18.218 Topics in Combinatorics Spring 2021 – problem set 2

1. Let $f\colon \{0,1\}^n \to \{0,1\}$, $0 < p < 1$, and let (J, z) be a p-random restriction.

   (a) Show that $\mathbb{E}_{(J,z)}\left[I[f_{\bar{J}\to z}]\right] = pI[f]$.

   (b) Define $W^{\approx d}[f] = \sum\limits_{d < j \leqslant 2d} W^{=j}[f]$. Show that there is a constant $c > 0$, such that if $p = \frac{1}{d}$, then

   $$\mathop{\mathbb{E}}_{(J,z)}\left[W^{=1}[f_{\bar{J}\to z}]\right] \geqslant cW^{\approx d}[f].$$

2. Let $f\colon \{0,1\}^n \to \{-1,1\}$ be a monotone function.

   (a) Prove that $\widehat{f}(\{i\}) = I_i[f]$.

   (b) Prove that $I[f] \leqslant \sqrt{n}$.

   (c) Prove that among all monotone functions, $I[f]$ is maximized by the majority function, i.e. by $f(x) = 1$ if $\sum_{i=1}^{n} x_i \geqslant 0$ and otherwise $f(x) = -1$.

3. In this question, we will think of an input $x \in \{0,1\}^{\binom{n}{2}}$ as representing a graph G_x: the vertices G_x of are $[n]$, the coordinates of x are thought of as subset of $[n]$ of size 2, and the edges of G_x are all e such that $x_e = 1$.

   A function $f\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ is called a graph property if it is invariant under vertex permutations, i.e. under S_n.

   (a) Show that there is an absolute constant $c > 0$, such that if $f\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ is a graph property, then $I[f] \geqslant c\,\mathsf{var}(f)\cdot \log n$.

   (b) Show that there is an absolute constant $C > 0$ such that if f is a monotone increasing graph property and $\mu_{1/2}(f) \geqslant 0.01$, then $\mu_{1/2+C/\log n}(f) \geqslant 0.99$. (You can use the fact that the KKL theorem holds for the p-biased measure μ_p for all $p \in [1/3, 2/3]$).

4. Design a polynomial time learning algorithm for the following classes:

   (a) $\mathcal{C}_1 = \left\{f\colon \{-1,1\}^n \to \{-1,1\} \mid I[f] \leqslant \sqrt{\log n}\right\}$ with membership queries.

   (b) $\mathcal{C}_2 = \left\{f\colon \{-1,1\}^n \to \{-1,1\} \mid f \text{ is monotone}, I[f] \leqslant \sqrt{\log n}\right\}$ with random queries (i.e. in the PAC model).

5. For a function $f\colon \{-1,1\}^n \to \mathbb{R}$ and $d \in \mathbb{N}$, we define the part of f of degree at most d, denoted as $f^{\leqslant d}$, by $f^{\leqslant d}(x) = \sum\limits_{|S| \leqslant d} \widehat{f}(S)\chi_S(x)$.

(a) Define the degree d influence of $i \in [n]$ on f as $I_i^{\leqslant d}[f] := I_i[f^{\leqslant d}]$. Show that $\sum_{i=1}^{n} I_i^{\leqslant d}[f] \leqslant d\|f\|_2^2$. Deduce that for all $\tau > 0$, the number of coordinates i such that $I_i^{\leqslant d}[f] \geqslant \tau$ is at most $\frac{d\|f\|_2^2}{\tau}$.

(b) Suppose $f\colon \{-1,1\}^n \to \{-1,1\}$ and that $\sum_{0<|S|\leqslant d} \widehat{f}(S)^2 \geqslant \delta$. Prove that there exists $i \in [n]$ such that

$$I_i^{\leqslant d}[f] \geqslant \frac{\delta^4}{9^d I[f]^4}.$$

6. (*) Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a function, $K \geqslant 1$ be such that $I[f] \leqslant K$, and let $\varepsilon > 0$. In this question, we will show that the Fourier spectrum of f is concentrated on $2^{O(K^2)}$ distinct Fourier coefficients.

(a) Show that for all $d \in \mathbb{N}$, $\delta, \tau > 0$ it holds that

$$\sum_{|S|\leqslant d} \widehat{f}(S)^2 1_{|\widehat{f}(S)|\leqslant \delta} \leqslant d\left(\frac{d}{\tau}\right)^d \delta + \sqrt{3}^d \tau^{1/4} K.$$

(b) Deduce that there is an absolute constant $C > 0$, such that $\sum_{|S|\leqslant K} \widehat{f}(S)^2 1_{|\widehat{f}(S)|\leqslant e^{-CK^2 \log(1/\varepsilon)}} \leqslant \varepsilon$.

(c) Deduce that there is an absolute constant $C > 0$ such that

$$\sum_{|S|\leqslant K} \widehat{f}(S)^2 \log\left(\frac{1}{\widehat{f}(S)^2}\right) \leqslant C \cdot K^2.$$

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