18.218 Topics in Combinatorics Spring 2021 – Lectures 11 - 12

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Our goal in this lecture as well as the next lecture will be to demonstrate the power of analytical tools in Extremal Combinatorics. More specifically, we will see an instance of the junta method, a successful method in the area that has been introduced in the early 2000's and has recently risen in popularity.

1 Erdős-Ko-Rado type theorem

For $k \leqslant n/2$, a family of subsets $\mathcal{F} \subseteq \binom{[n]}{k}$ is said to be intersecting if for any $A, B \in \mathcal{F}$ we have that $A \cap B \neq \emptyset$. Given this definition, one may wonder: (1) How large can an intersecting family $\mathcal{F}$ be? (2) What is the structure of the extremal families $\mathcal{F}$? How about stability results?

The solution to this particular question is the well know, Erdos-Ko-Rado theorem, which asserts that the size of the largest intersecting family is $\binom{n}{k-1}$, and furthermore the extremal families are precisely dictatorships, i.e. $\{\, A \mid |A| = k, i \in A\}$ for each $i \in [n]$. Can one prove a general structural result about intersecting families? Can they always be approximated by a considerably simpler families? This question is the main question we will consider in this as well as in the next lecture.

Before we dive into that, in order to get a feeling to where the method applies, we give several more examples of closely related problems.

1. **t-wise intersecting families.** Suppose $k \leqslant 0.49n$; how large can a family $\mathcal{F} \subseteq \binom{[n]}{k}$ be if for any $A, B \in \mathcal{F}$ it holds that $|A \cap B| \geqslant t$? Thinking about this problem for several minutes, one comes up with a candidate extremal example such as $\mathcal{F} = \{\, A \mid 1, \ldots, t \in A\}$, and with a few more minutes one realizes that a more general class of families is $\mathcal{F} = \{\, A \mid |A \cap [t+2r]| \geqslant t + r\}$ for any $r \in \mathbb{N}$. These turn out to be the extremal families and the junta method applies here; it is not a coincidence that these families are juntas.

2. Forbidden intersections. How large can a family $\mathcal{F} \subseteq \binom{[n]}{k}$ be if for any $A, B \in \mathcal{F}$ it holds that $|A \cap B| \neq t - 1$. Note that any t intersecting family is an immediate candidate, and it turns out that these are also the extremal examples. Both this question and the previous question have analog in different domains, such as larger alphabets $[m]^n$, S_n, vector spaces and more.

3. Suppose $n \geqslant sk$. How large can $\mathcal{F} \subseteq \binom{[n]}{k}$ be if it doesn't contain a matching of size s, i.e. $A_1, \ldots, A_s \in \mathcal{F}$ that are pairwise disjoint?

The solution to these problems is considerably more difficult and requires additional analytical tools as well as more advanced ideas in the junta method. In the example we show, we will only see the basic ideas and set-up.

Throughout this lecture, we will move back and fourth between the language of families of subsets $\mathcal{F} \subseteq P([n])$, and Boolean functions $f\colon \{0,1\}^n \to \{0,1\}$, by identifying a subset $A \subset [n]$ with its indicator vector $1_A \in \{0,1\}^n$, and a family of subsets $\mathcal{F}$ with the function $f(1_A) = 1_{A \in \mathcal{F}}$.

2 The p-biased cube

Instead of discussing uniform-sized families, it will be much more convenient for us to think of their product measure analogs. [1] That is, instead of thinking about families in $\binom{[n]}{k}$, we will think of families of subsets, or equivalently subsets of $\{0,1\}^n$, and our measure will be the p-biased measure with $p = k/n$. This is the measure on $\{0,1\}^n$ defined as $\mu_p(x) = p^{|x|}(1-p)^{n-|x|}$, where $|x|$ is the number of 1's in x. Intuitively, since most of the measure of μ_p lies on points whose number of ones is $pn \pm O(\sqrt{pn}) = k \pm \sqrt{k}$ we expect the p-biased measure of the largest intersecting family to be closely related to the size of the largest intersecting family in $\binom{[n]}{k}$. This is indeed the case, and hence from now on we will focus on the former problem.

Considering the measure space $(\{0,1\}^n, \mu_p)$, one can generalize much of what we've seen so far in the course with one important remark. Roughly speaking, the situation is vastly different depending on the range of p.

1. The range in which p is bounded away from 0 and 1, i.e. $0 < \zeta \leqslant p \leqslant 1-\zeta$ for some absolute constant ζ. In this case, almost everything behaves exactly as in the $p = 1/2$ case. In particular, the hypercontractive inequality holds (albeit with slightly worse constants), the KKL theorem and the Friedgut's junta theorem also hold (albeit with constants depending on ζ); we will not repeat the proofs of these results here, and refer the interested reader to the book.

2. The range in which p or $1-p$ decay with n, say $p = 1/\sqrt{n}$. This is a much more challenging range from the analytical perspective, and almost all of the results we've seen so far in this course completely break. Given time constraints, we will discuss this range later on in the course.

3 The main result

The key result we will prove in this and the next lecture is the following theorem, due to Dinur and Friedgut.

Theorem 3.1. *For all $\zeta > 0$, $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that the following holds. If $\mathcal{F} \subseteq \{0,1\}^n$ is an intersecting family, and $\zeta < p < \frac{1}{2} - \zeta$, then there exists an intersecting J-junta $\mathcal{J} \subseteq \{0,1\}^n$ such that $\mu_p(\mathcal{F} \setminus \mathcal{J}) \leqslant \varepsilon$.*

In words, the theorem asserts that an intersecting family is nearly contained in a special intersecting family, i.e. a junta. The proof of this theorem incorporates several components, some of which we have already seen, while others we have not:

1. First, we will define the notion of "monotonicity" of families/ functions, and argue that one may assume $\mathcal{F}$ to be monotone increasing.

2. Secondly, we will define a certain notion of "pseudo-randomness", and show that any family of subsets may be decomposed into a small number of sub-families that are pseudo-random (+junk sub-families that will be small in measure).

3. Thirdly, we will show that pseudo-random families contain intersections of any constant size (we will actually need, and prove, a stronger statement along these lines which deals with two pseudo-random families).

[1] There is a standard way to move between these two settings that we will not present here.

3.1 Upwards closure

Given a family $\mathcal{F} \subseteq P([n])$, the upwards closure of $\mathcal{F}$ is defined as

$$\mathcal{F}^{\uparrow} = \{\, A \subseteq [n] \mid \exists B \in \mathcal{F}, A \supseteq B \} \,.$$

Claim 3.2. *Suppose $\mathcal{F}$ is an intersecting family. Then $\mathcal{F}^{\uparrow}$ is also intersecting.*

Proof. Let $A, A' \in \mathcal{F}^{\uparrow}$. Then by definition there are $B, B' \in \mathcal{F}$ such that $B \subseteq A$, $B' \subseteq A'$. As $\mathcal{F}$ is intersecting, $B \cap B'$ is non-empty, and as $A \cap A' \supseteq B \cap B'$, we get that $A \cap A'$ is also non-empty. $\square$

Note that if we proved the theorem for $\mathcal{F}^{\uparrow}$, then we're done as $\mu_p(\mathcal{F} \setminus \mathcal{J}) \leqslant \mu_p(\mathcal{F}^{\uparrow} \setminus \mathcal{J})$ for any family $\mathcal{J}$. We thus assume that $\mathcal{F} = \mathcal{F}^{\uparrow}$ henceforth, i.e. that $\mathcal{F}$ is upwards closed.

3.2 Quasi-randomness

Next, we introduce a notion of quasi-randomness that will be useful for us in this context.

Definition 3.3. *Let $r \in \mathbb{N}$, $\varepsilon > 0$ and $0 < p < 1$. We say a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is (r, ε) quasi-random with respect to p if for any $R \subseteq [n]$ of size at most R, and any $z \in \{0,1\}^R$ it holds that*

$$|\mu_p(f_{R \to z}) - \mu_p(f)| \leqslant \varepsilon.$$

In words, a function f is (r, ε) quasi-random if any restriction of size at most r can change the average of the function by at most ε.

Remark 3.4. *In the homework assignment you will see a connection between this notion and a function having small Fourier coefficients on the low levels.*

Definition 3.5. *Let $r \in \mathbb{N}$, $\varepsilon > 0$ and $0 < p < 1$. We say $\mathcal{F}$ is (r, ε) quasi-random with respect to p if $1_{\mathcal{F}}$ is (r, ε) quasi-random with respect to p.*

We now state and prove a regularity lemma suitable for the notion of quasi-randomness we have just defined.

Lemma 3.6. *For all $r \in \mathbb{N}$, $\varepsilon > 0$, $\delta, \zeta > 0$ there exists $J \in \mathbb{N}$ such that the following holds. If $\zeta \leqslant p \leqslant 1 - \zeta$, and $f\colon \{0,1\}^n \to \{0,1\}$ is any Boolean function, then there exists a set $T \subseteq [n]$ of size at most J, such that*

$$\Pr_{z \sim \mu_p^T}\left[f_{T \to z} \text{ is not } (r, \varepsilon) \text{ quasi-random}\right] \leqslant \delta.$$

In words, the lemma asserts that for any function f we may find a constant size set of variables T, such that randomly restricting them, the resulting function is quasi-random.

Proof. The proof is iterative, and is based on the construction of an appropriate potential function. Starting with $T = \emptyset$, we define a potential function $p : P([n]) \to [0,1]$ by

$$p(T) = \mathop{\mathbb{E}}_{z \sim \mu_p^T}\left[(\mu_p(f_{T \to z}) - \mu_p(f))^2\right].$$

Our goal will show that if we have some $T \subseteq [n]$ for which the condition fails, i.e. for which many of the restrictions $f_{T \to z}$ are not quasi-random, then we may find T' which is a bit larger and $p(T')$ is substantially larger than $p(T)$. Thus, as $p(T)$ is always bounded by 1, the process would terminate in constantly many steps, in which case T necessarily satisfies the condition of the lemma.

Suppose we have T for which the condition fails, and let $Z = \{ z \mid f_{T\to z} \text{ is not } (r,\varepsilon) \text{ quasi-random}\}$. For each $z \in Z$, pick $R_z \subseteq [n] \setminus T$ of size at most r and $w \in \{0,1\}^{R_z}$ demonstrating that $f_{T\to z}$ is not (r,ε) quasi-random, and define $T' = T \cup \bigcup_{z\in Z} R_z$.

Bounding the size of T'. Note that $|Z| \leqslant 2^{|T|}$, and so $|T'| \leqslant |T|\,|Z|\,r \leqslant |T|\,2^{|T|} \leqslant r2^{2|T|}$.

Analyzing the potential function. Let us write $R = \bigcup_{z\in Z} R_z$ and

$$p(T') = \mathop{\mathbb{E}}_{z\sim\mu_p^T}\left[\mathop{\mathbb{E}}_{z'\sim\mu_p^R}\left[(\mu_p(f_{T\to z,R\to z'}) - \mu_p(f))^2\right]\right].$$

First, we note that by Cauchy-Schwarz, for each z

$$\mathop{\mathbb{E}}_{z'\sim\mu_p^R}\left[(\mu_p(f_{T\to z,R\to z'}) - \mu_p(f))^2\right] \geqslant (\mathop{\mathbb{E}}_{z'\sim\mu_p^R}\left[\mu_p(f_{T\to z,R\to z'}) - \mu_p(f)\right])^2 = (\mu_p(f_{T\to z}) - \mu_p(f))^2,$$

which immediately shows that $p(T') \geqslant p(T)$. The essence in our argument is to show that for $z \in Z$, there is substantial slack in the above inequality, which will give us the desired increase in the potential.

Fix $z \in Z$. As before, we may write

$$\begin{aligned}\mathop{\mathbb{E}}_{z'\sim\mu_p^R}\left[(\mu_p(f_{T\to z,R\to z'}) - \mu_p(f))^2\right] &= \mathop{\mathbb{E}}_{w\sim\mu_p^{R_z}}\left[\mathop{\mathbb{E}}_{w'\sim\mu_p^{R\setminus R_z}}\left[(\mu_p(f_{T\to z,R_z\to w,R\setminus R_z\to w'}) - \mu_p(f))^2\right]\right] \\ &\geqslant \mathop{\mathbb{E}}_{w\sim\mu_p^{R_z}}\left[(\mathop{\mathbb{E}}_{w'\sim\mu_p^{R\setminus R_z}}\left[\mu_p(f_{T\to z,R_z\to w,R\setminus R_z\to w'}) - \mu_p(f)\right])^2\right] \\ &= \mathop{\mathbb{E}}_{w\sim\mu_p^{R_z}}\left[(\mu_p(f_{T\to z,R_z\to w}) - \mu_p(f))^2\right].\end{aligned}$$

Consider the random variable $X_z\colon \{0,1\}^{R_z} \to [-1,1]$, whose value at w is $X_z(w) = \mu_p(f_{T\to z,R_z\to w}) - \mu_p(f)$. Then note that

$$\mathop{\mathbb{E}}_{w\sim\mu_p^{R_z}}\left[(\mu_p(f_{T\to z,R_z\to w}) - \mu_p(f))^2\right] - (\mu_p(f_{T\to z}) - \mu_p(f))^2 = \mathop{\mathbb{E}}_{w}\left[X_z(w)^2\right] - \mathop{\mathbb{E}}_{w}\left[X_z(w)\right]^2 = \mathsf{var}(X_z),$$

so it is enough to lower bound the variance of X. By definition,

$$\mathsf{var}(X_z) = \mathop{\mathbb{E}}_{w}\left[(X_z(w) - \mathbb{E}\left[X_z(w)\right])^2\right] = \mathop{\mathbb{E}}_{w}\left[(\mu_p(f_{T\to z,R_z\to w}) - \mu_p(f_{T\to z}))^2\right],$$

and as R_z is a witness that $f_{T\to z}$ is not (r,ε) quasi-random, there is some $w^\star \in \{0,1\}^{Rz}$ such that the inner difference is at least ε^2 in absolute value. As $\mu_p(w)^\star \geqslant \zeta^r$ (by assumption on p and the fact that the size of R_z is at most r), it follows that $\mathsf{var}(X_z) \geqslant \zeta^r\varepsilon^2$.

Combining everything, we get that

$$\begin{aligned}p(T') - p(T) &= \mathop{\mathbb{E}}_{z\sim\mu_p^T}\left[\mathop{\mathbb{E}}_{z'\sim\mu_p^R}\left[(\mu_p(f_{T\to z,R\to z'}) - \mu_p(f))^2\right] - (\mu_p(f_{T\to z}) - \mu_p(f))^2\right] \\ &\geqslant \mathop{\mathbb{E}}_{z\in\mu_p^T}\left[1_{z\in Z}\mathsf{var}(X_z)\right] \geqslant \mathop{\mathbb{E}}_{z\in\mu_p^T}\left[1_{z\in Z}\zeta^r\varepsilon^2\right] \geqslant \delta\zeta^r\varepsilon^2.\end{aligned}$$

Concluding, we get that the process terminates after at most $\frac{1}{\delta\zeta^r\varepsilon^2}$ steps and finds a set T whose size depends only on $r, \varepsilon, \delta, \zeta$ satisfying the condition of the lemma. $\square$

In the next lecture, we will use this regularity lemma in order to prove Theorem 3.1.

4 Properties of quasi-random families

4.1 Quasi-random families have a sharp threshold

We begin with a neat application of Friedgut's theorem, showing that quasi-random functions have a sharp threshold.

Lemma 4.1. *For all $\zeta, \alpha > 0$, there exists $r \in \mathbb{N}$, $\varepsilon > 0$ such that the following holds. Suppose $\zeta < p < 1-\zeta$ and $f\colon \{0,1\}^n \to \{0,1\}$ is monotone with $\mu_p(f) \geqslant \alpha$. If f is (r, ε) quasi-random, then $\mu_{p+\zeta/2}(f) \geqslant 0.9$.*

Proof. Suppose towards contradiction that $\mu_{p+\zeta/2}(f) \leqslant 0.9$. Then $\frac{\mu_{p+\zeta/2}(f)-\mu_p(f)}{(p+\zeta/2)-p} \leqslant \frac{2}{\zeta}$. By Lagrange's mean-value theorem, there is $p' \in (p, p+\zeta/2)$ such that

$$\frac{d\mu_q(f)}{dq}(p') = \frac{\mu_{p+\zeta/2}(f) - \mu_p(f)}{(p+\zeta/2)-p} \leqslant \frac{2}{\zeta}.$$

By the Russo-Margulis lemma, as f is monotone, $I[f;\mu_{p'}] = \frac{d\mu_q(f)}{dq}(p')$, so by Friedgut's junta theorem there is $J \in \mathbb{N}$ depending only on ζ, α, and $g\colon \{0,1\}^n \to \{0,1\}$ a J-junta such that $\Pr_{x\sim\mu_{p'}}[f(x) \neq g(x)] \leqslant \frac{\alpha}{1000}$.

We choose the parameters r, ε of the quasi-randomness now as $r = J$ and $\varepsilon = \frac{\alpha}{4}$.

Let R be the set of variables g depends on. We argue that

$$\Pr_{x\sim\mu_{p'}^R}[g_{R\to x} \equiv 0] > 10^{-2}.$$

Indeed,

$$0.9 \geqslant \mu_{p+\xi}(f) \geqslant \mu_{p'}(f) \geqslant \mu_{p'}(g) - \Pr_{x\sim\mu_{p'}}[f(x) \neq g(x)],$$

so $\mu_{p'}(g) \leqslant 0.9 + \frac{\alpha}{1000} < 0.99$.

Choose $x \sim \mu_{p'}^R$, and consider the following two events:

$$g_{R\to x} \equiv 0, \qquad\qquad \Pr_{w\sim\mu_{p'}^{[n]\setminus R}}[f_{R\to x}(w) \neq g_{R\to x}(w)] \leqslant \frac{\alpha}{2}.$$

The first event holds with probability $> 10^{-2}$ as we have just seen; as for the second event,

$$\mathop{\mathbb{E}}_x\left[\Pr_{w\sim\mu_{p'}^{[n]\setminus R}}[f_{R\to x}(w) \neq g_{R\to x}(w)]\right] = \Pr_{y\sim\mu_{p'}}[f(y) \neq g(y)] \leqslant \frac{\alpha}{1000},$$

so by Markov's inequality the second event holds with probability at least $1 - \frac{2}{10^3}$.

As the sum of the probabilities of the events exceeds 1, it follows that there is x for which both events hold. In that case, we get that

$$\Pr_{w\sim\mu_{p'}^{[n]\setminus R}}[f_{R\to x}(w) \neq 0] \leqslant \frac{\alpha}{2},$$

which by monotonicity implies that

$$\Pr_{w\sim\mu_{p}^{[n]\setminus R}}[f_{R\to x}(w) \neq 0] \leqslant \frac{\alpha}{2},$$

and in particular $\mu(f_{R\to x}) \leqslant \frac{\alpha}{2}$. The assignment (R, x) now contradicts the (r, ε) quasi-randomness of f. $\square$

4.2 Quasi-random families are not cross intersecting

Armed with Lemma 4.1, we are almost ready to prove Theorem 3.1. But first, we need a simple version of the Erdős-Ko-Rado theorem.

Claim 4.2. *Suppose $\mathcal{G}, \mathcal{H} \subseteq P([n])$ are such that $\mu_{1/2}(\mathcal{G}) + \mu_{1/2}(\mathcal{H}) > 1$. Then there are disjoint $F \in \mathcal{F}, G \in \mathcal{G}$.*

Proof. Sample $A \subseteq [n]$ uniformly at random, and write $\mu_{1/2}(\mathcal{G}) = \mathbb{E}_A [1_{A\in\mathcal{G}}]$, and $\mu_{1/2}(\mathcal{H}) = \mathbb{E}_A \left[1_{\bar{A}\in\mathcal{H}}\right]$; the last identity is true since the distribution of $\bar{A}$ is uniform among all subsets of $[n]$. Thus, by the premise

$$1 < \mu_{1/2}(\mathcal{G}) + \mu_{1/2}(\mathcal{H}) = \mathop{\mathbb{E}}_{A} \left[1_{A\in\mathcal{G}} + 1_{\bar{A}\in\mathcal{H}}\right],$$

so with positive probability $1_{A\in\mathcal{G}} + 1_{\bar{A}\in\mathcal{H}} > 1$, i.e. there is A such that $A \in \mathcal{G}$, $\bar{A} \in \mathcal{H}$, and we take $G = A$, $H = \bar{A}$. □

We now combine Lemma 4.1 and Claim 4.2 to show that quasi-random families are not cross intersecting.

Lemma 4.3. *For all $\alpha, \zeta > 0$ there exists $r \in \mathbb{N}$, $\varepsilon > 0$ such that the following holds. Suppose $\zeta < p \leqslant \frac{1}{2} - \zeta$ and $\mathcal{G}, \mathcal{H} \subseteq P([n])$ are monotone families such that $\mu_p(\mathcal{G}), \mu_p(\mathcal{H}) \geqslant \alpha$, and each one of them is (r, ε)-quasi-random.*

Then there are disjoint $G \in \mathcal{G}$, $H \in \mathcal{H}$.

Proof. Take $\xi = \frac{1}{2} - p \geqslant \zeta$, and find (r, ε) from Lemma 4.1. Then we get that

$$\mu_{p+\xi}(\mathcal{G}), \mu_{p+\xi}(\mathcal{H}) \geqslant 0.9,$$

and as $p + \xi = \frac{1}{2}$, we conclude from Claim 4.2 that $\mathcal{F}, \mathcal{G}$ cross contain disjoint sets, as desired. □

4.3 Proof of Theorem 3.1

Fix $\zeta, \varepsilon > 0$ as in the theorem.

As we argued last time, by moving from $\mathcal{F}$ to its upwards closure, we may assume $\mathcal{F}$ to be upwards closed.

Set $\alpha = \varepsilon/2$, and choose (r, ε') from Lemma 4.3. We now take J from Lemma 3.6 for r, ε', $\delta = \varepsilon/2$ and ζ. Applying Lemma 3.6 on $f = 1_{\mathcal{F}}$, we find a set $T \subseteq [n]$ of size at most J, such that

$$\Pr_{z\sim\mu_p^T} \left[f_{T\to z} \text{ is not } (r, \varepsilon') \text{ quasi-random}\right] \leqslant \delta.$$

Define

$$\mathcal{T} = \left\{ A \subseteq T \mid f_{T\to 1_A} \text{ is } (r, \varepsilon') \text{ quasi-random and } \mu_p(f_{T\to 1_A}) \geqslant \varepsilon/2 \quad ,\right.$$

and define the T-junta $\mathcal{J} = \{ A \subseteq [n] \mid A \cap T \in \mathcal{T}\}$. To complete the proof, we will show that $\mu_p(\mathcal{F} \setminus \mathcal{J})$ and that $\mathcal{J}$ is intersecting.

Bounding $\mu_p(\mathcal{F} \setminus \mathcal{J})$. Let $g\colon \{0,1\}^n \to \{0,1\}$ be the function of $\mathcal{J}$. Then

$$\mu_p(\mathcal{F} \setminus \mathcal{J}) = \mathop{\mathbb{E}}_{z \sim \{0,1\}^T} [\mu_p((\mathcal{F} \setminus \mathcal{J})_{T \to z})].$$

By definition of $\mathcal{J}$, each z for which $(\mathcal{F} \setminus \mathcal{J})_{T \to z})$ is non-empty either satisfies that $f_{T \to z}$ is not (r, ε') quasi-random, or that $\mu_p(f_{T \to z}) < \varepsilon/2$. Thus,

$$\mu_p(\mathcal{F} \setminus \mathcal{J}) \leqslant \mathop{\mathbb{E}}_{z \sim \{0,1\}^T} \left[1_{f_{T \to z} \text{ is } (r,\varepsilon') \text{ quasi-random}} + \mu_p(f_{T \to z}) 1_{\mu_p(f_{T \to z}) < \varepsilon/2}\right]$$

The expectation of the first indicator is at most $\delta \leqslant \varepsilon/2$, and for the second expectation we have that it is at most $\varepsilon/2$, hence $\mu_p(\mathcal{F} \setminus \mathcal{J}) \leqslant \varepsilon$.

Showing that $\mathcal{J}$ is intersecting. We will show that for any $A, A' \in \mathcal{T}$, we have that $A \cap A' \neq \emptyset$, which is clearly enough.

Assume towards contradiction otherwise, and take disjoint $A, A' \in \mathcal{T}$. Consider the families $\mathcal{G} = \mathcal{F}_{T \to A}$, $\mathcal{H} = \mathcal{F}_{T \to A'}$. By definition of $\mathcal{T}$, they are both (r, ε') quasi-random and have μ_p-measure at least α. Hence by Lemma 4.3 we may find disjoint $G \in \mathcal{G}$, $H \in \mathcal{H}$. Thus, we have $A \cup G \in \mathcal{F}$ and $A' \cup H \in \mathcal{F}$ which are disjoint, which contradicts the fact that $\mathcal{F}$ is intersecting.

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