18.218 Topics in Combinatorics Spring 2021 – Lecture 7

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Today we will talk about several strengthenings of the KKL-theorem, by Talagrand and Friedgut. We will then discuss isoperimetric inequalities over the hypercube in a broader term, and in particular several results of Talagrand, as well as a conjecture by him which was recently resolved by Eldan and Gross.

Recall that one version of the KKL theorem makes the following assertion.

Theorem 0.1. *Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a function such that $I[f] \leqslant K \cdot \mathsf{var}(f)$. Then there is an $i \in [n]$ such that $I_i[f] \geqslant e^{-O(K)}$.*

Can one prove stronger structural results for functions with low total influence? What are the sort of examples you can come up with?

1 Talagrand's version of the KKL theorem

The first result we will prove today seeks a strengthening of the KKL theorem, which reads as follows.

Theorem 1.1 (Talagrand)**.** *There exists an absolute constant $C > 0$, such that for any $f\colon \{-1,1\}^n \to \{-1,1\}$ it holds that*

$$C\sum_{i=1}^{n} \frac{I_i[f]}{\log(1/I_i[f])} \geqslant \mathsf{var}(f).$$

Exercise: show that this is indeed a strengthening of the KKL theorem.

The nice feature of this theorem is that it seems to tall us much more information about the function f than the KKL theorem itself. Namely, in some sense it tells that "on average" the influences have magnitude at least $e^{-O(K)}$ (as opposed to the largest one being of that magnitude), as i's such that $I_i[f] \leqslant e^{-M\cdot K}$ contribute at most $\mathsf{var}(f)/M$ to the left hand side.

Proof. If there is i such that $I_i[f] \geqslant 1/10$ there is nothing to prove, so assume otherwise. Write

$$\mathsf{var}(f) = \sum_{S\neq\emptyset} \widehat{f}(S)^2 = \sum_{i=1}^{n}\sum_{S\ni i} \frac{1}{|S|}\widehat{f}(S)^2 = \sum_{i=1}^{n} \|g_i\|_2^2,$$

where $g_i(x) = \sum\limits_{S\ni i} \frac{1}{\sqrt{|S|}}\widehat{f}(S)\chi_S(x)$. To bound the 2-norm of g_i, we set $d_i = \frac{1}{20}\log(1/I_i[f])$ and bound

$$\|g_i\|_2^2 = \sum_{\substack{S\ni i\\ |S|\leqslant d_i}} \frac{1}{|S|}\widehat{f}(S)^2 + \sum_{\substack{S\ni i\\ |S|> d_i}} \frac{1}{|S|}\widehat{f}(S)^2 \leqslant \sum_{\substack{S\ni i\\ |S|\leqslant d_i}} \widehat{f}(S)^2 + \frac{1}{d_i}\sum_{\substack{S\ni i\\ |S|> d_i}} \widehat{f}(S)^2.$$

For the first sum, we consider the function $\partial_i f(x)$, note it is $1,-1,0$ valued and non-zero on $I_i[f]$ fraction of the inputs, so by Claim 2 in the last lecture the first sum is at most $I_i[f]^{24/20} = I_i[f]^{6/5}$. As for the second sum, it is clearly bounded by $I_i[f]$, so overall we get that

$$\|g_i\|_2^2 \leqslant I_i[f]^{6/5} + \frac{I_i[f]}{d_i} \leqslant 40\frac{I_i[f]}{\log(1/I_i[f])},$$

and the proof is concluded. $\square$

2 The Friedgut junta theorem

One can hope to strengthen Theorem 0.1 to be a sort of (at least morally speaking) "if and only if" statement. The issue is that the structure Theorem 0.1, while very interesting, is still rather poor and leaves much to be desired. Luckily, for roughly the same effort, one may prove a much stronger statement.

Theorem 2.1. *Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Then for every $\varepsilon > 0$, there exists $J \subseteq [n]$ of size at most $2^{O\left(\frac{I[f]}{\varepsilon \mathsf{var}(f)}\right)}$ and a J-junta $g\colon \{-1,1\}^n \to \{-1,1\}$ such that $\|f-g\|_2 \leqslant \varepsilon$.*

Proof. Let $C > 0$ be an absolute constant to be determined later, let $\delta = 2^{-C\frac{I[f]}{\varepsilon \mathsf{var}(f)}}$ and take

$$J = \{\, i \mid I_i[f] \geqslant \delta \}\,.$$

Let $G(x) = \sum_{\substack{S\subseteq J\\ |S|\leqslant \frac{2I[f]}{\varepsilon}}} \widehat{f}(S)\chi_S(x)$ and $g(x) = \mathsf{sign}(G(x))$. Clearly, g is a J-junta (why?) and we next bound the size of J and the L^2 distance between it and f.

Bounding the size of J. We have $I[f] \geqslant |J|\, 2^{-C\frac{I[f]}{\varepsilon \mathsf{var}(f)}}$, so $|J| \leqslant 2^{(C+1)\frac{I[f]}{\varepsilon \mathsf{var}(f)}}$.

Bounding the distance between f and g. We have that $\|f-g\|_2 \leqslant 2\|f-G\|_2$, and we bound the latter norm.

$$\|f-G\|_2^2 = \sum_{\substack{S\\ S\not\subseteq J\\ \text{or } |S|>\frac{2I[f]}{\varepsilon}}} \widehat{f}(S)^2 \leqslant \sum_{S\not\subseteq J, |S|\leqslant \frac{2I[f]}{\varepsilon}} \widehat{f}(S)^2 + \sum_{|S|\geqslant \frac{2I[f]}{\varepsilon}} \widehat{f}(S)^2, \tag{1}$$

and we bound each sum separately. For the second one, we have

$$\sum_{|S|\geqslant \frac{2I[f]}{\varepsilon}} \widehat{f}(S)^2 \leqslant \frac{\sum_{|S|\geqslant \frac{2I[f]}{\varepsilon}} |S|\, \widehat{f}(S)^2}{2I[f]/\varepsilon} \leqslant \frac{I[f]}{2I[f]/\varepsilon} = \frac{\varepsilon}{2}. \tag{2}$$

For the first one, we denote $d = \frac{2I[f]}{\varepsilon}$. Fix $i \notin J$, and $g = \partial_i f$. As before, we get that

$$\sum_{|S|\leqslant d, i\in S} \widehat{f}(S)^2 = \sum_{|S|\leqslant d} \widehat{g}(S)^2 \leqslant \sum_{|S|\leqslant \frac{1}{20}\log(1/\delta)} \widehat{g}(S)^2 \leqslant I_i[f]^{24/20}.$$

Here we used the fact that C is large enough so that $d \leqslant \log(1/\delta)/20$. Summing over $i \notin J$ we get that

$$\sum_{|S|\leqslant d, S\not\subseteq J} \widehat{f}(S)^2 \leqslant \sum_{|S|\leqslant d} \left|S\cap \bar{J}\right| \widehat{f}(S)^2 \leqslant \sum_{i\notin J} I_i[f]^{24/20} \leqslant \delta^{1/5} I[f] \leqslant \frac{\varepsilon}{2}. \tag{3}$$

Plugging (2), (3) into (1) finishes the proof. □

3 Isoperimetric inequalities over the hypercube

The most basic isoperimetric inequality (as well as the weakest) we have seen in this class is Poincare's inequality, stating that $I[f] \geqslant \mathsf{var}(f)$. In general, this inequality is tight, but one may want to prover that stronger bounds hold for special classes of functions. One way to go about this goal is to inspect the equality cases, and see if one can prove stronger versions of it for functions that are "far" from the equality cases. For this purpose, we quickly recall the proof of Poincare's inequality we have seen earlier:

$$I[f] = \sum_{S} |S| \widehat{f}(S)^2 \geqslant \sum_{|S| \neq 0} \widehat{f}(S)^2 = \mathsf{var}(f).$$

We see that the equality cases are precisely those functions that have all of their mass on the empty character and the first level. Thus, among Boolean functions $f\colon \{-1,1\}^n \to \{-1,1\}$, the only equality cases are constant functions and dictatorships. In particular, ignoring constant functions, equality cases are achieved by balanced functions, which raises the question of whether there is an improvement of the inequality of highly unbalanced functions. Indeed, such strengthening exists:

Theorem 3.1 (The edge isoperimetric inequality)**.** *For all* $f\colon \{-1,1\}^n \to \{-1,1\}$ *it holds that*

$$I[f] \geqslant \Pr\left[f(x) = -1\right] \log\left(\frac{1}{\Pr\left[f(x) = -1\right]}\right).$$

3.1 Other notions of boundary

Recalling the interpretation of $I[f]$ as the edge boundary of the set $S = \{x \mid f(x) = -1\}$ in the hypercube graph, one may ask about different notions of boundaries. One notion that makes sense is the *vertex-boundary* of a set: $\mathsf{V}-\mathsf{boundary}(S) = \{x \mid s_f(x) > 0\}$. What can one say about it?

How small can it be (majority example; thm: this is best one can do, Kruskal-Katona). Note that in this case however, the edge boundary is much larger than promised by the Poincare inequality. Can it be the case that both the vertex boundary, and the edge boundary be simultaneously small?

Theorem 3.2 (Margulis)**.** *For all* $f\colon \{-1,1\}^n \to \{-1,1\}$,

$$\mu(\mathsf{V}-\mathsf{boundary}(S)) I[f] \geqslant \Omega(\mathsf{var}(f)^2).$$

Thus, this theorem tells us that indeed if the vertex boundary is exceptionally small (e.g. in the majority example, it is $O(1/\sqrt{n})$), it is necessarily the case that the edge boundary must be exceptionally high.

Shortly after establishing this result, Michel Talagrand had been looking into strengthening of this result. He came up with a quantity, that posteriori makes a lot of sense, but wasn't considered by earlier authors. Note that by Cauchy-Schwarz one has that

$$\mathop{\mathbb{E}}_{x}\left[\sqrt{s_f(x)}\right] = \mathop{\mathbb{E}}_{x}\left[\sqrt{s_f(x)} 1_{s_f(x)>0}\right] \leqslant \sqrt{\mathop{\mathbb{E}}_{x}\left[s_f(x)\right]}\sqrt{\Pr_{x}\left[s_f(x) > 0\right]} = \sqrt{I[f]\mu(\mathsf{V}-\mathsf{boundary}(S))}.$$

Thus, if it was the case that $\mathbb{E}_x\left[\sqrt{s_f(x)}\right] \geqslant \Omega(\mathsf{var}(f))$ for all Boolean f's, then one immediately gets Marguli's theorem as a corollary. Indeed, this is Talagrand's theorem.

Theorem 3.3. *For all* $f\colon \{-1,1\}^n \to \{-1,1\}$,

$$\mathop{\mathbb{E}}_{x}\left[\sqrt{s_f(x)}\right] \geqslant \Omega(\mathsf{var}(f)).$$

Talagrand's proof is a cute inductive proof, which we'll not see (but you are encouraged to look into it, it's really nice).

Talagrand then goes on and seeks a version of Theorem 3.3 which is stronger than the edge isoperimetric inequality (at least up to a constant factor). Indeed, Talagrand is able to prove:

Theorem 3.4. *For all* $f\colon \{-1,1\}^n \to \{-1,1\}$,

$$\mathop{\mathbb{E}}_{x}\left[\sqrt{s_f(x)}\right] \geqslant \Omega\left(\mathsf{var}(f)\sqrt{\log\left(\frac{1}{\mathsf{var}(f)}\right)}\right).$$

The proof of this result, once again, is inductive (though much less nice than the previous one).

3.2 KKL enters the picture

Recalling equality cases for Poincare's inequality, another way to think of functions "far from equality cases" is as the class of functions that have all individual influences being small. This is, in fact, a larger class than the class of highly unbalanced functions, and one may hope to improve Poincare's inequality for this class. Here, it is interesting to note that we have in fact already seen this improvement, which is nothing but the KKL theorem, stated as:

$$I[f] \geqslant c \min_i \log\left(\frac{1}{I_i[f]}\right) \mathsf{var}(f)$$

for some absolute constant $c > 0$. Thus, one gains a large factor provided all influences of f are small.

3.3 Mixing everything together

With the same motivation as behind Theorem 3.4, Talagrand seeked to prove an isoperimetric inequality that captures both Theorem 3.3 (and thereby Margulis inequality) and the KKL theorem. He was only partly successful; define a parameter $M(f) = \sum_{i=1}^{n} I_i[f]^2$. Talagrand showed that:

Theorem 3.5. *There exists $0 < \alpha < 1/2$ such that for all $f\colon \{-1,1\}^n \to \{-1,1\}$,*

$$\mathop{\mathbb{E}}_x\left[\sqrt{s_f(x)}\right] \geqslant \Omega\left(\mathsf{var}(f) \log^{1/2-\alpha}\left(\frac{1}{\mathsf{var}(f)}\right) \log^{\alpha}\left(\frac{1}{M(f)}\right)\right).$$

This result goes some of the way towards a result that encapsulates together Theorem 3.3 and the KKL theorem. For example it is an exercise to show that Theorem 3.5 implies that

$$\max_i I_i[f] \geqslant e^{-O\left(\left(\frac{I[f]}{\mathsf{var}(f)}\right)^{1/2\alpha}\right)},$$

which for $\alpha = 1/2$ would be KKL, but for $\alpha < 1/2$ is weaker. Talagrand then went on to conjecture that Theorem 3.4 holds for $\alpha = 1/2$, a conjecture that has only been resolved positively in 2020 by Eldan and Gross. For that, Eldan and Gross use tools from stochastic calculus, which have recently showed up in simultaneously a bunch of places in analysis of TCS (this is a potential topic for a final project).

We will not give the proofs of Theorems 3.3, 3.4, 3.5 today. Later on in the course we will see a recent simpler, unified proof for all of these results that only uses elementary tools.

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