Preface

Who Is This Book For?

This textbook is intended for graduate and advanced undergraduate students, as well as researchers in mathematics, computer science, and related areas. The material should appeal to anyone with an interest in combinatorics, theoretical computer science, analysis, probability, and number theory. It can be used as a textbook for a class or self-study, or as a research reference.

Why This Book?

There have been many exciting developments in graph theory and additive combinatorics in recent decades. It is the first introductory graduate level textbook to focus on a unifying set of topics connecting graph theory and additive combinatorics.

This textbook arose from a one-semester graduate-level course that I developed at MIT (and still teach regularly) to introduce students to a spectrum of beautiful mathematics in the field.

Lecture Videos

A complete set of video lectures from my Fall 2019 class is available for free through MIT OpenCourseWare and YouTube (search for *Graph Theory and Additive Combinatorics* and *MIT OCW*). The lecture videos are a useful resource and complement this book.

What Is This Book About?

This book introduces the readers to classical and modern developments in graph theory and additive combinatorics, with a focus on topics and themes that connect the two subjects.

A foundational result in additive combinatorics is **Roth's theorem**, which says that every subset of $\{1, 2, \ldots, \}$ without a 3-term arithmetic progression has at most $o(N)$ elements. We will see different proofs of Roth's theorem, using tools from graph theory and Fourier analysis. A key idea in both approaches is the *dichotomy of structure versus pseudorandomness*.

Roth's theorem laid the groundwork for many important later developments, such as

- **Szemerédi's theorem:** Every set of integers of positive density contains arbitrarily long arithmetic progressions; and
- **Green–Tao theorem:** The primes contain arbitrarily long arithmetic progressions.

A core thread throughout the book is the connection bridging graph theory and additive combinatorics. The book opens with Schur's theorem, which is an early example whose proof illustrates this connection. Graph theoretic perspectives are presented throughout the book.

Here are some of the topics and questions considered in this book:

Chapter 1: **Forbidding a subgraph**
What is the maximum number of edges in a triangle-free graph on n vertices? What if instead we forbid some other subgraph? This is known as the Turán problem.

Chapter 2: **Graph regularity method**
Szemerédi introduced this powerful tool that provides an approximate structural description for every large graph.

Chapter 3: **Pseudorandom graphs**
What does it mean for some graph to resemble a random graph?

Chapter 4: **Graph limits**
In what sense can a sequence of graphs, increasing in size, converge to some limit object?

Chapter 5: **Graph homomorphism inequalities**
What are possible relationships between subgraph densities?

Chapter 6: **Forbidding a 3-term arithemtic progression**
Roth's theorem and Fourier analysis in additive combinatorics.

Chapter 7: **Structure of set addition**
What can one say about a set of integer A with small sumset $A + A = \{a + b : a, b \in A\}$? Freiman's theorem is a foundational result that gives an answer.

Chapter 8: **Sum-product problem**
Can a set A simultaneously have both small sumset $A + A$ and product set $A \cdot A$?

Chapter 9: **Progressions in sparse pseudorandom sets**
Key ideas in the proof of the Green–Tao theorem. How can we apply a dense setting result, namely Szemerédi's theorem, to a sparse set?

For a more detailed list of topics, see the highlights and summary boxes at the beginning and the end of each chapter.

The book is roughly divided into two parts, with graph theory the focus of Chapters 1 to 5 and additive combinatorics the focus of Chapters 6 to 9. These are not disjoint and separate subjects. Rather, graph theory and additive combinatorics are interleaved throughout the book. We emphasize their interactions. Each chapter can be enjoyed independently, as there are very few dependencies between chapters, though one gets the most out of the book by appreciating the connections.

Using the Textbook for a Class

The contents may be taught as a fast-paced one-semester class, or as a two-semester sequence with each term focusing one half of the book: the first on extremal graph theory, and the second on additive combinatorics.

For a one-semester class (which is how I teach it at MIT; see my website or MIT OCW for syllabus, lecture videos, homework, and further information), I suggest skipping some more technical or advanced topics and proofs, such as the following: (Chapter 1) the proofs of the Erdős–Stone–Simonovits theorem, the $K_{s,t}$ construction, randomized algebraic construction; (Chapter 2) the proof of the graph counting lemma, induced graph removal and strong regularity, hypergraph regularity and removal; (Chapter 3) quasirandom groups, quasirandom Cayley graphs; (Chapter 4) most technical proofs on graph limits; (Chapter 5) Hölder, entropy; (Chapter 6) arithmetic regularity and popular common difference; (Chapter 7) proofs later in the chapter if short on time; (Chapter 9) proof details.

For a class focused on one part of the book, one may wish to explore further topics as suggested in *Further Reading* at the end of each chapter.

Prerequisites

The prerequisites are minimal – primarily mathematical maturity and an interest in combinatorics. Some basic concepts from abstract algebra, analysis, and linear algebra are assumed.

Exercises

The book contains around 150 carefully selected exercises. They are scattered throughout each chapter. Some exercises are embedded in the middle of a section – these exercises are meant as routine tests of understanding of the concepts just discussed. For example, they sometimes ask you to fill in missing proof details or think about easy generalizations and extensions. The exercises at the end of each section are carefully selected problems that reinforce the techniques discussed in the chapter. Hopefully they are all interesting. Most of them are intended to test your mastery of the techniques taught in the chapter. Many of these end-of-chapter exercises are quite challenging, with starred problems intended to be more difficult but still doable by a strong student given the techniques taught. Many of these exercises are adapted from lemmas and results from research papers. (I apologize for omitting references for the exercises so that they can be used as homework assignments.)

Spending time with the exercises is essential for mastering the techniques. I used many of these exercises in my classes. My students often told me that they thought that they had understood the material after a lecture, only to discover their incomplete mastery when confronted with the exercises. Struggling with these exercises led them to newfound insight.

Further Reading

This is a massive and rapidly expanding subject. The book is intended to be introductory and enticing rather than comprehensive. Each chapter concludes with recommendations for further reading for anyone who wishes to learn more. Additionally, references are given generously throughout the text for anyone who wishes to dive deeper and read the original sources.

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