18.218 Topics in Combinatorics Spring 2021 – problem set 5

1. In this question, we will establish Borel's theorem about noise stability in Gaussian space for some values of ρ. Let $f\colon \mathbb{R}^n \to \{-1,1\}$ be a balanced function, and let $M_\rho = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ be the covariance matrix of ρ-correlated Gaussians.

   (a) Let $\theta \in [0, 2\pi)$, and let X, Z be independent standard Gaussians. Argue that for $Y = \cos(\theta)X + \sin(\theta)Z$, the joint distribution of (X,Y) is of ρ-correlated Gaussians for $\rho = \cos(\theta)$.

   (b) Prove that for $\rho_1 = \cos(\theta)$, $\rho_2 = \cos(\theta/2)$, it holds that $\Pr_{(X,Y)\sim N(0,M_{\rho_1})^{\otimes n}}[f(X) \neq f(Y)] \leqslant 2\Pr_{(X,Y)\sim N(0,M_{\rho_2})^{\otimes n}}[f(X) \neq f(Y)]$.

   (c) Prove that if $\rho = \cos(\pi/2k)$ for $k \in \mathbb{N}$, then $\Pr_{(X,Y)\sim N(0,M_{\rho})^{\otimes n}}[f(X) \neq f(Y)] \geqslant \frac{1}{2k}$

   (d) Deduce that for $\rho = \cos(\pi/2k)$, it holds that $\mathsf{Stab}_\rho(f) \leqslant 1 - \frac{1}{k} = 1 - \frac{2}{\pi}\mathsf{Arccos}(\rho)$.

2. Let $\varepsilon > 0$, and consider the following variant of the linearity test we have seen early in the course. Given a function $f\colon \{-1,1\}^n \to \{-1,1\}$, sample $x, y \sim \{-1,1\}^n$ uniformly and sample $z \in \{-1,1\}^n$ by taking $z_i = -1$ with probability ε, and otherwise $z_i = 1$, for each $i \in [n]$ independently. Test that $f(x)f(y) = f(xyz)$, where as usual xyz denotes the coordinatewise product vector.

   Show that if f is a function that passes this test with probability at least $\frac{1}{2} + \delta$, then there exists $S \subseteq [n]$ such that $|S| \leqslant \frac{\log(1/\delta)}{2\varepsilon}$ and $\widehat{f}(S) \geqslant \delta$.

3. Recall that the $\text{3LIN}_{\mathbb{F}_2}$ is the problem wherein one is given as input a set of variables $X = \{x_1, \ldots, x_n\}$ and a collection of equations E, each one of the form $x_i x_j x_k = b_{i,j,k}$ where $b_{i,j,k} \in \{-1,1\}$. The goal is to find an assignment of ± 1 to the x_i's that satisfies as many of the equations as possible.

   (a) Let $\varepsilon, \delta > 0$. Design an instance of $\text{3LIN}_{\mathbb{F}_2}$ over the hypercube according to the dictatorship vs no-influential-coordinates paradigm, such that if $f\colon \{-1,1\}^n \to \{-1,1\}$ is a dictatorship, then it solves $1 - \varepsilon$ of the equations, and if $f\colon \{-1,1\}^n \to \{-1,1\}$ is balanced with no influential coordinates, then it solves at most $\frac{1}{2} + \delta$ of the equations.

   (b) Design a reduction from the Unique-Games problem to the Max-$\text{3LIN}_{\mathbb{F}_2}$ problem.

   (c) Using this reduction, show that assuming the Unique-Games Conjecture, Max-$\text{3LIN}_{\mathbb{F}_2}[1-\varepsilon, \frac{1}{2}+\delta]$ is NP-hard for all $\varepsilon, \delta > 0$.

   Remark 0.1. *This result is known to hold without assuming the Unique-Games Conjecture.*

4. (*) In this question, we will prove a strengthening of the Majority is Stablest theorem.

   (a) Show that for all $\varepsilon, \tau, \xi > 0$ there are $d, D \in \mathbb{N}$ and $\delta > 0$ such that the following holds. If $f\colon \{-1,1\}^n \to \{-1,1\}$ is balanced, and $\left|\widehat{f}(S)\right| \leqslant \delta$ for all $|S| \leqslant D$, then there is $J \subseteq [n]$ of size at most D such that

   $$\Pr_{z\in\{-1,1\}^J}\left[(\exists i \notin J, I_i^{\leqslant d}[f] \geqslant \tau) \bigvee |\mathbb{E}[f_{J\to z}]| \geqslant \xi\right] \leqslant \varepsilon.$$

   (b) Deduce that for all $\rho \in (0,1)$, $\varepsilon > 0$, there are $d, \delta > 0$, such that if if $f\colon \{-1,1\}^n \to \{-1,1\}$ is balanced, and $\left|\widehat{f}(S)\right| \leqslant \delta$ for all $|S| \leqslant d$, then $\mathsf{Stab}_\rho(f) \leqslant \mathsf{Stab}_\rho(\mathsf{Majority}) + \varepsilon$.

18.218 Topics in Combinatorics: Analysis of Boolean Functions
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