

# 18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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There will be no exams in this class; 18.212 is graded entirely on problem sets (one every two or three weeks).

There are two conflicting goals for this class: everyone should understand everything, so we shouldn't go too fast, but the class shouldn't be too slow. Tell the professor if things are too fast or too slow!

As the title suggests, this is a class on combinatorics. **Combinatorics** is the area of mathematics that studies discrete objects: graphs, permutations, and various diagrams. Basically, look at objects that we can count or list.

These days, there are two main flavors: **Stanley-style** and **Erdős-style**. Stanley-style or **enumerative / algebraic / geometric** combinatorics deals with counting objects or their connections with algebra and geometry. On the other hand, Erdős-style is also called **extremal / probabilistic** combinatorics. This class will deal with the first half: we'll get some of the other half in 18.218!

### Fact 1

In our class, we'll often have  $A = B$ : some number of objects is equal to some other number. For example, the number of labeled trees on  $n$  vertices is  $n^{n-2}$ : this is called the Cayley formula.

Meanwhile, Erdős-style combinatorics cares about things like "what is the probability a random graph is connected?". We'll get statements like  $A \sim n \log n$ . Notice that this is actually kind of related to the Cayley formula.

No textbooks are required for this class. The recommended textbooks for this class are

- Richard Stanley's *Algebraic Combinatorics*; we can see the pdf file on his website.
- Richard Stanley's *Enumerative Combinatorics*, volumes 1 and 2. This is a graduate-level book, and there is a lot of material there.
- von Lint and Wilson's *A Course in Combinatorics*. This is a very nice book but is a bit outdated.

We're going to start by doing something not from Richard Stanley's *Algebraic Combinatorics* (which directly contradicts what Professor Postnikov just said).

### Fact 2

Catalan numbers are the professor's favorite object in combinatorics.

On the other hand, Richard Stanley did compile hundreds of Catalan number interpretations separately from the book so he has an excuse.

**Definition 3**

The  $n$ th **Catalan number**  $C_n$  is the number of sequences  $(\epsilon_1, \epsilon_2, \dots, \epsilon_{2n})$  where all  $\epsilon_i = \pm 1$ , there are  $n$  1s and  $n$   $-1$ s, and the partial sum  $\epsilon_1 + \dots + \epsilon_i \geq 0$  for all  $i$ .

**Definition 4**

Such sequences are often represented by **Dyck paths**, which contain **up** and **down** edges. Each up step corresponds to a vector  $(1, 1)$ , and each down step corresponds to  $(1, -1)$ .

Lining up all of the vectors, we have a path from  $(0, 0)$  to  $(2n, 0)$ , where the entire path is contained in the upper-half plane (including the  $x$ -axis).

It is very easy to count this for small  $n$ :

$n$	0	1	2	3	4	5
$C_n$	1	1	2	5	14	42

There is an explicit formula that we will get to, but first, let's ask the question: practically, what is mathematics? If objects have no real-world applications, they might be meaningless, so here's a real-world application of Catalan numbers.

**Example 5 (Drunk man problem)**

This is not a problem that drunk people have (though drunk people have a lot of problems). A drunk person is on a road, and he cannot tell left from right, so he has a  $\frac{1}{2}$  chance of going to the left by 1 and a  $\frac{1}{2}$  chance of going to the right by 1. (This is the simplest example of a random walk.) However, he is at  $x = 1$ , and there is a cliff starting at  $x = 0$ . What is the probability he survives?

As a sidenote, the probability in this class is very different from the probabilistic method in professor Yufei Zhao's class.

So what's the probability he falls? He can make one step to the left, or one step to the right and two steps to the left, and so on. But these kind of paths correspond to Dyck paths! The drunk guy will take  $2n$  steps and then have a  $\frac{1}{2}$  chance of taking that last  $2n + 1$ th step, and those first  $2n$  steps must stay at  $x > 0$ . So our probability is

$$P = \frac{1}{2} \sum_{n \geq 0} \frac{C_n}{2^{2n}}$$

As a slight generalization, we can assume the road is going at some slope, so he goes left and right with different probabilities  $1 - p$  and  $p$ , respectively. This gives instead

$$P = p \sum_{n \geq 0} C_n p^n (1 - p)^n$$

and this is called a **biased random walk**: can we calculate this probability as well?

First of all, the solution to this problem doesn't require Catalan numbers. Here's an elementary proof:

*Solution.* As before, let's say the cliff is at  $x = 0$ , and the guy starts at  $x = 1$ . Let's say that there is a house where the drunk guy lives at  $x = N$ , and he is safe if he reaches it before falling.

Let  $P(i, N)$  denote the probability that the man survives if he starts at position  $i$  and his house is at position  $N$ . Our goal is actually to calculate the probability when  $N \rightarrow \infty$ . Notice that  $P(N, N) = 1$  and  $P(0, N) = 0$ .

**Fact 6**

Professor Postnikov is not implying that guys are more likely to be drunk. They might be a girl.

Now for all  $0 < i < N$ ,

$$P_i = \frac{1}{2}(P_{i-1} + P_{i+1})$$

since from position  $i$ , there's a one-half chance of going to  $i - 1$  and a one-half change of going to  $i + 1$ . So we actually have an arithmetic progression:

$$P(i, N) = \frac{i}{N}.$$

But at our initial problem, there is no house, so we can take the limit as  $N \rightarrow \infty$ , and this is obviously 0. So the guy will always fall off the cliff. □

**Corollary 7**

Don't drink if you are at the edge of a cliff, especially if you're under 21 years old.

On the other hand, we can also learn more about the generalized problem if we use Catalan numbers. Let's start by finding a way to express each Catalan number in terms of the previous ones:

**Proposition 8** (Recurrence relation for the Catalan numbers)

For all  $n \geq 1$ ,

$$C_n = \sum_{k=1}^n C_{k-1}C_{n-k}$$

and as a base case,  $C_0 = 1$ .

*Proof.* We want to somehow break a Dyck path into smaller Dyck paths. Dyck paths are allowed to touch the  $x$ -axis before  $(2n, 0)$ : find the first place where this happens, and call it  $(2k, 0)$ . (It will be an even  $x$ -coordinate because we need an equal number of up and down moves.)

The part of the path from 0 to  $2k$  is strictly above the  $x$ -axis, so it is an up move, a Dyck path on  $2k - 2$  moves ( $C_{k-1}$ ), and then a down move, followed by a Dyck path on  $2n - 2k$  moves ( $C_{n-k}$ ). This decomposition is unique, because we specified that we took the first point of contact with the  $x$ -axis! Since  $k$  can be anything from 1 to  $n$ , this gives our recurrence relation as desired. □

We can actually write this as a **generating function**: given any sequence  $C_0, C_1, \dots$ , we can make a generating function

$$f(x) = c_0 + c_1x + c_2x^2 + \dots$$

We can ask questions of convergence or divergence, but in combinatorics we avoid these questions by treating these as **formal power series**: infinite series with some coefficients, where we can add and multiply the series in a well-defined way.

**Proposition 9** (Generating function for the Catalan numbers)

The recurrence relation is equivalent to saying that

$$f(x) = xf(x)^2 - 1.$$

*Proof.* To show this, we'll extract coefficients on the left and right side and see that they are the same:

$$[x^n]f(x) = C_n$$

(where this notation means the  $x^n$  coefficient of  $f$ ), and

$$[x^n]xf(x) = C_{n-1}$$

since we're shifting our coefficients  $C_{k-1}x^{k-1} \rightarrow C_{k-1}x^k$ . Now

$$[x^n]f(x)^2 = C_n \cdot C_0 + C_{n-1} \cdot C_1 + \cdots + C_0 \cdot C_n$$

by expanding out our coefficients, and

$$[x^n]xf(x)^2 = C_{n-1} \cdot C_0 + C_{n-2} \cdot C_1 + \cdots + C_0 \cdot C_{n-1} = C_n$$

by our recurrence relation, and finally we just check our constant term (which is 1 on both sides). We're done!  $\square$

Finally, how do we solve our functional equation  $f(x) = xf(x)^2 + 1$ ? We solve as a quadratic in  $f$ ! We find, by the quadratic formula, that

$$xf^2 - f + 1 = 0 \implies f = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

There's some ambiguity about plus or minus, but  $f(x)$  is a power series with constant term 1, so  $f(x)$  can't diverge. Thus, we need to make sure the numerator is 0 at  $x = 0$  as well, and we have our final generating function

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

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