3 Alterations

3.1 Ramsey numbers

Recall from Section 1.1:

$R(s,t)$ = smallest n such that every red/blue edge coloring of K_n contains a red K_s or a blue K_t

Using the basic method (union bounds), we deduce

Theorem 3.1.1. If there exists $p \in [0,1]$ with

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

then $R(s,t) > n$.

Proof sketch. Color edge red with prob p and blue with prob $1-p$. LHS upper bounds the probability of a red K_s or a blue K_t. □

Using the alteration method, we deduce

Theorem 3.1.2. For all $p \in [0,1]$ and n,

$$R(s,t) > n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}$$

Proof sketch. Color edge red with prob p and blue with prob $1-p$ remove one vertex from each red K_s or blue K_t. RHS lower bounds the expected number remaining vertices. □

3.2 Dominating set in graphs

In a graph $G = (V, E)$, we say that $U \subset V$ is **dominating** if every vertex in $V \setminus U$ has a neighbor in U.

Theorem 3.2.1. Every graph on n vertices with minimum degree $\delta > 1$ has a dominating set of size at most $\left(\frac{\log(\delta+1)+1}{\delta+1}\right) n$.

Naive attempt: take out vertices greedily. The first vertex eliminates $1+\delta$ vertices, but subsequent vertices eliminate possibly fewer vertices.

Proof. Two-step process (alteration method):
1. Choose a random subset
2. Add enough vertices to make it dominating

Let $p \in [0,1]$ to be decided later. Let X be a random subset of V where every vertex is included with probability p independently.

Let $Y = V \setminus (X \cup N(X))$. Each $v \in V$ lies in Y with probability $\le (1-p)^{1+\delta}$.

Then $X \cup Y$ is dominating, and

$$\mathbb{E}[|X \cup Y|] = \mathbb{E}[|X|] + \mathbb{E}[|Y|] \le pn + (1-p)^{1+\delta} n \le (p + e^{-p(1+\delta)})n$$

using $1 + x \le e^x$ for all $x \in \mathbb{R}$. Finally, setting $p = \frac{\log(\delta+1)}{\delta+1}$ to minimize $p + e^{-p(1+\delta)}$, we bound the above expression by

$$\le \left(\frac{1+\log(\delta+1)}{\delta+1}\right). \qquad \square$$

3.3 Heilbronn triangle problem

Question 3.3.1. How can one place n points in the unit square so that no three points forms a triangle with small area?

Let

$$\Delta(n) = \sup_{\substack{S \subset [0,1]^2 \\ |S|=n}} \min_{\substack{p,q,r \in S \\ \text{distinct}}} \operatorname{area}(pqr)$$

Naive constructions fair poorly. E.g., n points around a circle has a triangle of area $\Theta(1/n^3)$ (the triangle formed by three consectutive points has side lengths $\asymp 1/n$ and angle $\theta = (1-1/n)2\pi$). Even worse is arranging points on a grid, as you would get triangles of zero area.

Heilbronn conjectured that $\Delta(n) = O(n^{-2})$.

Komlós, Pintz, and Szemerédi (1982) disproved the conjecture, showing $\Delta(n) \gtrsim n^{-2}\log n$. They used an elaborate probabilistic construction. Here we show a much simpler version probabilistic construction that gives a weaker bound $\Delta(n) \gtrsim n^{-2}$.

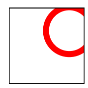
Remark 3.3.2. The currently best upper bound known is $\Delta(n) \le n^{-8/7+o(1)}$ (Komlós, Pintz, and Szemerédi 1981)

Theorem 3.3.3. For every positive integer n, there exists a set of n points in $[0,1]^2$ such that every triple spans a triangle of area $\ge cn^{-2}$, for some absolute constant $c > 0$.

Proof. Choose $2n$ points at random. For every three random points p, q, r, let us estimate

$$\mathbb{P}_{p,q,r}(\operatorname{area}(p, q, r) \leq \epsilon).$$

By considering the area of a circular annulus around p, with inner and outer radii x and $x + \Delta x$, we find



$$\mathbb{P}_{p,q}(|pq| \in [x, x + \Delta x]) \leq \pi((x + \Delta x)^2 - x^2)$$

So the probability density function satisfies

$$\mathbb{P}_{p,q}(|pq| \in [x, x + dx]) \leq 2\pi x dx$$

For fixed p, q

$$\mathbb{P}_r(\operatorname{area}(pqr) \leq \epsilon) = \mathbb{P}_r\left(\operatorname{dist}(pq, r) \leq \frac{2\epsilon}{|pq|}\right) \lesssim \frac{\epsilon}{|pq|}$$

Thus, with p, q, r at random

$$\mathbb{P}_{p,q,r}(\operatorname{area}(pqr) \leq \epsilon) \lesssim \int_0^{\sqrt{2}} 2\pi x \frac{\epsilon}{x}\, dx \asymp \epsilon.$$

Given these $2n$ random points, let X be the number of triangles with area $\leq \epsilon$. Then $\mathbb{E}X = O(\epsilon n^3)$.

Choose $\epsilon = c/n^2$ with $c > 0$ small enough so that $\mathbb{E}X \leq n$.

Delete a point from each triangle with area $\leq \epsilon$.

The expected number of remaining points is $\mathbb{E}[2n - X] \geq n$, and no triangles with area $\leq \epsilon = c/n^2$.

Thus with positive probability, we end up with $\geq n$ points and no triangle with area $\leq c/n^2$. $\square$

Algebraic construction. Here is another construction due to Erdős (in appendix of Roth (1951)) also giving $\Delta(n) \gtrsim n^{-2}$:

Let p be a prime. The set $\{(x, x^2) \in \mathbb{F}_p^2 : x \in \mathbb{F}_p\}$ has no 3 points collinear (a parabola meets every line in ≤ 2 points). Take the corresponding set of p points in $[p]^2 \subset \mathbb{Z}^2$. Then every triangle has area $\geq 1/2$ due to Pick's theorem. Scale back down to a unit square. (If n is not a prime, then use that there is a prime between n and $2n$.)

3.4 Markov's inequality

We note an important tool that will be used next.

Markov's inequality. Let $X \geq 0$ be random variable. Then for every $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Proof. $\mathbb{E}[X] \geq \mathbb{E}[X 1_{X \geq a}] \geq \mathbb{E}[a 1_{X \geq a}] = a\mathbb{P}(X \geq a)$ □

Take-home message: for r.v. $X \geq 0$, if $\mathbb{E}X$ is *very* small, then *typically* X is small.

3.5 High girth and high chromatic number

If a graph has a k-clique, then you know that its chromatic number is at least k.

Conversely, if a graph has high chromatic number, is it always possible to certify this fact from some "local information"?

Surprisingly, the answer is no. The following ingenious construction shows that a graph can be "locally tree-like" while still having high chromatic number.

The **girth** of a graph is the length of its shortest cycle.

Theorem 3.5.1 (Erdős 1959)**.** For all k, ℓ, there exists a graph with girth $> \ell$ and chromatic number $> k$.

Proof. Let $G \sim G(n,p)$ with $p = (\log n)^2/n$ (the proof works whenever $\log n/n \ll p \ll n^{-1+1/\ell}$). Here $G(n,p)$ is Erdős–Rényi random graph (n vertices, every edge appearing with probability p independently).

Let X be the number of cycles of length at most ℓ in G. By linearity of expectations, as there are exactly $\binom{n}{i}(i-1)!/2$ cycles of length i in K_n for each $3 \leq i \leq n$, we have (recall that ℓ is a constant)

$$\mathbb{E}X = \sum_{i=3}^{\ell} \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=3}^{\ell} n^i p^i = o(n).$$

By Markov's inequality

$$\mathbb{P}(X \geq n/2) \leq \frac{\mathbb{E}X}{n/2} = o(1).$$

(This allows us to get rid of all short cycles.)

How can we lower bound the chromatic number $\chi(\cdot)$? Note that $\chi(G) \geq |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number (the size of the largest independent set).

With $x = (3/p) \log n$,

$$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < n^x e^{-px(x-1)/2} = (ne^{-p(x-1)/2})^x = o(1).$$

Let n be large enough so that $\mathbb{P}(X \geq n/2) < 1/2$ and $\mathbb{P}(\alpha(G) \geq x) < 1/2$. Then there is some G with fewer than $n/2$ cycles of length $\leq \ell$ and with $\alpha(G) \leq (3/p) \log n$.

Remove a vertex from each cycle to get G'. Then $|V(G')| \geq n/2$, girth $> \ell$, and $\alpha(G') \leq \alpha(G) \leq (3/p) \log n$, so

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{np}{6 \log n} = \frac{\log n}{6} > k$$

if n is sufficiently large. □

Remark 3.5.2. Erdős (1962) also showed that in fact one needs to see at least a linear number of vertices to deduce high chromatic number: for all k, there exists $\epsilon = \epsilon_k$ such that for all sufficiently large n there exists an n-vertex graph with chromatic number $> k$ but every subgraph on $\lfloor \epsilon n \rfloor$ vertices is 3-colorable. (In fact, one can take $G \sim G(n, C/n)$; see "Probabilistic Lens: Local coloring" in Alon–Spencer)

3.6 Greedy random coloring

Recall $m(k)$ is the minimum number of edges in a k-uniform hypergraph that is not 2-colorable.

Earlier we proved that $m(k) \geq 2^{k-1}$. Indeed, given a k-graph with $< 2^{k-1}$ edges, by randomly coloring the vertices, the expected number of monochromatic numbers is < 1.

We also proved an upper bound $m(k) = O(k^2 2^k)$ by taking a random k-uniform hypergraph on k^2 vertices.

Here is the currently best known lower bound.

Theorem 3.6.1 (Radhakrishnan and Srinivasan (2000)). $m(k) \gtrsim \sqrt{\frac{k}{\log k}} 2^k$

Here we present a simpler proof, based on a **random greedy coloring**, due to Cherkashin and Kozik (2015), following an approach of Pluhaár (2009).

Proof. Suppose H is a k-graph with m edges.

Map $V(H) \to [0,1]$ uniformly at random.

Color vertices greedily from left to right: color a vertex blue unless it would create a monochromatic edge, in which case color it red (i.e., every red vertex is the final vertex in an edge with all earlier $k-1$ vertices have been colored blue).

The resulting coloring has no all-blue edges. What is the probability of seeing a red edge?

If there is a red edge, then there must be two edges e, f so that the last vertex of e is the first vertex of f. Call such pair (e, f) **conflicting**.

Want to bound probability of seeing a conflicting pair in a random $V(H) \to [0,1]$.

Here is an attempt (an earlier weaker result due to Pluhaár (2009)). Each pair of edges with exactly one vertex in common conflicts with probability $\frac{(k-1)!^2}{(2k-1)!} = \frac{1}{2k-1}\binom{2k-2}{k-1}^{-1} \asymp k^{-1/2}2^{-2k}$; union bounding over $< m^2$ pairs of edges, the probability of of getting a conflicting edge is $\lesssim m^2 k^{-1/2} 2^{-2k}$, which is < 1 for some $m \asymp k^{1/4}2^k$.

We'd like to do better by more carefully analyzing conflicting edges. Continuing ...

Write $[0,1] = L \cup M \cup R$ where (p to be decided)

$$L := \left[0, \frac{1-p}{2}\right) \qquad M := \left[\frac{1-p}{2}, \frac{1+p}{2}\right] \qquad R := \left(\frac{1+p}{2}, 1\right].$$

The probability that a given edge lands entirely in L is $(\frac{1-p}{2})^k$, and likewise with R

So probability that some edge of H is entirely contained in L or contained in R is $\le 2m(\frac{1-p}{2})^k$.

Suppose that no edge of H lies entirely in L or entirely in R. If (e, f) conflicts, then their unique common vertex $x_v \in e \cap f$ must lie in M. So the probability that (e, f) conflicts is (here we use $x(1-x) \le 1/4$)

$$\int_{(1-p)/2}^{(1+p)/2} x^{k-1}(1-x)^{k-1}\,dx \le p4^{-k+1}.$$

Thus the probability of seeing any conflicting pair is

$$\le 2m\left(\frac{1-p}{2}\right)^k + m^2 p 4^{-k+1} < 2^{-k+1} m e^{-pk} + (2^{-k+1}m)^2 p.$$

Set $p = \log(2^{-k+2}k/m)/k$, we find that the above probability is < 1 for $m = c2^k\sqrt{k/\log k}$, with $c > 0$ being a sufficiently small constant. $\square$

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Fall 2020

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