7 Correlation inequalities

Consider an Erdős–Rényi random graph $G(n,p)$. If we condition on it having a Hamiltonian cycle, intuitively, it seems that this conditioning would cause us to have more edges and thereby decreasing the likelihood that the random graph is planar. The main theorem of this chapter, the Harris–FKG inequality, makes this notion precise.

7.1 Harris–FKG inequality

Setup. We have n independent Bernoulli random variables $x_1, \dots, x_n$ (not necessarily identical, but independence is important).

An **increasing event** (or increasing property) A is defined by an upward closed subset of $\{0,1\}^n$ (an **up-set**), i.e.,

$$x \in A \text{ and } x \leq y \text{ (coordinatewise)} \quad \Longrightarrow \quad y \in A.$$

Examples in increasing properties of graphs:

- Having a Hamiltonian cycle
- Connected
- Average degree ≥ 4 (or: min degree, max degree, etc.)
- Having a triangle
- Not 4-colorable

Similarly, a **decreasing event** is defined by a downward closed collection of subset of $\{0,1\}^n$.

Note that $A \subset \{0,1\}^n$ is increasing if and only if its complement $\overline{A} \subset \{0,1\}^n$ is decreasing

The main theorem of this chapter is the following, which tells us that

increasing events of independent variables are positively correlated

Theorem 7.1.1 (Harris 1960)**.** If A and B are increasing events of independent boolean random variables, then

$$\mathbb{P}(A \wedge B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

Equivalently, we can write $\mathbb{P}(A \mid B) \geq \mathbb{P}(A)$.

Remark 7.1.2. Many of such inequalities were initially introduced for the problem of *percolations*, e.g., if we keep each edge of the infinite grid graph with vertex set $\mathbb{Z}^2$ with probability p, what is the probability that the origin is part of an infinite component (in which case we say that there is "percolation"). Harris showed that with probability 1, percolation does not occur for $p \le 1/2$. A later breakthrough of Kesten (1980) shows that percolation occurs with probability for all $p > 1/2$. Thus the "bond percolation threshold" for $\mathbb{Z}^2$ is exactly $1/2$. Such exact results are extremely rare.

We state and prove a more general result, which says that independent random variables possess **positive association**.

Let each Ω_i be a linearly ordered set (i.e., $\{0, 1\}$, $\mathbb{R}$) and $x_i \in \Omega_i$ with respect to some probability distribution independent for each i. We say that a function $f(x_1, \dots, x_n)$ is **monotone increasing** if

$$x \le y \text{ (coordinatewise)} \quad \Longrightarrow \quad f(x) \le f(y).$$

Theorem 7.1.3 (Harris)**.** If f and g are monotone increasing functions of independent random variables, then

$$\mathbb{E}[fg] \ge (\mathbb{E}f)(\mathbb{E}g).$$

This version of Harris inequality implies the earlier version by setting $f = 1_A$ and $g = 1_B$.

Remark 7.1.4. The inequality is often called the **FKG inequality**, attributed to Fortuin, Kasteleyn, Ginibre (1971), who proved a more general result in the setting of distributive lattices, which we will not discuss here.

Proof. We use induction on n by integrating out the inequality one variable at a time. For $n = 1$, for independent $x, y \in \Omega_1$, we have

$$0 \le \mathbb{E}[(f(x) - f(y)(g(x) - g(y)] = 2\mathbb{E}[fg] - 2(\mathbb{E}f)(\mathbb{E}g),$$

so $\mathbb{E}[fg] \ge \mathbb{E}[f]\mathbb{E}[g]$ (this is sometimes called Chebyshev's inequality/rearrangement inequality).

Now assume $n \ge 2$. Let $h = fg$. Define marginals $f_1, g_1, h_1 \colon \Omega_1 \to \mathbb{R}$ by

$$\begin{aligned} f_1(y_1) &= \mathbb{E}[f|x_1 = y_1] = \mathbb{E}_{(x_2,\dots,x_n)\in\Omega_2\times\cdots\times\Omega_n}[f(y_1, x_2, \dots, x_n)], \\ g_1(y_1) &= \mathbb{E}[g|x_1 = y_1] = \mathbb{E}_{(x_2,\dots,x_n)\in\Omega_2\times\cdots\times\Omega_n}[g(y_1, x_2, \dots, x_n)], \\ h_1(y_1) &= \mathbb{E}[h|x_1 = y_1] = \mathbb{E}_{(x_2,\dots,x_n)\in\Omega_2\times\cdots\times\Omega_n}[h(y_1, x_2, \dots, x_n)], \end{aligned}$$

Then f_1 and g_1 are 1-variable monotone increasing functions on Ω_1 (check!).

For every fixed $y_1 \in \Omega_1$, the function $(x_2, \dots, x_n) \mapsto f(y_1, x_2, \dots, x_n)$ is monotone increasing, and likewise with g. So applying the induction hypothesis for $n-1$, we have

$$h_1(y_1) \ge f_1(y_1) g_1(y_1). \tag{7.1}$$

Thus

$$\begin{aligned}
\mathbb{E}[fg] = \mathbb{E}[h] &= \mathbb{E}[h_1] \\
&\ge \mathbb{E}[f_1 g_1] && \text{[by (7.1)]} \\
&\ge (\mathbb{E} f_1)(\mathbb{E} g_1) && \text{[by the } n = 1 \text{ case]} \\
&= (\mathbb{E} f)(\mathbb{E} g).
\end{aligned}$$

□

Corollary 7.1.5. Let A and B be events on independent random variables.

(a) If A and B are decreasing, then $\mathbb{P}(A \wedge B) \ge \mathbb{P}(A)\mathbb{P}(B)$.

(b) If A is increasing and B is decreasing, then $\mathbb{P}(A \wedge B) \le \mathbb{P}(A)\mathbb{P}(B)$.

If $A_1, \dots, A_k$ are all increasing (or all decreasing) events on independent random variables, then

$$\mathbb{P}(A_1 \cdots A_k) \ge \mathbb{P}(A_1) \cdots \mathbb{P}(A_k).$$

Proof. For the second inequality, note that the complement $\overline{B}$ is increasing, so

$$\mathbb{P}(AB) = \mathbb{P}(A) - \mathbb{P}(A\overline{B}) \overset{\text{Harris}}{\le} \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(\overline{B}) = \mathbb{P}(A)\mathbb{P}(B).$$

The proof of the first inequality is similar. For the last inequality we apply the Harris inequality repeatedly. □

7.2 Applications to random graphs

7.2.1 Triangle-free probability

Question 7.2.1. What's the probability that $G(n, p)$ is triangle-free?

Harris inequality will allow us to prove a lower bound. In the next chapter, we will use Janson inequalities to derive upper bounds.

Theorem 7.2.2. $\mathbb{P}(G(n,p)$ is triangle-free$) \ge (1-p^3)^{\binom{n}{3}}$

Proof. For each triple of distinct vertices $i,j,k \in [n]$, let A_{ijk} be the event that ijk is a triangle in $G(n,p)$. Then A_{ijk} is increasing, and

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \ge \mathbb{P}\left(\bigwedge_{i<j<k} \overline{A}_{ijk}\right) \ge \prod_{i<j<k} \mathbb{P}(\overline{A}_{ijk}) = (1-p^3)^{\binom{n}{3}}. \qquad \square$$

Remark 7.2.3. How good is this bound? For $p \le 0.99$, we have $1 - p^3 = e^{-\Theta(p^3)}$, so the above bound gives

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \ge e^{-\Theta(n^3p^3)}.$$

Here is another lower bound

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \ge \mathbb{P}(G(n,p) \text{ is empty}) = (1-p)^{\binom{n}{2}} = e^{-\Theta(n^2p)}.$$

The bound from Harris is better when $p \ll n^{-1/2}$. Putting them together, we obtain

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \gtrsim \begin{cases} e^{-\Theta(n^3p^3)} & \text{if } p \lesssim n^{-1/2} \\ e^{-\Theta(n^2p)} & \text{if } n^{-1/2} \lesssim p \le 0.99 \end{cases}$$

(note that the asymptotics agree at the boundary $p \asymp n^{-1/2}$. In the next chapter, we will prove matching upper bounds using Janson inequalities.

7.2.2 Maximum degree

Question 7.2.4. What's the probability that the maximum degree of $G(n, 1/2)$ is at most $n/2$?

For each vertex v, $\deg(v) \le n/2$ is a decreasing event with probability just slightly over $1/2$. So by Harris inequality, the probability that every v has $\deg(v) \le n/2$ is at least $\ge 2^{-n}$.

It turns out that the appearance of high degree vertices is much more correlated than the independent case. The truth is exponentially more than the above bound.

Theorem 7.2.5 (Riordan and Selby 2000)**.**

$$\mathbb{P}(\operatorname{maxdeg} G(n,1/2) \le n/2) = (0.6102\cdots + o(1))^n$$

Instead of giving a proof, we consider an easier continuous model of the problem that motivates the numerical answer. Turning this continuous model paper into a rigorous proof about random graphs is more technical.

In a random graphs, we assign independent Bernoulli random variables on edges of a complete graph. Instead, let us assign independent standard normal random variables Z_{uv} to each edge uv of K_n.

Let $W_v = \sum_{u \neq v} Z_{uv}$, which models how much the degree of vertex v deviates from its expectation. In particular W_v is symmetric and mean 0, and $\mathbb{P}(W_v \leq 0)$.

The problem of estimating the probability that $\operatorname{maxdeg} G(n, 1/2) \leq n/2$ then should be modeled as

$$\mathbb{P}(\max_{v \in [n]} W_v \leq 0)$$

(Of course, other than intuition, there is no justification here that these two models actually mimic each other.)

Observe that $(W_v)_{v \in [n]}$ is a joint normal distribution, each coordinate has variance $n - 1$ and pairwise covariance 1. So $(W_v)_{v \in [n]}$ has the same distribution as

$$\sqrt{n-2}(Z_1', Z_2', \dots, Z_n') + Z_0'(1, 1, \dots, 1)$$

where $Z_0', \dots, Z_n'$ are iid standard normals.

Let Φ be the pdf and cdf of the standard normal $N(0, 1)$.

Thus

$$\mathbb{P}(\max_{v \in [n]} W_v \leq 0) = \mathbb{P}\left(\max_{i \in [n]} Z_i' \leq -\frac{Z_0'}{\sqrt{n-2}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \Phi\left(\frac{-z}{\sqrt{n-2}}\right)^n dz$$

where the final step is obtained by conditioning on Z_0'. Substituting $z = y\sqrt{n}$, the above quantity equals to

$$= \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} e^{nf(y)} \, dy \quad \text{where} \quad f(y) = -\frac{y^2}{2} + \log \Phi\left(y\sqrt{\frac{n}{n-2}}\right).$$

We can estimate the above integral for large n using the *Laplace method* (which can be justified rigorously by considering Taylor expansion around the maximum of f). We have

$$f(y) \approx g(y) := -\frac{y^2}{2} + \log \Phi(y)$$

and we can deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\max_{v\in[n]} W_v \le 0) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{nf(y)}\, dy = \max g = \log 0.6102\cdots.$$