18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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We will continue the discussion about Stirling numbers today. Recall that $s(n,k)$, the Stirling numbers of the first kind, are $(-1)^{n-k}$ times the number of permutations of S_n with exactly k cycles, including fixed points. Meanwhile, $S(n,k)$, the Stirling numbers of the second kind, count the number of set-partitions of $[n]$ with k blocks, where order doesn't matter for each.

Example 1

Take $n = 4, k = 2$. There are 8 ways to have S_n as a 3-cycle and a fixed point, and 3 ways to make it two transpositions. Thus,

$$s(4,2) = (-1)^{4-2} \cdot (8+3) = 11.$$

On the other hand, $S(4,2)$ is similar, but we don't care which orientation the 3-cycle goes in. So we have 4 3-cycles and 3 pairs of transpositions, and this means

$$S(4,2) = 4 + 3 = 7.$$

From last week, we found the following main result:

Theorem 2

We have

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$, and

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$

So consider the space of all polynomials $\mathbb{R}[x]$: this is a linear space, and it has a 2 linear bases.

Fact 3

$\{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis for $\mathbb{R}[x]$, but so is $\{1, x, x(x-1), \cdots, (x)_n, \cdots\}$. So the Stirling numbers are a "change of basis."

That means that we can construct matrices for both changes of bases, and they must be invertible!

Corollary 4

If we construct matrices $(s(n,k))_{n,k\geq 0}$ and $(S(n,k))_{n,k\geq 0}$, they are invertible.

These are infinite matrices, but we can always take all indices from 0 to some fixed N.

Example 5

For $0 \leq n, k \leq 3$, we have

$$(s(n,k)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix},$$

and

$$(S(n,k)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix}.$$

Both are lower triangular matrices, since $s(n,k) = S(n,k) = 0$ for $k > n$.

We've proved the first identity from the theorem above already, but what about the second identity?

Proof. We can prove this by induction, but here's a combinatorial proof. It is enough to prove that this holds for all positive integer x, since both sides are polynomials.

Consider the set of all functions

$$F = \{f : [n] \to [x]\}.$$

Each of $1, 2, \cdots, n$ has x options, so there are a total of x^n such functions in F.

On the other hand, given a function f, we can construct a set-partition π of $[n]$ as follows: put i and j in the same block if and only if $f(i) = f(j)$. So we group elements of our set based on the output, and we want to count the number of sets that produce a given set-partition.

Fix a set-partition π with k blocks $B_1, B_2, \cdots, B_k$. The number of functions that produce this specific set-partition is $x(x-1)\cdots(x-(k-1)) = (x)_k$, since B_1 has x options for its function value, then B_2 has $(x-1)$ options (since it can't be equal to the value on B_1), and so on.

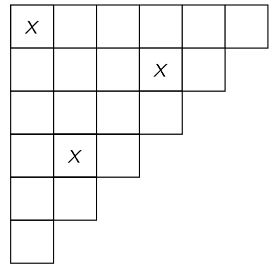
But now we're done: the number of such functions over all set partitions is

$$|F| = \sum_{k=0}^{n} S(n,k)(x)_k,$$

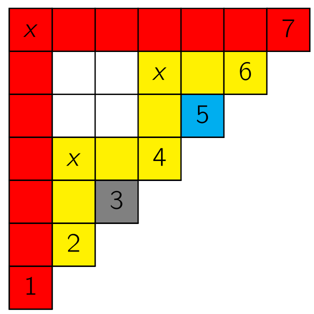
since we just pick a set-partition and then assign values to it, and we're done! □

Here's another combinatorial application of Stirling numbers of the second kind: **rook placements**. We're not allowed to place rooks on the same row and column, so they can't attack each other.

Consider the number of rook placements on a triangle board of Young tableau shape $((n-1),(n-2),\cdots,1)$. For example, here's a rook placement for $n = 7$:



Our goal is to put numbers in the corners to correspond to a set partition: for each rook, we place a hook, and then numbers connected by hooks are in the same part of the partition.



This corresponds to a set partition! Here, $2, 4, 6$ are connected, 1 and 7 are connected, and 3 and 5 are lonely, so this is the set partition

$$\pi = (1, 7 \mid 2, 4, 6 \mid 3 \mid 5).$$

Theorem 6

$S(n, k)$ is the number of non-attacking rook placements with $n - k$ rooks.

For example, notice that $S(n, n) = 1$ is the number of ways to place no rooks on the chessboard.

Definition 7

The **Bell number** $B_n = \sum_{k=0}^{n} S(n, k)$ is the total number of set-partitions of n, and this is also the total number of rook placements on $((n-1), (n-2), \cdots, 1)$.

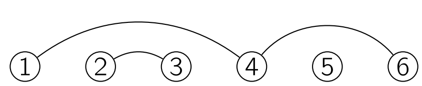
Example 8

For $n = 4$, there is 1 way to place no rooks, 6 ways to place 1 rook, 7 ways to place 2 rooks (do casework), and 1 way to place 3 rooks. This is a total of

$$B_4 = 1 + 6 + 7 + 1 = 15.$$

There are other categories of set partitions that we can use, and those come from "arc diagrams." Basically, place the numbers 1 through n on a number line, and draw an arc between adjacent entries in a block. (Arcs correspond to rooks: a to b is ath column, bth row.)

Courtesy of Agustin Garcia, here's the arc diagram for $(1, 4, 6 \mid 2, 3 \mid 5)$:



Definition 9

A set-partition is **non-crossing** if no two arcs intersect each other. Basically, if $i < j < k < l$, we can't have an arc between i and k and also between j and l.

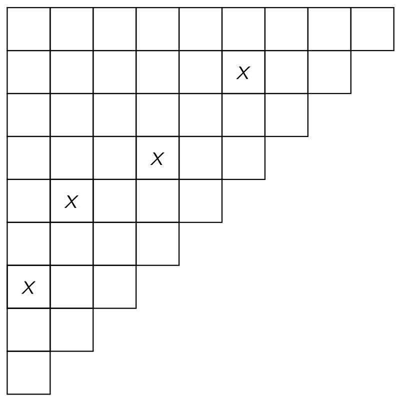
We also define a set-partition to be **non-nesting** if there is no arc "inside" another one. Basically, we can't have an arc between i and l and also between j and k.

There is an interesting duality between these two definitions:

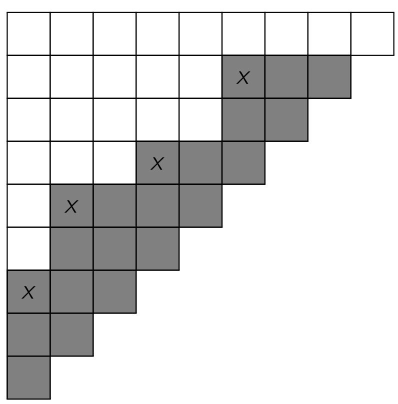
Theorem 10

The number of non-crossing set partitions of $[n]$ is equal to the number of non-nesting set partitions. This is just the Catalan number C_n.

Proof for non-nesting partitions. Use rook placements! π is non-nesting if and only if there is no rook to the right and down of another one. Basically, each rook is southwest or northeast of each other one. Here's an example:



Put a sun on the top left corner. Every rook casts a shadow:



Follow the border of the shadow. This is a Dyck path from the bottom left to the top right corner! □

As a bonus problem, figure out the analogous argument for non-crossing partitions.

Fact 11

We can draw a Pascal's triangle-like object for $S(n, k)$. There's a similar rule to calculate both $S(n, k)$ and $s(n, k)$: it's an exercise to find the recurrence relation!