18.218 Topics in Combinatorics Spring 2021 – problem set 1

1. Expand out the Fourier transform of the following functions:

   (a) AND function: we have a set $T \subseteq [n]$, and $f\colon \{-1,1\}^n \to \{0,1\}$ is defined as $f(x) = 1$ if and only if $x_i = -1$ for all $i \in T$.

   (b) The Equality function: $f\colon \{-1,1\}^n \to \{0,1\}$ is defined as $f(x) = 1$ if and only if $x_1 = x_2 = \ldots = x_n$.

2. A function $f\colon \{-1,1\}^n \to \mathbb{R}$ is called even if $f(x) = f(-x)$ for all $x \in \{-1,1\}^n$, and is called odd if $f(-x) = -f(x)$ for all $x \in \{-1,1\}^n$.

   (a) Prove that f is even if and only if its Fourier transform is supported only on even sized characters.

   (b) Prove that f is odd if and only if its Fourier transform is supported only on odd sized characters.

   (c) Show that any $f\colon \{-1,1\}^n \to \mathbb{R}$ may be written as a sum of an even and an odd function.

3. The degree of a function $f\colon \{-1,1\}^n \to \mathbb{R}$ is defined as the maximal size of $S \subseteq [n]$ such that $\widehat{f}(S) \neq 0$, and denoted by $\mathsf{deg}(f)$.

   (a) Prove that for any integer valued function $f\colon \{-1,1\}^n \to \mathbb{Z}$ and any $S \subseteq [n]$, the Fourier coefficient $\widehat{f}(S)$ is an integer multiple of $2^{-\mathsf{deg}(f)}$.

   (b) Deduce that the support size of the Fourier spectrum of a Boolean function $f\colon \{-1,1\}^n \to \{-1,1\}$ is at most $2^{2\mathsf{deg}(f)}$.

4. Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ satisfies $\mathsf{deg}(f) = 1$. Prove that there is an $i \in [n]$ and $b \in \{-1,1\}$ such that $f(x) = bx_i$ for all $x \in \{-1,1\}$ (such functions are often called dictatorships / anti-dictatorships).

5. Consider the following variant of the linearity tester we have seen in class. Sample $x, y, z \in \{-1,1\}^n$, and check that $f(x)f(y)f(z) = f(xyz)$.

   (a) Prove that for any $f\colon \{-1,1\}^n \to \{-1,1\}$, the probability that the test passes is at least $1/2$. Does this assertion hold for the version seen in class?

   (b) Let $\varepsilon < \frac{1}{2}$, and suppose that $f\colon \{-1,1\}^n \to \{-1,1\}$ satisfies that

   $$\Pr_{x,y,z\in\{-1,1\}^n}\left[f(x)f(y)f(z) = f(xyz)\right] \geqslant 1 - \varepsilon.$$

   Prove that there is $S \subseteq [n]$ and $b \in \{-1,1\}$ such that $\Pr_x\left[f(x) = b\chi_S(x)\right] \geqslant \sqrt{1-2\varepsilon}$.

6. (*) Given a function $f\colon \{0,1\}^n \to \mathbb{R}$ and a basis $B = \{b_1, \ldots b_r\}$ of a subspace $W \subseteq \mathbb{F}_2^n$, the restriction of f to that subspace according to the basis B is $f_{W,B}\colon \mathbb{F}_2^r \to \mathbb{R}$ defined as $f_{W,B}(y_1, \ldots, y_r) = f(\sum\limits_{i=1}^{r} y_i b_i)$.

(a) Prove that for any subspace W, if B, B' are two bases of W, then there is an invertible map $M\colon P([r]) \to P([r])$ such that $\widehat{f_{W,B}}(S) = \widehat{f_{W,B'}}(M(S))$.

(b) Let $\delta > 2 \cdot 2^{-n/10}$, and consider a uniformly chosen subspace $W \subseteq \mathbb{F}_2^n$ of dimension $n/2$. Suppose that for $f\colon \{0,1\}^n \to \{-1,1\}$ we have

$$\Pr_W \left[f_{W,B} \text{ has a Fourier coefficient of magnitude at least } \delta \text{ for some basis } B \text{ of } W\right] \geqslant \delta.$$

Show that there exists $S \subseteq [n]$, such that $\widehat{f}(S) \geqslant \frac{\delta^{2.5}}{\sqrt{2}}$.

18.218 Topics in Combinatorics: Analysis of Boolean Functions
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