_{ij} \right) \cdot \left(\prod_{f \text{ fixed point}} \sum_{j \ne f} x_{fj} \right).$$

Basically, this just writes our terms out in terms of the x_{ij}s by the definition of the Laplacian matrix!

Question 2. What do we know about the $\pm$?

Claim 2.1. For a permutation σ with a single cycle of size c,

$$(-1)^{\ell(\sigma)} = (-1)^{c-1}.$$

For example, we get a -1 from a transposition, 1 from a 3-cycle, and so on. But we also get an additional contribution of $(-1)^c$, because our Laplacian matrix has entries equal to $-x_{ij}$! This means that every cycle contributes a -1, and therefore we get a $\boxed{(-1)^m}$ at the front if we have m cycles.

How do we write this down combinatorially? Every cycle looks like some union of fixed points and cycles on the numbers 1 through $n-1$: let's say the cycles are colored in red. To represent the $\sum x_{fj}$ term, we pick an arbitrary green edge out from a fixed point into other vertices, potentially including n. So then we are summing over directed graphs H on the vertices $\{1, 2, \cdots, n\}$ with $n-1$ edges colored in red or green with the following conditions:

- For all vertices $i = 1, \cdots, n-1$, the outdegree is 1.
- The red edges form a disjoint union of cycles.

This means we can rewrite the sum as

$$\sum_{\text{graphs } H} (-1)^{\text{number of red cycles}} \prod_{i \to j \text{ edge}} x_{ij}$$

This looks a lot like what we want in the theorem. In-trees rooted at n exactly correspond to digraphs without cycles with $n-1$ edges, such that the outdegree of every non-root vertex is 1! Now the in-trees have zero red cycles, and basically our goal is for the sum of all the other terms! So our goal is to show a weight-preserving, sign-reversing involution ω on graphs H with at least one cycle.

That's not hard to create: first of all, fix a total ordering of edges $i \to j$. (We can just do this lexicographically.) Thus, we can also order all possible cycles lexicographically. Now find the first cycle in H and switch its color between green and red!

So our resulting graph has the same edges but a different parity of the number of red cycles, so those terms always cancel out. Thus our sum only leaves the contributions from in-trees, and we're done! □

Remember that we did this with the Euler's pentagonal number theorem as well, and we'll see it later on as well. This is called the **involution principle.**

Our next topic is about electrical networks, and this is related to the matrix tree theorem! Let's start by something we may have learned in physics class.

Given a graph G, think of edges as resistors - that's the only circuit element that we'll use today. Select two vertices A and B to be connected to some source of electricity.

Some current will then be generated through the edges: our goal is to find the voltages, currents, and so on.

To talk about this rigorously, we'll fix the orientation of edges. (The current is just negated if we flip the orientation, so the actual direction isn't particularly important.) Then for any edge e from vertices u to v, we have three numbers:

- A **current** I_e through the edge
- A **potential difference** or **voltage** V_e across the edge
- A **resistance** R_e which is a positive real number.

If we were physicists, we would mention units, but because this is mathematics, we don't need them.

Theorem 3 (First Kirchhoff's law)

The sum of the in-currents of any vertex is the same as the sum of the out-currents. In other words, for any vertex v,

$$\sum_{e:u\to v} I_e = \sum_{e' v\to w} I_{e'}.$$

The first Kirchhoff's law can be thought of as a "conservation of charge" statement.

Theorem 4 (Second Kirchhoff's law)

For any cycle C in our graph with edges $e_1, \cdots, e_m$, the sum of the voltages is 0 if we direct all the signs. Specifically,

$$\sum_{i=1}^{m} \pm V_{e_i} = 0$$

where we fix the orientation of our cycle, and we have a $-$ exactly when an edge disagrees with the orientation of the cycle.

Fact 5

The second law is equivalent to the existence of a **potential function**

$$U : \{\text{vertices}\} \to \mathbb{R}$$

such that for any edge $e : u \to v$, $V_e = U(v) - U(u)$.

Specifically, this means we can "ground" one vertex at voltage 0, and then we can iteratively figure out the voltages of the whole graph by following edges arbitrarily. The second law then tells us that there aren't any contradictions, since two paths will always have the same voltage!

Theorem 6 (Ohm's law)

For any edge e,

$$V_e = I_e R_e.$$

This is really a statement that defines resistance, so it doesn't really warrant a proof: it explains the setup of why we have resistances, currents, and voltages.

A typical problem is of the form "given all resistances, find all the currents and voltages." Basically, we can write a bunch of linear equations, and we can use those to calculate the answer. It turns out we can represent this as a Kirchhoff matrix: it's actually the same thing as the Laplacian matrix! Our weights will basically look like the conductances, which are inversely proportional to the resistances.