8 Janson inequalities

We present a collection of inequalities, known collectively as Janson inequalities (Janson 1990). These tools allow us to estimate **lower tail** large deviation probabilities.

8.1 Probability of non-existence

Question 8.1.1. What is the probability that $G(n, p)$ is triangle-free?

As in indicated in the previous chapter, Janson inequalities will allow us upper bound such probabilities.

The following setup should be a reminiscent of both the second moment method as well as Lovász local lemma (the random variable model).

Setup 8.1.2. Let R be a random subset of $[N]$ with each element included independently (possibly with different probabilities).

Let $S_1, \ldots, S_k \subseteq [N]$. Let A_i be the event that $S_i \subseteq R$. Let

$$X = \sum_i 1_{A_i}$$

be the number of events that occur. Let

$$\mu = \mathbb{E}[X] = \sum_i \mathbb{P}(A_i).$$

Write $i \sim j$ if $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Let (as in the second moment method)

$$\Delta = \sum_{(i,j):i\sim j} \mathbb{P}(A_i \wedge A_j)$$

(note that (i, j) and (j, i) is each counted once).

The following inequality was proved by Janson, Łuczak, and Ruciński (1990).

Theorem 8.1.3 (Janson inequality I)**.** Assuming Setup 8.1.2,

$$\mathbb{P}(X = 0) \leq e^{-\mu+\Delta/2}.$$

Remark 8.1.4. When $\mathbb{P}(A_i) = o(1)$, Harris inequality gives us

$$\mathbb{P}(X=0) = \mathbb{P}(\overline{A}_1 \cdots \overline{A}_k) \geq \mathbb{P}(\overline{A}_1)\cdots\mathbb{P}(\overline{A}_k) = \prod_{i=1}^{k}(1-\mathbb{P}(A_i)) = e^{-(1+o(1))\sum_{i=1}^{k}\mathbb{P}(A_i)} = e^{-(1+o(1))\mu}.$$

If furthermore $\Delta = o(\mu)$, then two bounds match to give $\mathbb{P}(X=0) = e^{-(1+o(1)\mu}$.

(Not Janson's original proof, which was by analytic interpolation. The following proof is by Boppana and Spencer (1989), with a modification by Warnke[1]. It has some similarities to the proof of Lovász local lemma)

Proof. Let

$$r_i = \mathbb{P}(A_i | \overline{A}_1 \cdots \overline{A}_{i-1}).$$

We have

$$\begin{aligned}
\mathbb{P}(X=0) &= \mathbb{P}(\overline{A}_1 \cdots \overline{A}_k) \\
&= \mathbb{P}(\overline{A}_1)\mathbb{P}(\overline{A}_2|\overline{A}_1)\cdots\mathbb{P}(\overline{A}_k|\overline{A}_1\cdots\overline{A}_{k-1}) \\
&= (1-r_1)\cdots(1-r_k) \\
&\leq e^{-r_1-\cdots-r_k}
\end{aligned}$$

It suffices now to prove that:

Claim. For each $i \in [k]$

$$r_i \geq \mathbb{P}(A_i) - \sum_{j<i:j\sim i}\mathbb{P}(A_iA_j).$$

Summing the claim over $i \in [k]$ would then yield

$$\sum_{i=1}^{k} r_i \geq \sum_i \mathbb{P}(A_i) - \frac{1}{2}\sum_i\sum_{j\sim i}\mathbb{P}(A_iA_j) = \mu - \frac{\Delta}{2}$$

and thus

$$\mathbb{P}(X=0) \leq \exp\left(-\sum_i r_i\right) \leq \exp\left(-\mu + \frac{\Delta}{2}\right)$$

Proof of claim. Let

$$D_0 = \bigwedge_{j<i:j\not\sim i}\overline{A}_j \qquad \text{and} \qquad D_1 = \bigwedge_{j<i:j\sim i}\overline{A}_j$$

[1] Personal communication

Then

$$\begin{aligned} r_i = \mathbb{P}(A_i|\overline{A}_1 \cdots \overline{A}_{i-1}) = \mathbb{P}(A_i|D_0D_1) &= \frac{\mathbb{P}(A_iD_0D_1)}{\mathbb{P}(D_0D_1)} \\ &\geq \frac{\mathbb{P}(A_iD_0D_1)}{\mathbb{P}(D_0)} \\ &= \mathbb{P}(A_iD_1|D_0) \\ &= \mathbb{P}(A_i|D_0) - \mathbb{P}(A_i\overline{D}_1|D_0) \\ &= \mathbb{P}(A_i) - \mathbb{P}(A_i\overline{D}_1|D_0) \quad \text{[by independence]} \end{aligned}$$

Since A_i and $\overline{D}_1$ are both increasing events, and D_0 is a decreasing event, by Harris inequality (Corollary 7.1.5),

$$\mathbb{P}(A_i\overline{D}_1|D_0) \leq \mathbb{P}(A_i\overline{D}_1) = \mathbb{P}\left(A_i \wedge \bigvee_{j<i:j\sim i} A_j\right) \leq \sum_{j<i:j\sim i} \mathbb{P}(A_iA_j)$$

And the claim follows. □

In Setup 8.1.2 (as well as subsequent Janson inequalities by extension), one can actually allow A_i to be any increasing events, not simply events of the form $S_i \subseteq R$ (known as "principal up-sets").

Theorem 8.1.5 (Riordan and Warnke 2015)**.** Theorem 8.1.3 remains true if Setup 8.1.2 is modified as follows. The events A_i are allowed to any increasing events independent boolean random variables. We write $i \sim j$ if A_i and A_j are not independent (this is initially a pairwise condition, though see lemma below).

In most applications of Janson inequalities, it is easiest to work with principal up-sets. Note that Janson's inequality is false for general events.

Here to how to modify the above proof for work for arbitrary increasing events A_i. The only place we used independence is the "by independence" step above. The next statement shows that the this step remains valid for general up-sets.

Proposition 8.1.6. Let A and $B_1, \ldots, B_k$ be increasing events of independent boolean random variables. If A is independent of B_i for every $i \in [k]$, then A is independent of $\{B_1, \ldots, B_k\}$.

Proof. We first prove the statement for $k = 2$. Writing $B = B_1$ and $C = B_2$, we have

$$\mathbb{P}(A \cap (B \cap C)) + \mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) = \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C))$$

By Harris inequality, since $B \cap C$ and $B \cup C$ are increasing,

$$\mathbb{P}(A \cap (B \cap C)) \geq \mathbb{P}(A)\mathbb{P}(B \cap C) \quad \text{and} \quad \mathbb{P}(A \cap (B \cup C)) \geq \mathbb{P}(A)\mathbb{P}(B \cup C)$$

Summing the above two gives the previous equality, so the above two inequalities must be equalities. In particular, A is independent of $B \cap C$.

Since the intersection of two up-sets is an up-set, we see that A is independent of the intersection of any subset of $\{B_1, \ldots, B_k\}$, which then implies that A is independent of $\{B_1, \ldots, B_k\}$. □

Now let us return to the probability that $G(n, p)$ is triangle-free. In Setup 8.1.2, let $[N]$ with $N = \binom{n}{2}$ be the set of edges of K_n, and let $S_1, \ldots, S_{\binom{n}{3}}$ be 3-element sets where each S_i is the edge-set of a triangle. As in the second moment calculation in Section 4.1, we have

$$\mu = \binom{n}{3} p^3 \asymp n^3 p^3 \qquad \text{and} \qquad \Delta \asymp n^4 p^5.$$

(where Δ is obtained by considering all appearances of a pair of triangles glued along an edge).

If $p \ll n^{-1/2}$, then $\Delta = o(\mu)$, in which case Janson inequality I (Theorem 8.1.3 and Remark 8.1.4) gives the following.

Theorem 8.1.7. If $p = o(n^{-1/2})$, then

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) = e^{-(1+o(1))\mu} = e^{-(1+o(1))n^3p^3/6}.$$

Corollary 8.1.8. For a constant $c > 0$,

$$\lim_{n \to \infty} \mathbb{P}(G(n, c/n) \text{ is triangle-free}) = e^{-c^3/6}.$$

In fact, the number of triangles in $G(n, c/n)$ converges to a Poisson distribution with mean $c^3/6$. On the other hand, when $p \gg 1/n$, the number of triangles is asymptotically normal.

What about if $p \gg n^{-1/2}$, so that $\Delta \gg \mu$. Janson inequality I does not tell us anything nontrivial. Do we still expect the triangle-free probability to be $e^{-(1+o(1))\mu}$, or even $\leq e^{-c\mu}$?

As noted earlier in Remark 7.2.3, another way to obtain a lower bound on the probability triangle-freeness is to consider the probability the $G(n, p)$ is empty (or contained in some fixed complete bipartite graph), in which case we obtain

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \geq (1-p)^{\Theta(n^2)} = e^{-\Theta(n^2 p)}$$

(the second step assumes that p is bounded away from 1. If $p \gg n^{-1/2}$, so the above lower bound better than the previous one: $e^{-\Theta(n^2 p)} \gg e^{-(1+o(1))\mu}$.

Nevertheless, we'll still use Janson to bootstrap an upper bound on the triangle-free probability. More generally, the next theorem works in the complement region of the Janson inequality I, where now $\Delta \geq \mu$.

Theorem 8.1.9 (Janson inequality II)**.** Assuming Setup 8.1.2, if $\Delta \geq \mu$, then

$$\mathbb{P}(X = 0) \leq e^{-\mu^2/(2\Delta)}.$$

The proof idea is to applying the first Janson inequality on a randomly sampled subset of events. This sampling technique might remind you of some earlier proofs, e.g., the proof of the crossing number inequality (Theorem 2.4.2), where we first proved a "cheap bound" that worked in a more limited range, and then used sampling to obtain a better bound.

Proof. For each $T \subseteq [k]$, let $X_T := \sum_{i \in T} A_i$ denote the number of occurring events in T. We have

$$\mathbb{P}(X = 0) \leq \mathbb{P}(X_T = 0) \leq e^{-\mu_T + \Delta_T/2}$$

where

$$\mu_T = \sum_{i \in T} \mathbb{P}(A_i)$$

and

$$\Delta_T = \sum_{(i,j) \in T^2 : i \sim j} \mathbb{P}(A_i A_j)$$

Choose $T \subset [k]$ randomly by including every element with probability $q \in [0,1]$ independently. We have

$$\mathbb{E}\mu_T = q\mu \qquad \text{and} \qquad \mathbb{E}\Delta_T = q^2 \Delta$$

and so

$$\mathbb{E}(-\mu_T + \Delta_T/2) = -q\mu + q^2\Delta/2.$$

By linearity of expectations, thus there is some choice of $T \subseteq [k]$ so that

$$-\mu_T + \Delta_T/2 \leq -q\mu + q^2\Delta/2$$

so that

$$\mathbb{P}(X = 0) \leq e^{-q\mu + q^2\Delta/2}$$

for every $q \in [0,1]$. Since $\Delta \geq \mu$, we can set $q = \mu/\Delta \in [0,1]$ to get the result. $\square$

To summarize, the first two Janson inequalities tell us that

$$\mathbb{P}(X=0) \le \begin{cases} e^{-\mu+\Delta/2} & \text{if } \Delta < \mu \\ e^{-\mu^2/(2\Delta)} & \text{if } \Delta \ge \mu. \end{cases}$$

Remark 8.1.10. If $\mu \to \infty$ and $\Delta \ll \mu^2$, then Janson inequality II implies $\mathbb{P}(X=0) = o(1)$, which we knew from second moment method. However Janson's inequality gives an exponentially decaying tail bound, compared to only a polynomially decaying tail via the second moment method. The exponential tail will be important in an application below to determining the chromatic number of $G(n,1/2)$.

Let us revisit the example of estimating the probability that $G(n,p)$ is triangle-free, now in the regime $p \gg n^{-1/2}$. We have

$$n^3p^3 \asymp \mu \ll \Delta \asymp n^4p^5.$$

So so for large enough n, Janson inequality II tells us

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \le e^{-\mu^2/(2\Delta)} = e^{-\Theta(n^2p)}$$

Since

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) \ge \mathbb{P}(G(n,p) \text{ is empty}) \ge (1-p)^{\binom{n}{2}} = e^{-\Theta(n^2p)}$$

where the final step assumes that p is bounded away from 1, we conclude that

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) = e^{-\Theta(n^2p)}$$

We summarize the results below (strictly speaking we have not yet checked the case $p \asymp n^{-1/2}$, which we can verify by applying Janson inequalities; note that the two regimes below match at the boundary).

Theorem 8.1.11. Suppose $p = p_n \le 0.99$. Then

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) = \begin{cases} \exp\left(-\Theta(n^2p)\right) & \text{if } p \gtrsim n^{-1/2} \\ \exp\left(-\Theta(n^3p^3)\right) & \text{if } p \lesssim n^{-1/2} \end{cases}$$

Remark 8.1.12. Sharper results are known. Here are some highlights.

1. The number of triangle-free graphs on n vertices is $2^{(1+o(1))n^2/4}$. In fact, an even stronger statement is true: almost all (i.e., $1-o(1)$ fraction) n-vertex triangle-free graphs are bipartite (Erdős, Kleitman, and Rothschild 1976).

2. If $m \geq Cn^{3/2}\sqrt{\log n}$ for any constant $C > \sqrt{3}/4$ (and this is best possible), then almost all all n-vertex m-edge triangle-free graphs are bipartite (Osthus, Prömel, and Taraz 2003). This result has been extended to K_r-free graphs for every fixed r (Balogh, Morris, Samotij, and Warnke 2016).

3. For $n^{-1/2} \ll p \ll 1$, (Łuczak 2000)

$$-\log \mathbb{P}(G(n,p) \text{ is triangle-free}) \sim -\log \mathbb{P}(G(n,p) \text{ is bipartite}) \sim n^2p/4.$$

   This result was generalized to general H-free graphs using the powerful recent method of hypergraph containers (Balogh, Morris, and Samotij 2015).

8.2 Lower tails

Previously we looked at the probability of non-existence. Now we would like to estimate lower tail probabilities. Here is a model problem.

Question 8.2.1. Fix a constant $0 < \delta \leq 1$. Let X be the number of triangles of $G(n,p)$. Estimate

$$\mathbb{P}(X \leq (1-\delta)\mathbb{E}X).$$

We will bootstrap Janson inequality I, $\mathbb{P}(X = 0) \leq \exp(-\mu + \Delta/2)$, to an upper bound on lower tail probabilities.

Theorem 8.2.2 (Janson inequality III)**.** Assume Setup 8.1.2. For any $0 \leq t \leq \mu$,

$$\mathbb{P}(X \leq \mu - t) \leq \exp\left(\frac{-t^2}{2(\mu+\Delta)}\right)$$

Note that setting $t = \mu$ we basically recover the first two Janson inequalities (up to an unimportant constant factor in the exponent):

$$\mathbb{P}(X = 0) \leq \exp\left(\frac{-\mu^2}{2(\mu+\Delta)}\right). \tag{8.1}$$

(Note that this form of the inequality conveniently captures Janson inequalities I & II.)

Proof. (Lutz Warnke[2]) Let $q \in [0,1]$. Let $T \subset [k]$ where each element is included with probability q independently.

Let $X_T = \sum_{i \in T} 1_{A_i}$. Note that this is the same as $\sum_i 1_{A_i} W_i$ where each $W_i \sim \text{Bernoulli}(q)$.

[2]Personal communication

We have

$$\mathbb{P}(X_T = 0|X) = (1-q)^X$$

Taking expectation and applying Janson inequality I to X_T, we obtain

$$\mathbb{E}[(1-q)^X] = \mathbb{P}(X_T = 0) \le e^{-\mu' + \Delta'/2} = e^{-q\mu + q^2\Delta/2}$$

where

$$\mu' = q\mu \quad \text{and} \quad \Delta' = q^2\Delta.$$

By Markov's inequality,

$$\begin{aligned}
\mathbb{P}(X \le \mu - t) &= \mathbb{P}\left((1-q)^X \le (1-q)^{\mu - t}\right) \\
&\le (1-q)^{-\mu+t}\mathbb{E}[(1-q)^X] \\
&\le (1-q)^{-\mu+t} e^{-q\mu + q^2\Delta/2}.
\end{aligned}$$

It remains to show that there is a choice of q so that $RHS \le \exp\left(\frac{-t^2}{2(\mu+\Delta)}\right)$.

Let $1 - q = e^{-\lambda}$, $\lambda \ge 0$. Then

$$\lambda - \frac{\lambda^2}{2} \le q \le \lambda$$

So

$$\begin{aligned}
\mathbb{P}(X \le -\mu + t) &\le (1-q)^{\mu - t} e^{-q\mu + q^2\Delta/2} \\
&\le \exp\left(\lambda(\mu - t) - \left(\lambda - \frac{\lambda^2}{2}\right)\mu + \lambda^2\frac{\Delta}{2}\right) \\
&= \exp\left(\lambda t - \frac{\lambda^2}{2}(\mu + \Delta)\right)
\end{aligned}$$

Setting $\lambda = 1/(\mu + \Delta)$ yields the result. □

Example 8.2.3 (Lower tails for triangle counts)**.** Let X be the number of triangles in $G(n,p)$. We have $\mu \asymp n^3p^3$ and $\Delta \asymp n^4p^5$. Fix a constant $\delta \in (0,1]$. Let $t = \delta\mathbb{E}X$. We have

$$\mathbb{P}(X \le (1-\delta)\mathbb{E}X) \le \exp\left(-\Theta\left(\frac{-\delta^2n^6p^6}{n^3p^3 + n^4p^5}\right)\right) = \begin{cases} \exp\left(-\Theta_\delta(n^2p)\right) & \text{if } p \gtrsim n^{-1/2}, \\ \exp\left(-\Theta_\delta(n^3p^3)\right) & \text{if } p \lesssim n^{-1/2}. \end{cases}$$

The bounds are tight up to a constant in the exponent, since

$$\mathbb{P}(X \le (1-\delta)\mathbb{E}X) \ge \mathbb{P}(X = 0) = \begin{cases} \exp\left(-\Theta(n^2p)\right) & \text{if } p \gtrsim n^{-1/2}, \\ \exp\left(-\Theta(n^3p^3)\right) & \text{if } p \lesssim n^{-1/2}. \end{cases}$$

Example 8.2.4 (No corresponding Janson inequality for upper tails)**.** Continuing with X being the number of triangles of $G(n,p)$, abased on the above lower tails, naively we might expect $\mathbb{P}(X \geq (1+\delta)\mathbb{E}X) \leq \exp(-\Theta_\delta(n^2 p))$, but actually this is false!

By planting a clique of size $\Theta(np)$, we can force $X \geq (1+\delta)\mathbb{E}X$. Thus

$$\mathbb{P}(X \geq (1+\delta)\mathbb{E}X) \geq p^{\Theta_\delta(n^2p^2)}$$

which is much bigger than $\exp\left(-\Theta(n^2p)\right)$. The above is actually the truth (Kahn–DeMarco 2012 and Chatterjee 2012):

$$\mathbb{P}(X \geq (1+\delta)\mathbb{E}X) = p^{\Theta_\delta(n^2p^2)} \quad \text{if } p \gtrsim \frac{\log n}{n},$$

but the proof is much more intricate. Recent results allow us to understand the exact constant in the exponent though new developments in large deviation theory. The current state of knowledge is summarized below.

Theorem 8.2.5 (Harel, Mousset, Samotij 2019+)**.** Let X be the number of triangles in $G(n,p)$ with $p = p_n$ satisfying $n^{-1/2} \ll p \ll 1$,

$$-\log \mathbb{P}(X \geq (1+\delta)X) \sim \min\left\{\frac{\delta}{3}, \frac{\delta^{2/3}}{2}\right\} n^2p^2\log(1/p),$$

and for $n^{-1}\log n \ll p \ll n^{-1/2}$,

$$-\log \mathbb{P}(X \geq (1+\delta)X) \sim \frac{\delta^{2/3}}{2} n^2p^2\log(1/p).$$

Remark 8.2.6. The leading constants were determined by Lubetzky and Zhao (2017) by solving an associated variational problem. Earlier results, starting with Chatterjee and Varadhan (2011) and Chatterjee and Dembo (2016) prove large deviation frames that gave the above theorem for sufficiently slowly decaying $p \geq n^{-c}$.

For the corresponding problem for lower tails, the exact leading constant is known only for sufficiently small $\delta > 0$, where the answer is given by "replica symmetry", meaning that the exponential rate is given by a uniform decrement in edge densities for the random graph. In contrast, for δ close to 1, we expect (though cannot prove) that the typical structure of a conditioned random graph is close to a two-block model (Zhao 2017).

8.3 Clique and chromatic number of $G(n, 1/2)$

In Section 4.3, we used the second moment method to find the clique number ω of $G(n, 1/2)$. We saw that, with probability $1 - o(1)$, the clique number is concentrated on two values, and

$$\omega(G(n, 1/2)) \sim 2\log_2 n \qquad \text{whp.}$$

Let use recall the proof using the second moment method. Let X denote the number of k-cliques in $G(n, 1/2)$. Then

$$\mu := \mu_k = \mathbb{E}[X] = \binom{n}{k} 2^{-\binom{k}{2}}.$$

Here $k = k_n$ depends on n.

If $\mu \to 0$, then Markov gives $X = 0$ whp.

If $\mu \to \infty$, then one checks that $\Delta \ll \mu^2$, so that Chebyshev's inequality gives $X > 0$ whp.

Let $k_0 = k_0(n)$ be the largest possible k so that $\mu_k \geq 1$. We have $\mu_{k_0} \geq 1 > \mu_{k_0+1}$ and

$$k_0 \sim 2\log_2 n.$$

We have

$$\frac{\mu_{k+1}}{\mu_k} = n^{-1+o(1)} \quad \text{for } k \sim 2\log_2 n$$

Thus $\omega(G(n, 1/2)) \sim 2\log_2 n$ whp. In fact, this proof gives more, namely that the clique number is concentrated on at most two values

$$\omega(G(n, 1/2)) \in \{k_0 - 1, k_0\} \qquad \text{whp.}$$

Can two point concentration of $\omega(G(n, 1/2))$ really occur? (As opposed to being always concentrated on a single value with high probability.) It turns out that the answer is yes.

Theorem 8.3.1. Fix $\lambda \in (-\infty, \infty)$. Let $n_0(k)$ be the minimum n satisfying $\binom{n}{k}2^{-\binom{k}{2}} \geq 1$. Then, as $k \to \infty$, and for

$$n = n_0(k)\left(1 + \frac{\lambda + o(1)}{k}\right),$$

one has

$$\mathbb{P}(\omega(G(n,1/2)) = k-1) = e^{-e^{\lambda}} + o(1)$$
$$\text{and} \quad \mathbb{P}(\omega(G(n,1/2)) = k) = 1 - e^{-e^{\lambda}} + o(1).$$

Proof. Let X denote the number of k-cliques in $G(n,1/2)$. Using the notation of Setup 8.1.2 for Janson inequalities, one can check that

$$\mu = \binom{n}{k}2^{-\binom{k}{2}} \sim \left(1 + \frac{\lambda + o(1)}{k}\right)^k = e^{\lambda} + o(1)$$

and (details omitted)

$$\Delta \sim \mu^2 \frac{k^4}{n^2} + \mu \frac{2kn}{2^k} = o(1).$$

Then, by Harris inequality (lower bound) and Janson inequality I (upper bound), we have

$$e^{-(1+o(1))\mu} = (1 - 2^{-\binom{k}{2}})^{\binom{n}{k}} \leq \mathbb{P}(X = 0) \leq e^{-\mu + \Delta/2} = e^{-(1+o(1))\mu}.$$

Thus

$$\mathbb{P}(\omega(G(n,1/2)) < k) = \mathbb{P}(X = 0) = e^{-(1+o(1))\mu} = e^{-e^{\lambda}} + o(1).$$

At this point, we can use two-point concentration to conclude. Alternatively, note that $n_0(k) = 2^{(1+o(1))k/2}$, and thus $n = n_0(k-1)(1 + \frac{\lambda'}{k-1})$ for some $\lambda' \to \infty$, and so that the above bound also gives

$$\mathbb{P}(\omega(G(n,1/2)) < k-1) \leq e^{-e^{\lambda'}} + o(1) = o(1).$$

This again proves two-point concentration, and hence the conclusion. $\square$

Thus one has genuine two-point concentration (i.e., with $\mathbb{P}(\omega(G(n,1/2)) = k_0)$ bounded away from 0 and 1) if

$$n = n_0(k)\left(1 + \frac{O(1)}{k}\right)$$

for some k. Noting that $n_0(k) = 2^{(1+o(1))k/2}$. The intervals $[n_0(k)\,(1 - K/k)\,, n_0(k)\,(1 + K/k)]$ are disjoint for large enough k. We see that the number of integers n up to N with twopoints concentration is asymptotically

$$\sum_{k:n_0(k)\le N} O\left(\frac{n_0(k)}{k}\right) = O\left(\frac{N}{\log N}\right).$$

Thus for almost all integers we actually have one-point concentration.

The next statement tells us we have an exponentially small probability of having cliques of size $\sim 2\log_2 n$. This estimate will be important in the following theorem where we determine the chromatic number of $G(n, 1/2)$.

Theorem 8.3.2. Let $k_0 = k_0(n)$ be the largest possible k so that $\mu_k := \binom{n}{k}2^{-\binom{k}{2}} \ge 1$. Then

$$\mathbb{P}(\omega(G(n,1/2)) < k_0 - 3) \le e^{-n^{2-o(1)}}$$

Note that there is a trivial lower bound of $2^{-\binom{n}{2}}$ coming from an empty graph.

Proof. We have $\mu_{k+1}/\mu_k = n^{-1+o(1)}$ whenever $k \sim k_0(n) \sim 2\log_2 n$.

Writing $k = k_0 - 3$ and using the notation of Setup 8.1.2 for Janson inequalities for X being the number of k-cliques, we have

$$\mu = \mu_k > n^{3-o(1)}.$$

One can check that (again details omitted on Δ; the second step uses $2^k = n^{2+o(1)}$),

$$\Delta \sim \mu^2 \frac{k^4}{n^2} + \mu\frac{2kn}{2^k} \sim \mu^2\frac{k^4}{n^2}$$

So $\Delta > \mu$ for sufficiently large n, and we can apply Janson inequality II:

$$\mathbb{P}(X=0) = \mathbb{P}(\omega(G(n,1/2)) < k) \le e^{-\mu^2/(2\Delta)} < e^{-(1/2+o(1))n^2/k^4} = e^{-\Omega(n^2/(\log n)^4)}. \qquad \square$$

Since $G(n,1/2)$ and its graph complement are identically distributed, and $\omega(G) = \alpha(\overline{G})$, the independence number α satisfies

$$\alpha(G(n,1/2)) \sim 2\log_2 n \qquad \text{whp.}$$

It follows that the chromatic number of $G \sim G(n,1/2)$ satisfies

$$\chi(G) \ge \frac{n}{\alpha(G)} \ge (1+o(1))\frac{n}{2\log_2 n} \quad \text{whp.}$$

The following landmark remark of Bollobás pins down the asymptotics of the chromatic number of the random graph.

Theorem 8.3.3 (Bollobás 1988). With probability $1 - o(1)$,

$$\chi(G(n, 1/2)) \sim \frac{n}{2 \log_2 n}.$$

Proof. The lower bound proof was discussed before the theorem statement. For the upper bound we will give a strategy to properly color the graph with not too many colors. We will proceed by taking out independent sets of size $\sim 2 \log_2 n$ iteratively until $o(n/\log n)$ vertices remain, at which point we can use a different color for each remaining vertex.

Note that after taking out the first independent set of size $\sim 2 \log_2 n$, we cannot claim that the remaining graph is still distributed as $G(n, 1/2)$. It is not. Our selection of the vertices was dependent on the random graph. We are not allowed to "resample" the edges after the first selection. Instead, we will use the previous theorem to tell us that, in $G(n, 1/2)$, with high probability, every not-too-small subset of vertices has an independent set of size $\sim 2 \log_2 n$.

Let $G \sim G(n, 1/2)$. Let $m = \lfloor n/(\log n)^2 \rfloor$, say. For any set S of m vertices, the induced subgraph $G[S]$ has the distribution of $G(m, 1/2)$. By Theorem 8.3.2, for

$$k = k_0(m) \sim 2 \log_2 m \sim 2 \log_2 n,$$

we have

$$\mathbb{P}(\alpha(G[S]) < k) = e^{-m^{2-o(1)}} = e^{-n^{2-o(1)}}.$$

Taking a union bound over all $\binom{n}{m} < 2^n$ such sets S,

$$\mathbb{P}(\exists \text{ an } m\text{-vertex subset } S \text{ with } \alpha(G[S]) < k) < 2^n e^{-n^{2-o(1)}} = o(1).$$

Thus, with probability $1 - o(1)$ every m-vertex subset contains a k-vertex independent set. Assume that G has this property. Now we execute our strategy at the beginning of the proof:

- While $\geq m$ vertices remain:
  - Find an independent set of size k, and let it form its own color class
  - Remove these k vertices
- Color the remaining $< m$ vertices each with a new color.

Thus we obtain a proper coloring using at most

$$\frac{n}{k} + m = (1 + o(1))\frac{n}{2\log_2 n}$$

colors. $\square$