GRAPH THEORY AND ADDITIVE COMBINATORICS
MIT 18.225 (FALL 2023)
PROBLEM SET

A. Ramsey

A1. *Upper bound on Ramsey numbers.* Let s and t be positive integers. Show that if the edges of a complete graph on $\binom{s+t-2}{s-1}$ vertices are colored with red and blue, then there must be either a red K_s or a blue K_t.

A2. *Ramsey's theorem.* Show that for every s and r there exists some $N = N(s, r)$ such that every coloring of the edges of K_N using r colors, there exists some monochromatic copy clique on s vertices. Also, generalize this statement to hypergraphs.

ps1 A3. Prove that it is possible to color $\mathbb{N}$ using two colors so that there is no infinitely long monochromatic arithmetic progression.

ps1 A4. *Many monochromatic triangles*

(a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored uniformly at random.)

(b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

B. Forbidding a Subgraph

ps1 B1. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$ triangles.

B2. Prove that every n-vertex nonbipartite triangle-free graph has at most $(n-1)^2/4 + 1$ edges.

ps1 B3. Show that every n-vertex triangle-free graph with minimum degree greater than $2n/5$ is bipartite.

ps1 B4. *Mantel stability.* Let G be an n-vertex triangle-free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.

ps1⋆ B5. *Turán stability.* Let G be an n-vertex K_{r+1}-free graph with at least $e(T_{n,r}) - k$ edges, where $T_{n,r}$ is the Turán graph. Prove that G can be made r-partite by removing at most k edges.

ps1⋆ B6. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

ps1⋆ B7. Let G be an n-vertex graph with $\lfloor n^2/4 \rfloor - k$ edges (here $k \in \mathbb{Z}$) and t triangles. Prove that G can be made bipartite by removing at most $k + 6t/n$ edges, and that this constant 6 is best possible.

ps1⋆ B8. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is best possible.

ps1⋆ B9. *Large induced bipartite subgraph.* Prove that for every $\varepsilon > 0$, there exist $\delta, C > 0$ so that the following holds. If G is an n-vertex graph with at least $n^2/4$ edges such that every edge of G lies in at most $(1/2 - \varepsilon)n$ triangles, and the number of triangles t of G is at most δn^3, then there is an induced bipartite subgraph containing all but at most Ct/n^2 vertices of G.

B10. Let G be a K_{r+1}-free graph. Prove that there is another graph H on the same vertex set as G such that $\chi(H) \le r$ and $d_H(x) \ge d_G(x)$ for every vertex x (here $d_H(x)$ is the degree of x in H, and likewise with $d_G(x)$ for G). Give another proof of Turán's theorem from this fact.

B11. Let X and Y be independent and identically distributed random vectors in $\mathbb{R}^d$ according to some arbitrary probability distribution. Prove that

$$\mathbb{P}(|X+Y| \ge 1) \ge \frac{1}{2}\mathbb{P}(|X| \ge 1)^2.$$

ps1 B12. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).

B13. *Density Ramsey.* Prove that for every s and r, there exist $c > 0$ and n_0 such that for all $n > n_0$, if the edges of K_n are colored using r colors, then at least c fraction of all copies of K_s are monochromatic.

ps2 B14. *Density Szemerédi.* Let $k \ge 3$. Assuming Szemerédi's theorem for k-term arithmetic progressions (i.e., every subset of $[N]$ without a k-term arithmetic progression has size $o(N)$), prove the following density version of Szemerédi's theorem:

For every $\delta > 0$ there exist $c > 0$ and N_0 (both depending only on k and δ) such that for every $A \subseteq [N]$ with $|A| \ge \delta N$ and $N \ge N_0$, the number of k-term arithmetic progressions in A is at least cN^2.

ps2 B15. (How *not* to define density in a product set) Let $S \subseteq \mathbb{Z}^2$. Define

$$d_k(S) = \max_{\substack{A,B \subseteq \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A||B|}.$$

Show that $\lim_{k\to\infty} d_k(S)$ exists and is always either 0 or 1.

(Note: You are only allowed to invoke theorems we proved in class.)

B16. Show that a C_4-free bipartite graph between two vertex parts of sizes a and b has at most $ab^{1/2} + b$ edges.

B17. Show that, for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph with n vertices and at least ϵn^2 edges contains a copy of $K_{s,t}$ where $s \ge \delta \log n$ and $t \ge n^{0.99}$.

B18. *Density version of Kővári–Sós–Turán.* Prove that for every positive integers $s \le t$, there are constants $C, c > 0$ such that every n-vertex graph with $p\binom{n}{2}$ edges contains at least $cp^{st}n^{s+t}$ copies of $K_{s,t}$, provided that $p \ge Cn^{-1/s}$.

(Note: check that you are using $p \ge Cn^{-1/s}$ to avoid a common mistake.)

B19. *Erdős–Stone theorem for hypergraphs.* Let H be an r-graph. Show that $\pi(H[s]) = \pi(H)$, where $H[s]$, the s-blow-up of H, is obtained by replacing every vertex of H by s duplicates of itself.

ps2 B20. Let T be a tree with k edges. Show that $\mathrm{ex}(n,T) \le kn$.

ps2 B21. Find a graph H with $\chi(H) = 3$ and $\mathrm{ex}(n, H) > \frac{1}{4}n^2 + n^{1.99}$ for all sufficiently large n.

The next two problems concern the dependent random choice technique.

ps2 B22. Let $\epsilon > 0$. Show that, for sufficiently large n, every K_4-free graph with n vertices and at least ϵn^2 edges contains an independent set of size at least $n^{1-\epsilon}$.

ps2⋆ B23. *Extremal numbers of degenerate graphs*

(a) Prove that there is some absolute constant $c > 0$ so that for every positive integer r, every n-vertex graph with at least $n^{2-c/r}$ edges contains disjoint non-empty vertex subsets A and B such that every subset of at most r vertices in A has at least n^c common neighbors in B and every subset of at most r vertices in B has at least n^c neighbors in A.

(b) We say that a graph H is *r-degenerate* if its vertices can be ordered so that every vertex has at most r neighbors that appear before it in the ordering. Show that for every r-degenerate bipartite graph H there is some constant $C > 0$ so that $\mathrm{ex}(n, H) \le Cn^{2-c/r}$, where c is the same absolute constant from part (a) (c should not depend on H or r).

C. Graph Regularity Method

You are welcome to apply the equitable version of the graph regularity lemma.

C1. *Basic inheritance of regularity.* Let G be a graph and $X, Y \subseteq V(G)$. If (X, Y) is an $\epsilon\eta$-regular pair, then (X', Y') is ϵ-regular for all $X' \subseteq X$ with $|X'| \ge \eta |X|$ and $Y' \subseteq Y$ with $|Y'| \ge \eta |Y|$.

C2. *An alternate definition of regular pairs.* Let G be a graph and $X, Y \subseteq V(G)$. Say that (X, Y) is *ϵ-homogeneous* if for all $A \subseteq X$ and $B \subseteq Y$, one has

$$|e(A, B) - |A| |B| d(X, Y)| \le \epsilon |X| |Y| .$$

Show that if (X, Y) is ϵ-regular, then it is ϵ-homogeneous. Also, show that if (X, Y) is ϵ^3-homogeneous, then it is ϵ-regular.

C3. *Robustness of regularity.* Prove that for every $\epsilon' > \epsilon > 0$, there exists $\delta > 0$ so that given an ϵ-regular pair (X, Y) in some graph, if we modify the graph by adding/deleting $\le \delta |X|$ vertices to/from X, adding/deleting $\le \delta |Y|$ vertices to/from Y, and adding/deleting $\le \delta |X| |Y|$ edges, then the resulting new (X, Y) is still ϵ'-regular.

ps2 C4. *Existence of a large regular pair.* Show that there is some absolute constant $C > 0$ such that for every $0 < \epsilon < 1/2$, every graph on n vertices contains an ϵ-regular pair of vertex subsets each with size at least δn, where $\delta = 2^{-\epsilon^{-C}}$.

C5. *Existence of a regular set.* Given a graph G, we say that $X \subseteq V(G)$ is *ϵ-regular* if the pair (X, X) is ϵ-regular, i.e., for all $A, B \subseteq X$ with $|A|, |B| \ge \epsilon|X|$, one has $|d(A, B) - d(X, X)| \le \epsilon$. This problem asks for two different proofs of the claim: for every $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex graph contains an ϵ-regular subset of vertices of size at least δn.

ps2 (a) Prove the claim using the graph regularity lemma by showing that an ϵ-regular subset can be produced by combining parts from some regularity partition.

ps2⋆ (b) Give an alternative proof of the claim showing that one can take $\delta = \exp(-\exp(\epsilon^{-C}))$ for some constant C.

ps2⋆ C6. *Regularity partition into regular sets.* Prove that for every $\epsilon > 0$ there exists M so that every graph has an ϵ-regular partition into at most M parts, with each part ϵ-regular with itself.

ps2⋆ C7. *Making each part regular to nearly all other parts.* Prove that for all $\epsilon > 0$ and m_0, there exists a constant M so that every graph has an equitable vertex partition into k parts, with $m_0 \le k \le M$, such that each part is ϵ-regular with all but at most ϵk other parts.

ps3 C8. *Unavoidability of irregular pairs.* Let the *half-graph* H_n be the bipartite graph on $2n$ vertices $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with edges $\{a_i b_j : i \le j\}$.

(a) For every $\epsilon > 0$, explicitly construct an ϵ-regular partition of H_n into $O(1/\epsilon)$ parts.

(b) Show that there is some $\epsilon > 0$ such that for every integer k and sufficiently large multiple n of k, every partition of the vertices of H_n into k equal-sized parts contains at least ϵk pairs of parts none of which are ϵ-regular.

ps3 C9. *Diamond-free redux.* Prove that each of the following statements is equivalent to the diamond-free lemma (GTAC Corollary 2.3.3).

(a) *The (6,3) theorem.* Let H be an n-vertex 3-uniform hypergraph without a subgraph having 6 vertices and 3 edges. Then H has $o(n^2)$ edges.

(b) *Induced matchings.* Every n-vertex graph that is a union of n induced matchings has $o(n^2)$ edges.

(An *induced matching* is an induced subgraph that is also a matching. For example, $K_{2,2}$ is not the union of two induced matchings, but it is the union of four induced matchings each being a single edge.)

ps3⋆ C10. *Arithmetic triangle removal lemma.* Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subseteq [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subseteq A$ with $|A \setminus B| \le \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.

ps3⋆ C11. *Avoiding length-5 quadratic configurations.* Show that there is some constant $C > 0$ so that for every N there is a subset $S \subseteq [N]$ with $|S| \ge N e^{-C\sqrt{\log N}}$ such that there does not exist a nonconstant quadratic polynomial P with $P(0), P(1), P(2), P(3), P(4) \in S$.

ps3 C12. *Ramsey–Turán.*

(a) Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with independence number at most δn has at most $(\frac{1}{8} + \epsilon)n^2$ edges.

(b) Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{8} - \delta)n^2$ edges and independence number at most δn can be made bipartite by removing at most ϵn^2 edges.

ps3 C13. *Nearly homogeneous subset.* Show that for every H and $\epsilon > 0$, there exists $\delta > 0$ such that every graph on n vertices without an induced copy of H contains an induced subgraph on at least δn vertices whose edge density is at most ϵ or at least $1 - \epsilon$.

ps3 C14. *Ramsey numbers of bounded degree graphs.* Show that for every Δ, there exists a constant C_Δ so that if H is a graph with maximum degree at most Δ, then every 2-edge-coloring of a complete graph on at least $C_\Delta v(H)$ vertices contains a monochromatic copy of H.

ps3⋆ C15. *Counting H-free graphs.* Let H be a graph. Show that the number of H-free graphs on an n-vertex set is $2^{\mathrm{ex}(n,H)+o(n^2)}$.

ps3⋆ C16. *Induced Ramsey.* Show that for every graph H, there is some graph G such that every 2-edge-coloring of G has an induced monochromatic copy of H.

ps3⋆ C17. Show that for every $\alpha > 0$, there exists $\beta > 0$ such that every n-vertex graph with at least αn^2 edges contains a d-regular subgraph for some $d \geq \beta n$ (here *d-regular* refers to every vertex having degree d).

ps3 C18. *Multidimensional Szemerédi for four-point patterns.* For this problem, you should assume GTAC Corollary 2.10.2 (of the tetrahedron removal lemma).

(a) *Three-dimensional corners.* Suppose $A \subseteq [N]^3$ contains no four points of the form

$$(x, y, z),\ (x + d, y, z),\ (x, y + d, z),\ (x, y, z + d), \quad \text{with } d > 0.$$

Show that $|A| = o(N^3)$.

(b) *Axis-aligned squares.* Suppose $A \subseteq [N]^2$ contains no four points of the form

$$(x, y),\ (x + d, y),\ (x, y + d),\ (x + d, y + d), \quad \text{with } d > 0.$$

Show that $|A| = o(N^2)$.

D. Pseudorandom Graphs

ps4 D1. Let q be a prime. Let $S \subseteq \mathbb{F}_q \cup \{\infty\}$. Construct a graph G on vertex set $\mathbb{F}_q^2$ where two points are joined if the slope of the line connecting them lies in S. Viewed as a sequence of graphs as $q \to \infty$, prove that G is quasirandom as long as $|S|/q$ converges to a limit.

D2. *Nearly optimal C_4-free graphs are sparse quasirandom.* Let G_n be a sequence of n-vertex C_4-free graphs with $(1/2 - o(1))n^{3/2}$ edges. Prove that $e_{G_n}(A, B) = n^{-1/2} |A| |B| + o(n^{3/2})$ for every $A, B \subseteq V(G_n)$.

Hint: Revisit the CODEG $\implies$ DISC proof and the proof of the KST theorem.

ps4⋆ D3. *Quasirandomness through fixed sized subsets.* Fix $p \in [0, 1]$. Let (G_n) be a sequence with $v(G_n) = n$. Write $G = G_n$.

(a) Fix a single $\alpha \in (0, 1)$. Suppose

$$e(S) = \frac{p\alpha^2 n^2}{2} + o(n^2) \qquad \text{for all } S \subseteq V(G) \text{ with } S = \lfloor \alpha n \rfloor .$$

Prove that G is quasirandom.

(b) Fix a single $\alpha \in (0, 1/2)$. Write $\overline{S} = V(G) \setminus S$. Suppose

$$e(S, \overline{S}) = p\alpha(1 - \alpha)n^2 + o(n^2) \qquad \text{for all } S \subseteq V(G) \text{ with } S = \lfloor \alpha n \rfloor .$$

Prove that G is quasirandom. Furthermore, show that the conclusion is false for $\alpha = 1/2$.

D4. *Quasirandomness and regularity partitions.* Fix $p \in [0, 1]$. Let (G_n) be a sequence of graphs with $v(G_n) \to \infty$. Suppose that for every $\epsilon > 0$, there exists $M = M(\epsilon)$ so that each G_n has an ϵ-regular partition where all but ϵ-fraction of vertex pairs lie between pairs of parts with edge density $p + o(1)$ (as $n \to \infty$). Prove that G_n is quasirandom.

ps4⋆ D5. *Triangle counts on induced subgraphs.* Fix $p \in (0, 1]$. Let (G_n) be a sequence of graphs with $v(G_n) = n$. Let $G = G_n$. Suppose that for every $S \subseteq V(G)$, the number of triangles in the induced subgraph $G[S]$ is $p^3\binom{|S|}{3} + o(n^3)$. Prove that G is quasirandom.

D6. Prove that there are constants $\beta, \epsilon > 0$ such that for every even positive integer n and real $p \geq n^{-\beta}$, if G is an n-vertex graph where every vertex has degree $(1 \pm \epsilon)pn$ (meaning within ϵpn of pn) and every pair of vertices has codegree $(1 \pm \epsilon)p^2 n$, then G has a perfect matching.

The next two exercises ask you to prove *Cheeger's inequality*:

$$\kappa/2 \leq h \leq \sqrt{2d\kappa}$$

for every d-regular graph with *spectral gap* $\kappa = d - \lambda_2$ and *edge-expansion ratio*

$$h := \min_{\substack{S \subseteq V \\ 0 < |S| \leq |V|/2}} \frac{e_G(S, V \setminus S)}{|S|}.$$

ps4 D7. *Spectral gap implies expansion.* Prove that every d-regular graph with spectral gap κ has edge-expansion ratio $\geq \kappa/2$.

ps4 D8. *Expansion implies spectral gap.* Let $G = (V, E)$ be a connected d-regular graph with spectral gap κ. Let $x = (x_v)_{v \in V} \in \mathbb{R}^V$ be an eigenvector associated to the second largest eigenvalue $\lambda_2 = d - \kappa$ of the adjacency matrix of G. Assume that $x_v > 0$ on at most half of the vertex set (or else we replace x by $-x$). Let $y = (y_v)_{v \in V} \in \mathbb{R}^V$ be obtained from x by replacing all its negative coordinates by zero.

(a) Prove that

$$d - \frac{\langle y, Ay \rangle}{\langle y, y \rangle} \leq \kappa.$$

Hint: recall that $\lambda_2 x_v = \sum_{u \sim v} x_u$.

(b) Let

$$\Theta = \sum_{uv \in E} \left| y_u^2 - y_v^2 \right|.$$

Prove that

$$\Theta^2 \leq 2d(d \langle y, y \rangle - \langle y, Ay \rangle) \langle y, y \rangle.$$

Hint: $y_u^2 - y_v^2 = (y_u - y_v)(y_u + y_v)$. Apply Cauchy–Schwarz.

(c) Relabel the vertex set V by $[n]$ so that $y_1 \geq y_2 \cdots \geq y_t > 0 = y_{t+1} = \cdots = y_n$. Prove

$$\Theta = \sum_{k=1}^{t} (y_k^2 - y_{k+1}^2)\ e([k], [n] \setminus [k]).$$

(d) Prove that for some $1 \leq k \leq t$,

$$\frac{e([k], [n] \setminus [k])}{k} \leq \frac{\Theta}{\langle y, y \rangle}.$$

(e) Prove the G has edge-expansion ratio $\leq \sqrt{2d\kappa}$.

D9. *Independence number.* Prove that every independent set in a (n, d, λ)-graph has size at most $n\lambda/(d + \lambda)$.

D10. *Diameter.* Prove that the diameter of an (n, d, λ)–graph is at most $\lceil \log n / \log(d/\lambda) \rceil$. (The *diameter* of a graph is the maximum distance between a pair of vertices.)

ps4⋆ D11. *Counting cliques.* For each part below, prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that the conclusion holds for every (n, d, λ)-graph G with $d = pn$.

(a) If $\lambda \le \delta p^2 n$, then the number of triangles of G is within a $1 \pm \epsilon$ factor of $p^3 \binom{n}{3}$.

(b) If $\lambda \le \delta p^3 n$, then the number of K_4's in G is within a $1 \pm \epsilon$ factor of $p^6 \binom{n}{4}$.

ps4 D12. Let p be an odd prime and $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$. Show that

$$\left| \sum_{a \in A} \sum_{b \in B} \left(\frac{a+b}{p} \right) \right| \le \sqrt{p \, |A| \, |B|}$$

where (a/p) is the Legendre symbol defined by

$$\left(\frac{a}{p} \right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod p, \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p. \end{cases}$$

D13. *No spectral gap if too few generators.* Prove that for every $\epsilon > 0$ there is some $c > 0$ such that for every $S \subseteq \mathbb{Z}/n\mathbb{Z}$ with $0 \notin S = -S$ and $|S| \le c \log n$, the second largest eigenvalue of the adjacency matrix of $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ is at least $(1-\epsilon)\,|S|$.

ps4★ D14. Let p be a prime and let S be a multiplicative subgroup of $\mathbb{F}_p^\times$. Suppose $-1 \in S$. Prove that all eigenvalues of the adjacency matrix of $\mathrm{Cay}(\mathbb{Z}/p\mathbb{Z}, S)$, other than the top one, are at most $\sqrt{p}$ in absolute value.

D15. *Growth and expansion in quasirandom groups.* Let Γ be a finite group with no non-trivial representations of dimension less than K. Let $X, Y, Z \subseteq \Gamma$. Suppose $|X|\,|Y|\,|Z| \ge |\Gamma|^3 / K$. Then $XYZ = \Gamma$ (i.e., every element of Γ can be expressed as xyz for some $(x, y, z) \in X \times Y \times Z$).

ps4 D16. Prove that for every positive integer d and real $\epsilon > 0$, there is some constant $c > 0$ so that every n-vertex d-regular graph has at least cn eigenvalues greater than $2\sqrt{d-1} - \epsilon$.

(Full credit will be awarded for proving the weaker statement that $\ge cn$ eigenvalues have *absolute value* $> 2\sqrt{d-1} - \epsilon$.)

ps4★ D17. Show that for every d and r, there is some $\epsilon > 0$ such that if G is a d-regular graph, and $S \subseteq V(G)$ is such that every vertex of G is within distance r of S, then the top eigenvalue of the adjacency matrix of $G - S$ (i.e., remove S and its incident edges from G) is at most $d - \epsilon$.

E. Graph Limits

E1. *Zero-one valued graphons.* Let W be a $\{0,1\}$-valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_\square \to 0$ as $n \to \infty$. Show that $\|W_n - W\|_1 \to 0$ as $n \to \infty$.

ps4 E2. Define $W \colon [0,1]^2 \to \mathbb{R}$ by $W(x,y) = 2\cos(2\pi(x-y))$. Let F be a graph. Show that $t(F, W)$ is the number of ways to orient all edges of F so that every vertex has the same number of incoming edges as outgoing edges.

ps4 E3. *Weak regularity decomposition.* The following exercise offers alternate approach to the weak regularity lemma. It gives an approximation of a graphon as a linear combination of $\le \epsilon^{-2}$ indictor functions of boxes. The polynomial dependence of ϵ^{-2} is important for designing efficient approximation algorithms.

(a) Let $\epsilon > 0$. Show that for every graphon W, there exist measurable $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0,1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < \epsilon^{-2}$, such that

$$\left\| W - \sum_{i=1}^{k} a_i \mathbf{1}_{S_i \times T_i} \right\|_\square \le \epsilon.$$

The rest of the exercise shows how to recover a regularity partition from the above approximation.

(b) Show that the stepping operator is contractive with respect to the cut norm, in the sense that if $W\colon [0,1]^2 \to \mathbb{R}$ is a measurable symmetric function, then $\|W_\mathcal{P}\|_\square \le \|W\|_\square$.

(c) Let $\mathcal{P}$ be a partition of $[0,1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. Show that for every graphon W, one has

$$\|W - W_\mathcal{P}\|_\square \le 2\|W - U\|_\square.$$

(d) Use (a) and (c) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W, there exists partition $\mathcal{P}$ of $[0,1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_\mathcal{P}\|_\square \le \epsilon$.

ps4⋆ E4. *Second neighborhood distance.* Let W be a graphon. Define $\tau_{W,x}\colon [0,1] \to [0,1]$ by

$$\tau_{W,x}(z) = \int_{[0,1]} W(x,y)W(y,z)\, dy.$$

(This models the second neighborhood of x.) Let $0 < \epsilon < 1/2$. Prove that if a finite set $S \subseteq [0,1]$ satisfies

$$\|\tau_{W,s} - \tau_{W,t}\|_1 > \epsilon \qquad \text{for all distinct } s, t \in S,$$

then $|S| \le (1/\epsilon)^{C/\epsilon^2}$, where C is some absolute constant.

E5. Show that for every $0 < \epsilon < 1/2$, every graphon lies within cut distance at most ϵ from some graph on at most C^{1/ϵ^2} vertices, where C is some absolute constant.

ps5 E6. *Inverse counting lemma.* Using the compactness of the graphon space and the uniqueness of moments theorem, deduce that for every $\epsilon > 0$ there exist $\eta > 0$ and integer $k > 0$ such that if U and W are graphons with

$$|t(F,U) - t(F,W)| \le \eta \quad \text{whenever } v(F) \le k,$$

then $\delta_\square(U,W) \le \epsilon$.

ps5⋆ E7. *Generalized maximum cut.* For symmetric measurable functions $W, U\colon [0,1]^2 \to \mathbb{R}$, define

$$\mathcal{C}(W,U) := \sup_\phi \langle W, U^\phi \rangle = \sup_\phi \int W(x,y)U(\phi(x),\phi(y))\, dxdy,$$

where ϕ ranges over all invertible measure preserving maps $[0,1] \to [0,1]$. Extend the definition of $\mathcal{C}(\cdot,\cdot)$ to graphs by $\mathcal{C}(G,\cdot) := \mathcal{C}(W_G,\cdot)$, etc.

(a) Is $\mathcal{C}(U,W)$ continuous jointly over pairs (U,W) of graphons with respect to the cut norm? Is it continuous in U if W is held fixed?

(b) *(Key part of the problem)* Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1,U) = \mathcal{C}(W_2,U)$ for all graphons U, then $\delta_\square(W_1,W_2) = 0$.

(c) Let $G_1, G_2, \dots$ be a sequence of graphs such that $\mathcal{C}(G_n, U)$ converges as $n \to \infty$ for every graphon U. Show that $G_1, G_2, \dots$ is convergent.

(d) Can the hypothesis in (c) be replaced by "$\mathcal{C}(G_n, H)$ converges as $n \to \infty$ for every graph H"?

E8. (a) Let G_1 and G_2 be two graphs such that $\hom(F, G_1) = \hom(F, G_2)$ for every graph F. Show that G_1 and G_2 are isomorphic.

(b) Let G_1 and G_2 be two graphs such that $\hom(G_1, H) = \hom(G_2, H)$ for every graph H. Show that G_1 and G_2 are isomorphic.

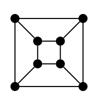
F. Graph Homomorphism Inequalities

Recall some definitions. A graph F is said to be

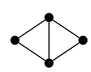
- *Sidorenko* if $t(F, W) \ge t(K_2, W)^{e(F)}$ for all graphons W;
- *forcing* if every graphon W with $t(F, W) = t(K_2, W)^{e(F)}$ is a constant graphon;
- *common* if $t(F, W) + t(F, 1 - W) \ge 2^{-e(F)+1}$ for all graphons W.

F1. Prove that C_6 is Sidorenko.

ps5 F2. Prove that Q_3, the skeleton of the 3-cube, shown below, is Sidorenko.



ps5 F3. Prove that K_4^- is common, where K_4^- is K_4 with one edge removed (a.k.a. diamond).



F4. Prove that every forcing graph is bipartite and has at least one cycle.

F5. Prove that every forcing graph is Sidorenko.

F6. *Forcing and quasirandomness.* Show that a graph F is forcing if and only if for every constant $p \in [0, 1]$, every sequence of graphs $G = G_n$ with

$$t(K_2, G) = p + o(1) \qquad \text{and} \qquad t(F, G) = p^{e(F)} + o(1)$$

is quasirandom.

F7. *Forcing and stability.* Show that a graph F is forcing if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if a graph G satisfies $t(F, G) \le t(K_2, G)^{e(F)} + \delta$, then $\delta_\square(G, p) \le \epsilon$.

F8. *Tensor power trick.* Let F be a bipartite graph. Suppose there is some constant $c > 0$ such that

$$t(F, G) \ge c\, t(K_2, G)^{e(F)} \quad \text{for all graphs } G.$$

Show that F is Sidorenko.

F9. Prove that $K_{s,t}$ is forcing whenever $s, t \ge 2$.

F10. *A lower bound on clique density.* Show that for every positive integer $r \ge 3$, and graphs G, writing $p = t(K_2, W)$,

$$t(K_r, W) \ge p(2p - 1)(3p - 2) \cdots ((r - 1)p - (r - 2)).$$

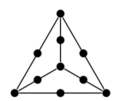
Note that this inequality is tight when W is the associated graphon of a clique.

ps5⋆ F11. Prove there is a function $f\colon [0,1] \to [0,1]$ with $f(x) \ge x^2$ and $\lim_{x\to 0} f(x)/x^2 = \infty$ such that

$$t(K_4^-, G) \ge f(t(K_3, G))$$

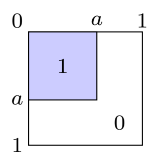
for all graphs G. Here K_4^- is K_4 with one edge removed.

ps5 F12. Let F be the 3-graph with 10 vertices and 6 edges illustrated below (with each line denoting an edge). Prove that the hypergraph Turán density of F is $2/9$.



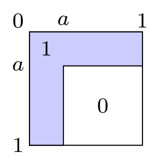
ps5⋆ F13. *Cliquey edges.* Let n, r, t be nonnegative integers. Show that every n-vertex graph with at least $(1 - \frac{1}{r})\frac{n^2}{2} + t$ edges contains at least rt edges that belong to a K_{r+1}.

ps5⋆ F14. *Maximizing $K_{1,2}$ density.* Prove that, for every $p \in [0,1]$, among all graphons W with $t(K_2, W) = p$, the maximum possible value of $t(K_{1,2}, W)$ is attained by either a "clique" or a "hub" graphon, illustrated below.



clique graphon

$W(x,y) = 1_{\max\{x,y\} \le a}$



hub graphon

$W(x,y) = 1_{\min\{x,y\} \le a}$

G. Forbidding 3-term Arithmetic Progressions

ps5 G1. *Fourier uniformity does not control 4-AP counts.* Let

$$A = \{x \in \mathbb{F}_5^n : x \cdot x = 0\}.$$

Prove that

(a) $|A| = (5^{-1} + o(1))5^n$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$

Hint: Gauss sum

(b) $|\{(x,y) \in \mathbb{F}_5^n : x, x+y, x+2y, x+3y \in A\}| \neq (5^{-4} + o(1))5^{2n}$.

ps5⋆ G2. *Fourier uniformity does not control 4-AP counts.* Fix $0 < \alpha < 1$. Let

$$A = \left\{x \in \mathbb{Z}/N\mathbb{Z} : (x^2 \bmod N) \in [0, \alpha N]\right\}.$$

Prove that, for every sufficiently small α and as $N \to \infty$ along primes,

(a) $|A| = (\alpha + o(1))N$ and $\max_{r\neq 0}|\widehat{1_A}(r)| = o(1)$;

(b) $|(x,y) \in \mathbb{Z}/N\mathbb{Z} : x, x+y, x+2y, x+3y \in A| \neq (\alpha^4 + o(1))N^2$.

ps5 G3. *Linearity testing.* Show that for every prime p there is some $C_p > 0$ such that if $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ satisfies

$$\mathbb{P}_{x,y\in\mathbb{F}_p^n}(f(x) + f(y) = f(x+y)) = 1 - \epsilon$$

then there exists some $a \in \mathbb{F}_p^n$ such that

$$\mathbb{P}_{x\in\mathbb{F}_p^n}(f(x) = a \cdot x) \ge 1 - C_p\epsilon.$$

In the above $\mathbb{P}$ expressions x and y are chosen i.i.d. uniform from $\mathbb{F}_p^n$.

G4. *Gowers U^2 uniformity norm.* Let $f\colon \mathbb{F}_p^n \to \mathbb{C}$. Define

$$\|f\|_{U^2} := \left(\mathbb{E}_{x,y,y'\in\mathbb{F}_p^n} f(x)\overline{f(x+y)f(x+y')}f(x+y+y')\right)^{1/4}.$$

(a) Show that the expectation above is always a nonnegative real number, so that the above expression is well defined. Also, show that $\|f\|_{U^2} \geq |\mathbb{E}f|$.

(b) (Gowers Cauchy–Schwarz) For $f_1, f_2, f_3, f_4\colon \mathbb{F}_p^n \to \mathbb{C}$, let

$$\langle f_1, f_2, f_3, f_4\rangle = \mathbb{E}_{x,y,y'\in\mathbb{F}_p^n} f_1(x)\overline{f_2(x+y)f_3(x+y')}f_4(x+y+y').$$

Prove that

$$|\langle f_1, f_2, f_3, f_4\rangle| \leq \|f_1\|_{U^2}\,\|f_2\|_{U^2}\,\|f_3\|_{U^2}\,\|f_4\|_{U^2}\,.$$

(c) (Triangle inequality) Show that

$$\|f+g\|_{U^2} \leq \|f\|_{U^2} + \|g\|_{U^2}.$$

Conclude that $\|\ \|_{U^2}$ is a norm.

Hint: Note that $\langle f_1, f_2, f_3, f_4\rangle$ is multilinear. Apply (b).

(d) (Relation with Fourier) Show that

$$\|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}.$$

Furthermore, deduce that if $\|f\|_\infty \leq 1$, then

$$\|\widehat{f}\|_\infty \leq \|f\|_{U^2} \leq \|\widehat{f}\|_\infty^{1/2}.$$

(The second inequality gives a so-called "inverse theorem" for the U^2 norm: if $\|f\|_{U^2} \geq \delta$ then $|\widehat{f}(r)| \geq \delta^2$ for some $r \in \mathbb{F}_p^n$. Informally, if f is not U^2-uniform, then f correlates with some exponential phase function of the form $x \mapsto \omega^{r\cdot x}$.)

ps5⋆ G5. *Gowers U^3 uniformity norm.* Let $f\colon \mathbb{F}_p^n \to \mathbb{C}$. Define

$$\|f\|_{U^3} := \Bigg(\mathbb{E}_{x,y_1,y_2,y_3} f(x)\overline{f(x+y_1)f(x+y_2)f(x+y_3)}\cdots \cdot f(x+y_1+y_2)f(x+y_1+y_3)f(x+y_2+y_3)\overline{f(x+y_1+y_2+y_3)}\Bigg)^{1/8}.$$

Alternatively, for each $y \in \mathbb{F}_p^n$, define the multiplicative finite difference $\Delta_y f\colon \mathbb{F}_p^n \to \mathbb{C}$ by $\Delta_y f(x) := f(x)\overline{f(x+y)}$. We can rewrite the above expression in terms of the U^2 uniformity norm from the previous exercise as

$$\|f\|_{U^3}^8 = \mathbb{E}_{y\in\mathbb{F}_p^n}\,\|\Delta_y f\|_{U^2}^4\,.$$

You should convince yourself that the above two definitions for $\|f\|_{U^3}$ coincide and give well-defined nonnegative real numbers.

(a) (Monotonicity) Also, show that

$$\|f\|_{U^2} \leq \|f\|_{U^3}\,.$$

(b) (Separation of norms) Let p be odd and $f\colon \mathbb{F}_p^n \to \mathbb{C}$ be defined by $f(x) = e^{2\pi i x\cdot x/p}$. Prove that $\|f\|_{U^3} = 1$ and $\|f\|_{U^2} = p^{-n/4}$.

(c) (Triangle inequality) Prove that

$$\|f+g\|_{U^3} \le \|f\|_{U^3} + \|g\|_{U^3}.$$

Conclude that $\|\ \|_{U^3}$ is a norm.

(d) (U^3 norm controls 4-APs) Let $p \ge 5$ be a prime, and $f_1, f_2, f_3, f_4\colon \mathbb{F}_p^n \to \mathbb{C}$ all taking values in the unit disk. We write

$$\Lambda(f_1, f_2, f_3, f_4) := \mathbb{E}_{x,y\in\mathbb{F}_p^n} f_1(x) f_2(x+y) f_3(x+2y) f_4(x+3y).$$

Prove that

$$|\Lambda(f_1, f_2, f_3, f_4)| \le \min_s \|f_s\|_{U^3}.$$

Furthermore, deduce that if $f, g\colon \mathbb{F}_p^n \to [0,1]$, then

$$|\Lambda(f,f,f,f) - \Lambda(g,g,g,g)| \le 4\|f-g\|_{U^3}.$$

Hint: Re-parameterize and repeatedly apply Cauchy–Schwarz.

ps5 G6. *Counting solutions to a single linear equation.*

(a) Given a function $f\colon \mathbb{Z} \to \mathbb{C}$ with finite support, define $\widehat{f}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n\in\mathbb{Z}} f(n) e^{-2\pi i n t}.$$

Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subseteq \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1,\dots,a_k) \in A^k : c_1 a_1 + \cdots + c_k a_k = 0\}| = \int_0^1 \widehat{1_A}(c_1 t)\widehat{1_A}(c_2 t)\cdots\widehat{1_A}(c_k t)\, dt.$$

(b) Show that if a finite set A of integers contains $\beta|A|^2$ solutions $(a,b,c) \in A^3$ to $a+2b = 3c$, then it contains at least $\beta^2|A|^3$ solutions $(a,b,c,d) \in A^4$ to $a+b = c+d$.

ps6 G7. Let $a_1, \dots, a_m, b_1, \dots, b_m, c_1, \dots, c_m \in \mathbb{F}_2^n$. Suppose that the equation $a_i + b_j + c_k = 0$ holds if and only if $i = j = k$. Show that there is some constant $c > 0$ such that $m \le (2-c)^n$ for all sufficiently large n.

G8. *Sunflower-free subset.* Three sets A, B, C form a *sunflower* if $A\cap B = B\cap C = A\cap B = A\cap B\cap C$. Prove that there exists some $c > 0$ such that if $\mathcal{F}$ is a collection of subsets of $[n]$ without a sunflower, then $|\mathcal{F}| \le (3-c)^n$ provided that n is sufficiently large.

H. Structure of Set Addition

ps6 H1. Show that for every real $K \ge 1$ there is some C_K such that for every finite set A of an abelian group with $|A+A| \le K|A|$, one has $|nA| \le n^{C_K}|A|$ for every positive integer n.
(You may not quote Freiman's theorem for abelian groups, which we did not prove.)

Hint: Review the proof of Freiman's theorem in groups with bounded exponent.

ps6⋆ H2. Show that there is some constant C so that if S is a finite subset of an abelian group, and k is a positive integer, then $|2kS| \le C^{|S|}|kS|$.

ps6⋆ H3. Show that for every sufficiently large K there is there some finite set $A \subseteq \mathbb{Z}$ such that $|A+A| \le K|A|$ and $|A-A| \ge K^{1.99}|A|$.

ps6⋆ H4. *Loomis–Whitney for sumsets.* Show that for every finite subsets A, B, C in an abelian group, one has

$$|A+B+C|^2 \le |A+B|\,|A+C|\,|B+C|\,.$$

H5. *Sumset versus difference set.* Let $A \subseteq \mathbb{Z}$. Prove that $|A-A|^{2/3} \le |A+A| \le |A-A|^{3/2}$.

ps6⋆ H6. *Another covering lemma.* Let A and B be finite sets in an abelian group satisfying $|A+A| \le K|A|$ and $|A+B| \le K'|B|$. Show that there exist some set X in the abelian group with $|X| = O(K\log(KK'))$ so that $A \subseteq \Sigma X + B - B$, where ΣX denotes the set of all elements that can be written as the sum of a subset of elements of X (including zero as the sum of the empty set).

Hint: Try first finding $2K$ disjoint translates $a + B$.

ps6 H7. *Modeling arbitrary sets of integers.* Let $A \subseteq \mathbb{Z}$ with $|A| = n$.

(a) Let p be a prime. Show that there is some integer t relatively prime to p such that $\|at/p\|_{\mathbb{R}/\mathbb{Z}} \le p^{-1/n}$ for all $a \in A$.

(b) Show that A is Freiman 2-isomorphic to a subset of $[N]$ for some $N = (4+o(1))^n$.

(c) Show that (b) cannot be improved to $N = 2^{n-2}$.

(You may use the fact that the smallest prime larger than m has size $m + o(m)$.)

ps6 H8. *Sumset with 3-AP-free set.* Let A and B be n-element subsets of the integers. Suppose A is 3-AP free. Prove that $|A+B| \ge n(\log\log n)^{1/100}$ provided that n is sufficiently large.

Hint: Ruzsa triangle ineq, Plünnecke ineq, Ruzsa model, Roth. Leave Freiman alone.

ps6 H9. *3-AP-free subsets of arbitrary sets of integers.* Prove that there is some constant $C > 0$ so that every set of n integers has a 3-AP-free subset of size at least $ne^{-C\sqrt{\log n}}$.

H10. *Bogolyubov with 3-fold sums.* Let $A \subseteq \mathbb{F}_p^n$ with $|A| = \alpha p^n$. Prove that $A+A+A$ contains a translate of a subspace of codimension $O(\alpha^{-3})$.

ps6⋆ H11. *Slightly better bounds on Bogolyubov.* Let $A \subseteq \mathbb{F}_2^n$ with $|A| = \alpha 2^n$.

(a) Show that if $|A+A| < 0.99 \cdot 2^n$, then there is some $r \in \mathbb{F}_2^n \setminus \{0\}$ such that $|\widehat{1_A}(r)| > c\alpha^{3/2}$ for some constant $c > 0$.

(b) By iterating (a), show that $A+A$ contains 99% of a subspace of codimension $O(\alpha^{-1/2})$.

(c) Deduce that $4A$ contains a subspace of codimension $O(\alpha^{-1/2})$ (i.e., Bogolyubov's lemma with better bounds than the one shown in class)

ps6 H12. *Approximate homomorphism.* Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \colon \mathbb{F}_2^n \to \mathbb{F}_2^m$ is a map satisfying

$$\mathbb{P}_{x,y,z}(\phi(x) + \phi(x+y+z) = \phi(x+y) + \phi(x+z)) \ge \epsilon,$$

(here $x, y, z \in \mathbb{F}_2^n$ are chosen uniformly and independently at random) then there exists some homomorphism $\psi \colon \mathbb{F}_2^n \to \mathbb{F}_2^m$ and an element $b \in \mathbb{F}_2^m$ such that $\mathbb{P}_x(\phi(x) = \psi(x) + b) \ge \delta$.

Hint: Consider the additive energy of the "graph" $\{(x, \phi(x)) : x \in \mathbb{F}_2^n\}$ of ϕ