18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Quick review: last time we defined the Catalan numbers and bijected them to Dyck paths. Next, we discovered a recurrence relation for these Catalan numbers, and from that, we found our generating function

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Let's do an application of this generating function idea to the biased random walk problem. Remember that we are at the edge of a cliff, and we have a probability p of walking to the right and $1-p$ of walking to the left. We start at the very edge.

Proposition 1

The probability that the man falls is

$$(1-p)f(p(1-p)) = \begin{cases} 1, p \le \frac{1}{2} \\ \frac{1-p}{p}, p > \frac{1}{2} \end{cases}.$$

Fact 2

If you're going to drink at the edge of a cliff, make sure it slopes down to the right.

We'll start with a fact that many of us probably know well:

Proposition 3 (Binomial Theorem)

For a real number y and nonnegative integer a,

$$(1+y)^a = 1 + \binom{a}{1}y + \binom{a}{2}y^2 + \binom{a}{3}y^3 + \cdots.$$

This is actually true for any value of a at all! So Pascal's triangle is part of the picture, but it doesn't tell us the whole story. We just need to define (for a nonnegative integer k)

$$\binom{a}{k} \equiv \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!}.$$

This follows by writing out a Taylor expansion of the left hand side: the $a(a-1)(a-2)\cdots$ comes from the power rule.

So we can apply this to our generating function by setting $y = -4x$, $a = \frac{1}{2}$: we can ignore the constant term, and for all $n \geq 1$,

$$f(x) = \frac{1-(1-4x)^{-1/2}}{2x} \implies [x^n]f(x) = -\frac{1}{2}[x^{n+1}](1-4x)^{1/2}$$

which is (by the binomial theorem)

$$[x^n]f(x) = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1}$$

and after expansion, this finally gives us a nice formula:

Theorem 4 (Closed form for the Catalan numbers)
The nth Catalan number is

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

But this is a very nice expression: is there a combinatorial proof instead?

Proof by reflection principle. Recall that C_n counts the number of Dyck paths that are above the x-axis from $(0,0)$ to $(2n,0)$.

First of all, if we remove the "above the x-axis" condition, we just want the number of paths that have n ups and n downs: this is just $\binom{2n}{n}$. We now want to remove all paths that were overcounted: how many non-Dyck paths are there?

Each of these paths must intersect the line $y=-1$ at least once. Pick the first intersection, and take the portion of the path from that intersection to the end $(2n,0)$. Reflect it over $y=-1$, and now the endpoint of our path is $(2n,-2)$.

This new path has $n-1$ up steps and $n+1$ down steps, and such paths are in bijection with the non-Dyck paths! This is because we can find the first intersection with $y=-1$ and do the reflection again. Since there are $\binom{2n}{n-1}$ such paths, the number of Dyck paths is

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

□

But as a combinatorialist, professor Postnikov doesn't like subtraction. Here's a proof that doesn't require subtraction!

Proof by necklaces. Start with the binomial coefficient $\binom{2n+1}{n}$, which is the number of sequences $\varepsilon_1, \cdots, \varepsilon_{2n+1}$ with n +1s and $n+1$ -1s.

For example, for $n=2$, consider the sequence $(+,-,-,+,-)$. Do a **cyclic shift** of the sequence: move the first term to the end to get $(-,-,+,-,+)$. Keep doing this: we get $(-,+,-,+,-)$, $(+,-,+,-,-)$, and $(-,+,-,-,+)$. In general, there are $2n+1$ cyclic shifts.

So we can arrange these on a necklace (as black and white beads). The necklace has n +s and $n+1$ -s, and each of the cyclic shifts corresponds to cutting the necklace at one spot.

Claim 4.1. All cyclic shifts are different.

This is left as an exercise! The idea is that if two cyclic shifts were the same, we'd have subgroups that are identical, which is bad because the difference in the number of +s and -s is 1.

So this means the number of total necklaces that exists is

$$\frac{\binom{2n+1}{n}}{2n+1}.$$

Claim 4.2. For each necklace, there is a unique sequence $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2n+1})$ among all $2n+1$ ways such that $\varepsilon_1, \cdots, \varepsilon_{2n}$ form a Catalan sequence and $\varepsilon_{2n+1} = -1$.

This was also left as an exercise! The idea is to find the "lowest point" on the Dyck path.

Once that's resolved, notice that the number of Catalan sequences is therefore also equal to

$$\frac{\binom{2n+1}{n}}{2n+1} = \frac{1}{n+1}\binom{2n}{n},$$

as desired. □

This is a very good proof, because we're actually doing some kind of natural division: the cyclic shifts explain where the binomial coefficient is reduced.

A few more things about Catalan numbers: we'll find that they appear everywhere!

Fact 5

C_n is the number of triangulations of an $n+2$-gon.

For example, given a hexagon, there are $C_4 = 14$ ways to cut it into 4 triangles. (There are 6 of them where all three diagonals drawn come from one vertex, 6 in a zig-zag pattern, and 2 by drawing an equilateral triangle.)

Fact 6

C_n is also the number of valid parenthesizations of $n+1$ letters.

For $n = 4$, we want to put some parentheses in the expression

$$a \;\; b \;\; c \;\; d \;\; e.$$

One way is $((a(bc))(de))$.

Fact 7

Finally, C_n is the number of plane binary trees with n non-leaf vertices and $n+1$ leaves.

Basically, start with a root, and repeatedly take a leaf and draw a left and right child for it. We can show that the number of leaves and non-leaves will always differ by 1.

Proposition 8

There's a pretty simple bijection between all three of these examples!

Proof. Starting with an $n+2$-gon, the binary tree will be the dual graph to the triangulation: pick one of the sides of our polygon to start constructing our tree. Non-leaf vertices will correspond to triangles, and the triangle starting to the edge is our root. This means we can convert any triangulation to a tree and vice versa.

Then we can take our binary tree and use "common neighbor" as a set of parentheses, where we take leaves from left to right! This finishes the bijection between trees and parenthesizations. □

But the bijection to Dyck paths is less obvious; this will be an exercise!