18.218 Topics in Combinatorics Spring 2021 – Lecture 4

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In this lecture, we will define influences of Boolean functions, and give several important interpretations of them.

1 Motivation: Boolean functions as voting rules

Suppose a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is thought of as an aggregation rule for a certain vote. Namely, we have n voters that are supposed to decide between two options, 0 and 1. The input to the function is the vector of opinions x of the voters, wherein x_i denotes the opinion of the ith voter. The output of the function, $f(x)$, then stands for the outcome of the vote.

In light of this view, a few natural functions come to mind, as well as the names we associate them with.

1. Dictatorship, i.e. a function $f\colon \{0,1\}^n \to \{0,1\}$ of the form $f(x) = x_1$. Here, only the vote of the first participant counts.

2. Majority, i.e. the function $f\colon \{0,1\}^n \to \{0,1\}$ defined as $f(x) = 1_{x_1+\ldots+x_n>n/2}$.

3. Juntas. Here, we have a small set of participants, $J \subseteq [n]$, and some $g\colon \{0,1\}^J \to \{0,1\}$, and our function f is defined as $f(x) = g(x_J)$. Here, only the votes of the participants from the set J count. In this context, J is thought of as a small set, hence the name "junta" makes sense.

Equipped with this intuition, one may try to define parameters of Boolean functions in order to decide which type of voting rules are more "fair" [1], or more modestly to understand their features in a more precise sense.

For the following discussion, we will assume the distribution of the vote of each player is distributed uniformly and independently of the others (which is of course, not realistic, but nevermind), so that the distribution of x is uniform over $\{-1,1\}^n$. How much did the vote of participant i "mattered"?

Definition 1.1. *For a function $f\colon \{-1,1\}^n \to \{0,1\}$ and a coordinate $i \in [n]$, the influence of i is defined as*

$$I_i[f] = \Pr_{x\in\{-1,1\}^n}\left[f(x) \neq f(x^{\oplus i})\right].$$

Here, $x^{\oplus}$ is the vector x in which the ith coordinate has been flipped.

Let's examine influences in the above examples. For dictatorship, one clearly has $I_1[f] = 1$, whereas $I_i[f] = 0$ for all $i \neq 1$, so influences capture well the intuition we intended. Analyzing the majority function is more challenging, but the symmetry clearly implies that all influences are the same, and a direct calculation shows that they are all of the order $1/\sqrt{n}$. For the junta example, one easily sees that $I_j[f] = 0$ for all $j \notin J$, so the intuition is again captured well.

[1] In this course, we will not attempt to give an answer to the (difficult) question of what qualifies as "fair".

Definition 1.2. *For a function $f\colon \{-1,1\}^n \to \{0,1\}$, the total influence of f is defined as*

$$I[f] = \sum_{i=1}^{n} I_i[f].$$

The total influence of the function is one of the most important parameters associated with a Boolean function, and in this lecture we will see some of its basic interpretations and properties.

2 Generalizing the notion of influences

The notion of influences may be generalized to arbitrary real valued functions in the following way.

Definition 2.1 (Discrete derivatives). *Given a function $f\colon \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$, the discrete derivative of f along i is a function $\partial_i f\colon \{-1,1\}^{n-1} \to \mathbb{R}$ defined as*

$$\partial_i f(y) = \frac{1}{2}(f(x_i = 1, x_{-i} = y) - f(x_i = -1, x_{-i} = y)).$$

Definition 2.2. *Given a function $f\colon \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$, the L^2 influence of i on f is defined as $I_i[f] = \|\partial_i f\|_2^2$. The total L^2 influence of f is $I[f] = \sum\limits_{i=1}^{n} I_i[f]$.*

It is instructive to check that the two definitions of influences coincide for $\{0,1\}$ valued functions up to a factor of 4.

In the rest of this lecture, we will develop Fourier analytic formulas for derivatives, influences and derive the basic isoperimetric inequality for Boolean functions (also known as Poincaré's inequality). We will then spend some time dwelling on these and give more interpretations to the total influence of a function.

3 A combinatorial view of the total influence

The total influence has the following important combinatorial interpretation. Consider the graph whose vertices are $\{-1,1\}^n$, and two vertices are connected by an edge if they differ in exactly one coordinate. Thus, a Boolean function $f\colon \{-1,1\}^n \to \{1,-1\}$ can be identified with the subset of vertices $F = \{x \in \{-1,1\}^n \mid f(x) = -1\}$.

For $x \in \{-1,1\}^n$, let $s_f(x)$ be the number of edges adjacent to x that cross the bi-partition $(F, \bar{F})$, i.e. the number of $i \in [n]$ such that $f(x) \neq f(x \cdot e_i)$. The quantity $s_f(x)$ often goes by the name of "the sensitivity of f at x" in the literature.

Claim 3.1. *If $f\colon \{-1,1\}^n \to \{-1,1\}$, then $I[f] = \mathbb{E}_x\left[s_f(x)\right]$.*

Proof. For $x \in \{-1,1\}^n$ and $i \in [n]$, denote by $Z_{i,x}$ the random variable which is 1 if and only if $f(x) \neq f(x^{\oplus i})$. Then $s_f(x) = \sum\limits_{i=1}^{n} Z_{i,x}$, and so by linearity of expectation

$$\mathop{\mathbb{E}}_x\left[s_f(x)\right] = \mathop{\mathbb{E}}_x\left[\sum_{i=1}^{n} Z_{i,x}\right] = \sum_{i=1}^{n} \mathop{\mathbb{E}}_x\left[Z_{i,x}\right].$$

The proof is now concluded by noting that $\mathbb{E}_x\left[Z_{i,x}\right] = I_i[f]$. $\square$

Thus, the total influence of f also clearly deserves the name average sensitivity.

4 Sharp thresholds and the total influence

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a monotone function, i.e. if $x_i \leqslant y_i$ for all i, then $f(x) \leqslant f(y)$. For example, one can think of $n = \binom{N}{2}$ and as the input as specifying the adjacency matrix of some graph on N vertices. In this case, the function f could be any monotone graph property, such as (1) being connected, (2) containing a clique of size $\log N$, (3) containing at least $\log n$ triangles etc. For such properties, it is often known that they exhibit a sharp threshold. [2]

It turns out that understanding the total influence of function is often useful to shed further light on such questions. Towards this end, we define the p-biased distribution over $\{0,1\}^n$, denoted by $\mu_p^{\otimes n}$, as: for each $i \in [n]$, sample $x_i = 1$ with probability p, and otherwise set $x_i = 0$. The quantity we wish to study is thus $\mu_p(f) = \mathbb{E}_{x\sim\mu_p^{\otimes n}}[f(x)]$, and in particular the way this quantity varies when we increase p. Towards this end, the p-biased analogs of influences as well as the total influence can be defined in the natural way:

$$I_i[f;\mu_p^{\otimes n}] = \mathop{\mathbb{E}}_{x\sim\mu_p^{\otimes n}}\left[|\partial_i f(x)|^2\right], \qquad I[f;\mu_p^{\otimes n}] = \sum_{i=1}^n I_i[f;\mu_p^{\otimes n}].$$

We have the following basic result, asserting that large total influence implies a sharp threshold.

Lemma 4.1 (Russo-Margulis)**.** *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a monotone function. Then*

$$\frac{d}{dp}\mu_p(f) = I[f;\mu_p^{\otimes n}].$$

Proof. Take ε to be very small, and let us sample (x,y) in a coupled way so that marginally $x \sim \mu_p^{\otimes n}$, $y \sim \mu_{p+\varepsilon}^{\otimes n}$ and $x \leqslant y$ always. This can be done by sampling $x \sim \mu_p^{\otimes n}$, and then for each i, if $x_i = 1$ take $y_i = 1$, and if $x_i = 0$ take $y_i = 1$ with probability $\varepsilon/(1-p)$.

Then
$$\mu_{p+\varepsilon}(f) - \mu_p(f) = \mathop{\mathbb{E}}_{(x,y)}[f(x) - f(y)] = \mathop{\mathbb{E}}_{(x,y)}[(f(x) - f(y))1_{x\neq y}].$$

Note that the probability that x and y differ in more than a single coordinate is at most $n^2\varepsilon^2$, so

$$\mu_{p+\varepsilon}(f) - \mu_p(f) - \sum_{i=1}^n \mathop{\mathbb{E}}_{(x,y)}\left[(f(x) - f(y))1_{x \text{ and } y \text{ differ only at } i}\right] \leqslant n^2\varepsilon^2$$

Observe that

$$\mathop{\mathbb{E}}_{(x,y)}\left[(f(x) - f(y))1_{x \text{ and } y \text{ differ only at } i}\right] = \Big(\varepsilon - \Pr\left[x \text{ and } y \text{ differ in at least two coordinates}\right]\Big)\, I_i[f;\mu_p^{\otimes n}],$$

so we get
$$\mu_{p+\varepsilon}(f) - \mu_p(f) - \sum_{i=1}^n \varepsilon I_i[f;\mu_p^{\otimes n}] \leqslant n^2\varepsilon^2 + n^3\varepsilon^2.$$

Dividing by ε and sending it to 0 gives the result. □

Remark 4.2. *There are arguably simpler proofs in the literature, but we give this one since we think it nicely highlights the intuition behind.*

[2] That is, looking at the Erdos Reyni graph model, there is a critical edge density p such that below it the property holds with probability $o(1)$, whereas above it the property holds with probability $1 - o(1)$.

5 Fourier analytic formulas for derivatives and influences

Claim 5.1. *For a function $f\colon \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$, we have that*

$$\partial_i f(y) = \sum_{S \ni i} \widehat{f}(S)\chi_{S\setminus\{i\}}(y)$$

Proof. By definition,

$$\partial_i f(y) = \frac{1}{2}(f(x_i = 1, y) - f(x_i = -1, y)) = \frac{1}{2}\sum_{S} \widehat{f}(S)(\chi_S(x_i = 1, y) - \chi_S(x_i = -1, y)).$$

Note that if $i \notin S$, then $\chi_S(x_i = 1, y) = \chi_S(x_i = -1, y)$ and these terms cancel. Otherwise, if $i \in S$, then $\chi_S(x_i = 1, y) = \chi_{S\setminus\{i\}}(y)$ and $\chi_S(x_i = -1, y)) = -\chi_{S\setminus\{i\}}(y)$. Therefore, we get that

$$\partial_i f(y) = \frac{1}{2}\sum_{S} \widehat{f}(S)(\chi_S(x_i = 1, y) - \chi_S(x_i = -1, y)) = \sum_{S} \widehat{f}(S)\chi_S(y).$$ □

In particular, we see that if f has degree at most d, then $\partial_i f$ has degree at most $d-1$.

Corollary 5.2. $I_i[f] = \sum\limits_{S \ni i} \widehat{f}(S)^2$.

Proof. As $I_i[f] = \|\partial_i f\|_2^2$, the corollary follows from the last claim and Parseval. □

Corollary 5.3. $I[f] = \sum\limits_{S} |S|\, \widehat{f}(S)^2$.

Proof. By definition and Corollary 5.2,

$$I[f] = \sum_{i=1}^{n} I_i[f] = \sum_{i=1}^{n}\sum_{S \ni i} \widehat{f}(S)^2 = \sum_{S} |S|\, \widehat{f}(S)^2.$$ □

Though nearly trivial, the last statement gives us an important interpretation of the total influence of a function. Note that the degree of a character χ_S is just $|S|$, and we can think of the square of the coefficients $\widehat{f}(S)^2$ as a distribution over characters if f is ± 1 valued. Thus, the above formula asserts that $I[f]$ can be thought of, in a sense, as the average degree of f, according to these weights. This relaxation of the notion is degree is a very important one, and in the upcoming lectures we will be interested in characterizing functions with low average degree (which will later play important roles in several applications).

An immediate implication of the previous corollary is the so-called Poincaré inequality.

Corollary 5.4 (Poincaré inequality)**.** *For any $f\colon \{-1,1\}^n \to \mathbb{R}$ we have that $I[f] \geqslant \mathsf{var}(f)$.*

Proof. This is immediate by the Fourier analytic formulas for $\mathsf{var}(f)$ and $I[f]$. □

Poincaré's inequality holds for general real-valued functions, and an interesting question is if it can be improved for Boolean functions. It is a nice exercise to check for which Boolean functions one has the equality $I[f] = \mathsf{var}(f)$, and later on in the course we will see several improvements of this result.

One immediate consequence of Poincaré inequality, is that if $f\colon \{-1,1\}^n \to \{-1,1\}$ is balanced, i.e. if $\mathbb{E}[f] = 0$, then there is i such that $I_i[f] \geqslant \frac{1}{n}$. Is there such function f such that $I_i[f] = \frac{1}{n}$? This would be desirable in the sense of voting, since intuitively we would like to minimize the individual influence of each one of the participants.

A landmark result in the area, which we will prove in a couple of lectures, asserts that this is impossible. In fact, there is always a coordinate whose influence beats $1/n$ substantially.

Theorem 5.5 (KKL theorem)**.** *There is an absolute constant $c > 0$, such that for any $f\colon \{-1,1\}^n\{-1,1\}$, there is $i \in [n]$ such that $I_i[f] \geqslant c\frac{\log n}{n}\mathsf{var}(f)$.*

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