18.218 Topics in Combinatorics Spring 2021 – problem set 4

1. Let $f\colon \{-1,1\}^n \to \{0,1\}$ be a function, let $\alpha, \varepsilon > 0$ and assume that $\Pr_{x\sim\{-1,1\}^n}[f(x) = 1] = \alpha$.

   (a) Suppose that $S_1, \ldots, S_r$ are characters whose indicator vectors $1_{S_1}, \ldots, 1_{S_r}$ are linearly independent over $\mathbb{F}_2$. Show that there is $g\colon \{-1,1\}^n \to \{0,1\}$ with $\Pr_{x\sim\{-1,1\}^n}[g(x) = 1] = \alpha$ and $\widehat{g}(\{i\}) = \widehat{f}(S_i)$ for all $i = 1, \ldots, r$.

   (b) Consider the set of indicator vectors of heavy Fourier coefficients of f,

   $$H = \left\{ 1_S \in \{0,1\}^n \mid S \subseteq [n],\ \widehat{f}(S) \ \geqslant \varepsilon \right\}.$$

   Prove that $\mathsf{dim}_{\mathbb{F}_2}(\mathsf{Span}_{\mathbb{F}_2}(H)) \leqslant O\left(\frac{\alpha^2 \log(1/\alpha)}{\varepsilon^2}\right)$.

2. Let $f\colon \{0,1\}^n \to \{0,1\}$, $r \in \mathbb{N}$ and $\varepsilon > 0$ and suppose that $p = 1/2$.

   (a) Show that if f is (r, ε) quasi-random with respect to μ_p, then for any non-empty $S \subseteq [n]$ of size at most r, it holds that $\widehat{f}(S) \ \leqslant 2^r \varepsilon$.

   (b) Show that if $\max_{|S|\leqslant r} \ \widehat{f}(S) \ \leqslant \varepsilon$, then f is $(r, 2^r\varepsilon)$ quasi-random with respect to μ_p.

3. In this question we prove the 2 function version of the majority is stablest theorem. Prove that for all $\rho \in (0,1)$, $\varepsilon > 0$ there are $\tau > 0$ and $d \in \mathbb{N}$, such that if $f, g\colon \{-1,1\}^n \to [-1,1]$ are balanced functions such that $I_i^{\leqslant d}[f], I_i^{\leqslant d}[g] \leqslant \tau$ for all $i \in [n]$, then

   $$\langle \mathrm{T}_\rho f, g\rangle \leqslant \mathsf{Stab}_\rho(\mathsf{Majority}) + \varepsilon.$$

4. In this question we will establish a multi-dimensional form of the invariance principle. Let $\rho \in (0,1)$ and let μ be a probability measure on $(x_1, y_1) \in \{-1,1\} \times \{-1,1\}$ whose marginal on each one of x_1, y_1 is uniform and $\mathbb{E}[x_1 y_1] = \rho$ (i.e. x_1 and y_1 are ρ-correlated). Let (z_1, w_1) be distributed as standard Gaussians with correlation ρ, i.e. (z, w) has covariance matrix $M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

   (a) Prove that for all $d \in \mathbb{N}$, and $C > 0$ there is $K > 0$ such that if $f(x_1, \ldots, x_n), g(y_1, \ldots, y_n)$ are multilinear polynomials of degree at most d, and $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ is smooth whose 3rd order partial derivatives are all bounded by C, then

   $$\mathop{\mathbb{E}}_{(x,y)\sim\mu^{\otimes n}}[\psi(f(x), g(y))] - \mathop{\mathbb{E}}_{(z,w)\sim N(0,M)^{\otimes n}}[\psi(f(z), g(w))] \ \leqslant K \sum_{i=1}^{n} I_i[f]^{3/2} + I_i[g]^{3/2}.$$

   (b) Deduce that for all $d \in \mathbb{N}$, $\varepsilon, C > 0$ there is $\tau > 0$ such that if $f(x_1, \ldots, x_n), g(y_1, \ldots, y_n)$ are multilinear polynomials of degree at most d, such that $\|f\|_2, \|g\|_2 \leqslant 1$ and $\max(I_i[f], I_i[g]) \leqslant \tau$ for all $i \in [n]$, then for all smooth $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ whose 3rd order partial derivatives are bounded by C it holds that

   $$\mathop{\mathbb{E}}_{(x,y)\sim\mu^{\otimes n}}[\psi(f(x), g(y))] - \mathop{\mathbb{E}}_{(z,w)\sim N(0,M)^{\otimes n}}[\psi(f(z), g(w))] \ \leqslant \varepsilon.$$

5. (*)

   (a) A family of subsets $\mathcal{F} \subseteq P([n])$ is called t-intersecting if for any $A, B \in \mathcal{F}$ it holds that $|A \cap B| \geqslant t$. Prove that for all $t \in \mathbb{N}, \zeta, \varepsilon > 0$ there is $N, J \in \mathbb{N}$, such that if $\zeta < p < \frac{1}{2} - \zeta$, $n \geqslant N$ and $\mathcal{F} \subseteq P([n])$ is t-intersecting, then there exists a J-junta $\mathcal{J} \subseteq P([n])$ such that (a) $\mu_p(\mathcal{F} \setminus \mathcal{J}) \leqslant \varepsilon$, and (b) $\mathcal{J}$ is t-intersecting.

   (b) A family of subsets $\mathcal{F} \subseteq P([n])$ is called $(t-1)$-avoiding if for any $A, B \in \mathcal{F}$ it holds that $|A \cap B| \neq t-1$. In this question, we will prove that the assertion of the previous question continues to hold for $(t-1)$-avoiding families.

      i. Explain why the proof from the previous question no longer works in this case.

      ii. Show how to fix that proof so that it works for $(t-1)$-avoiding families.

6. (*) In this question, we will prove a regularity lemma for low-degree influences. For $\rho \in (0,1)$, a function $f\colon \{0,1\}^n \to \{0,1\}$ and a coordinate $i \in [n]$, the ρ-noisy influence of i on f is $I_i^{(\rho)}[f] = I_i[T_\rho f]$.

   (a) Show that for every $\rho \in (0,1)$ and $\varepsilon, \tau > 0$, there exists $D \in \mathbb{N}$ such that the following holds. For any $f\colon \{-1,1\}^n \to \{-1,1\}$, there is $J \subseteq [n]$ of size at most D such that

$$\Pr_{z \in \{-1,1\}^J}\left[\exists i \notin J, I_i^{(\rho)}[f_{J \to z}] \geqslant \tau\right] \leqslant \varepsilon.$$

   (b) Deduce that for every $\varepsilon > 0$ and $d \in \mathbb{N}$, there is $D \in \mathbb{N}$, such that the following holds. For any $f\colon \{-1,1\}^n \to \{-1,1\}$, there is $J \subseteq [n]$ of size at most D such that

$$\Pr_{z \in \{-1,1\}^J}\left[\exists i \notin J, I_i^{\leqslant d}[f_{J \to z}] \geqslant \tau\right] \leqslant \varepsilon.$$