18.218 Topics in Combinatorics Spring 2021 – problem set 3

1. Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a function, and $\rho > 0$.

   (a) Show that for all $k \in \mathbb{N}$, $\frac{d^k \mathsf{Stab}_\rho f}{d^k \rho}(0) = k! W^{=k}[f]$.

   (b) Use this formula to deduce an asymptotic formula for $W^{=k}[\mathsf{Majority}]$ of the form $a_k + \mathsf{err}(k,n)$, where $\lim_{n\to\infty} \mathsf{err}(k,n) = 0$ for all $k \in \mathbb{N}$.

2. (a) Does the KKL theorem hold for $f\colon \{-1,1\}^n \to \mathbb{R}$?

   (b) How about $f\colon \{-1,1\}^n \to [-1,1]$?

3. In this question, we will establish a result known as Peres' theorem. A linear threshold function (LTF in short) is a function $f\colon \{-1,1\}^n \to \{-1,1\}$ of the form $f(x) = \mathsf{sign}(\sum_{i=1}^{n} a_i x_i)$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

   (a) Show that if f is an LTF, then $I[f] \leqslant \sqrt{n}$.

   (b) Let $m \in \mathbb{N}$ be such that $m \leqslant n$, let $\pi\colon [n] \to [m]$ and let $z \in \{-1,1\}^n$. The projection of f along π with signing z is defined as $f|_{\pi,z}\colon \{-1,1\}^m \to \{-1,1\}$ by $f|_{\pi,z}(y) = f(zx)$, where $x_i = y_{\pi(i)}$ for all $i \in [n]$, and zx represents the coordinatewise product of z and x.

   Show that $\mathsf{NS}_{1/m}[f] = \mathbb{E}_{\pi,z}\left[\frac{I[f|_{\pi,z}]}{m}\right]$, where the distribution over π is uniform over all $\pi\colon [n] \to [m]$ and z is uniform in $\{-1,1\}^n$.

   (c) Deduce that if f is an LTF, then for all $\delta > 0$ we have that $\mathsf{NS}_\delta[f] \leqslant \sqrt{2\delta}$.

4. An instance of the 3-LIN$_{\mathbb{F}_2}$ problem is given by a set of variables $X = \{x_1, \ldots, x_n\}$, and a set of equations $E = \{e_1, \ldots, e_m\}$ such that each equation e_i has the form $x_j x_k x_\ell = b_{j,k,\ell}$ for some $j,k,\ell \in [n]$ and $b_{j,k,\ell} \in \{-1,1\}$. The goal in the problem is to find a $\{-1,1\}$-assignment to the variables in X and satisfy as many of the equations as possible.

   Given an instance of 3-LIN$_{\mathbb{F}_2}$, for each (j,k,ℓ) such that $x_j x_k x_\ell = b_{j,k,\ell}$ is not an equation, define $b_{j,k,\ell} = 0$, and associate with the system the cubic function $f(x) = \frac{1}{\sqrt{m}} \sum_{j,k,\ell \in [n]} b_{j,k,\ell} x_j x_k x_\ell$.

   (a) Show that for all $x \in \{-1,1\}^n$,

   $$f(x) = \frac{1}{\sqrt{m}}\big(|\{ i \mid e_i \text{ is satisfied by } x\}| - |\{ i \mid e_i \text{ is not satisfied by } x\}| \big).$$

   (b) Prove that there is an assignment satisfying at least $\frac{1}{2}m$ of the equations.

   (c) Prove that there is an absolute constant $c > 0$, such that there is an assignment satisfying at least $\frac{1}{2}m + c\sqrt{m}$ of the equations.

5. (*) In this question, we will introduce the decoupling operation for functions over the hypercube. For $S, T \subseteq [n]$ and $x, y \in \{-1,1\}^n$ we denote $\chi_{S,T}(y,x) = \chi_S(y)\chi_T(x)$. Suppose a function $f\colon \{-1,1\}^n \to \mathbb{R}$ is homogenous of degree 3, i.e. $f(x) = \sum_{|S|=3} \widehat{f}(S)\chi_S(x)$. We define the function $g\colon \{-1,1\}^n \times \{-1,1\}^n \to \mathbb{R}$ by replacing each character $\chi_S(x) = \prod_{i\in S} x_i$ with $\sum_{j\in S} \chi_{\{j\},S\setminus j}(y,x)$, i.e.

$$g(y,x) = \sum_{|S|=3} \sum_{j\in S} \widehat{f}(S)\chi_{\{j\},S\setminus\{j\}}(y,x).$$

For example, if $f(x) = \frac{1}{2}(x_1x_2x_3 + x_1x_5x_6)$, then

$$g(y,x) = \frac{1}{2}\left(y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 + y_1x_5x_6 + x_1y_5x_6 + x_1x_5y_6\right).$$

Prove that there exists an absolute constant $c > 0$, such that $\|f\|_\infty \geqslant c\|g\|_\infty$ for all homogenous f of degree 3.

6. (*) In this question, we will give an improvement of the result proved in question 4. The number of occurrences of a variable x_j in a 3-$\text{LIN}_{\mathbb{F}_2}$ instance is the number of equations in the system it appears in. Prove that there is an absolute constant $c > 0$ such that if (X, E) is an instance of 3-$\text{LIN}_{\mathbb{F}_2}$ in the number of occurrences of each x_j is at most D, then there is an assignment $x \in \{-1,1\}^n$ satisfying at least $\left(\frac{1}{2} + \frac{c}{\sqrt{D}}\right) m$ of the equations.