# 3 Alterations

We saw the alterations method in Section 1.1 to give lower bounds to Ramsey numbers. The basic idea is to first make a random construction, and then fix the blemishes.

## 3.1 Dominating set in graphs

In a graph $G = (V, E)$, we say that $U \subseteq V$ is ***dominating*** if every vertex in $V \setminus U$ has a neighbor in $U$.

**Theorem 3.1.1**

Every graph on $n$ vertices with minimum degree $\delta > 1$ has a dominating set of size

$$\leq \left(\frac{\log(\delta+1)+1}{\delta+1}\right) n.$$

Naive attempt: take out vertices greedily. The first vertex eliminates $1+\delta$ vertices, but subsequent vertices eliminate possibly fewer vertices.

*Proof.* Two-step process (alteration method):

1. Choose a random subset
2. Add enough vertices to make it dominating

Let $p \in [0, 1]$ to be decided later. Let $X$ be a random subset of $V$ where every vertex is included with probability $p$ independently.

Let $Y = V \setminus (X \cup N(X))$. Each $v \in V$ lies in $Y$ with probability $\leq (1-p)^{1+\delta}$.

Then $X \cup Y$ is dominating, and

$$\mathbb{E}[|X \cup Y|] = \mathbb{E}[|X|] + \mathbb{E}[|Y|] \leq pn + (1-p)^{1+\delta} n \leq (p + e^{-p(1+\delta)})n$$

using $1 + x \leq e^x$ for all $x \in \mathbb{R}$. Finally, setting $p = \frac{\log(\delta+1)}{\delta+1}$ to minimize $p + e^{-p(1+\delta)}$, we bound the above expression by

$$\leq \left(\frac{1+\log(\delta+1)}{\delta+1}\right).$$

□

## 3.2 Heilbronn triangle problem

**Question 3.2.1**

How can one place $n$ points in the unit square so that no three points forms a triangle with small area?

Let

$$\Delta(n) = \sup_{\substack{S \subseteq [0,1]^2 \\ |S|=n}} \min_{\substack{p,q,r \in S \\ \text{distinct}}} \operatorname{area}(pqr)/$$

Naive constructions fair poorly. E.g., $n$ points around a circle has a triangle of area $\Theta(1/n^3)$ (the triangle formed by three consectutive points has side lengths $\asymp 1/n$ and angle $\theta = (1 - 1/n)2\pi$). Even worse is arranging points on a grid, as you would get triangles of zero area.

Heilbronn conjectured that $\Delta(n) = O(n^{-2})$.

Komlós, Pintz, and Szemerédi (1982) disproved the conjecture, showing $\Delta(n) \gtrsim n^{-2} \log n$. They used an elaborate probabilistic construction. Here we show a much simpler version probabilistic construction that gives a weaker bound $\Delta(n) \gtrsim n^{-2}$.

***Remark* 3.2.2** (Upper bounds)**.** For a long time, the best upper bound known was $\Delta(n) \leq n^{-8/7+o(1)}$ due to Komlós, Pintz, and Szemerédi (1981). This was recently improved to $\Delta(n) \leq n^{-8/7-c}$ by Cohen, Pohoata, and Zakharov (2023+).

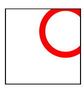
**Theorem 3.2.3** (Many points without small area triangles)

For every positive integer $n$, there exists a set of $n$ points in $[0, 1]^2$ such that every triple spans a triangle of area $\geq cn^{-2}$, for some absolute constant $c > 0$.

*Proof.* Choose $2n$ points at random. For every three random points $p, q, r$, let us estimate

$$\mathbb{P}_{p,q,r}(\operatorname{area}(p, q, r) \leq \varepsilon).$$

By considering the area of a circular annulus around $p$, with inner and outer radii $x$ and $x + \Delta x$, we find



$$\mathbb{P}_{p,q}(|pq| \in [x, x + \Delta x]) \leq \pi((x + \Delta x)^2 - x^2).$$

So the probability density function satisfies

$$\mathbb{P}_{p,q}(|pq| \in [x, x + dx]) \leq 2\pi x dx.$$

For fixed $p, q$

$$\mathbb{P}_r(\text{area}(pqr) \leq \varepsilon) = \mathbb{P}_r\left(\text{dist}(pq, r) \leq \frac{2\varepsilon}{|pq|}\right) \lesssim \frac{\varepsilon}{|pq|}.$$

Thus, with $p, q, r$ at random

$$\mathbb{P}_{p,q,r}(\text{area}(pqr) \leq \varepsilon) \lesssim \int_0^{\sqrt{2}} 2\pi x \frac{\varepsilon}{x}\, dx \asymp \varepsilon.$$

Given these $2n$ random points, let $X$ be the number of triangles with area $\leq \varepsilon$. Then $\mathbb{E}X = O(\varepsilon n^3)$.

Choose $\varepsilon = c/n^2$ with $c > 0$ small enough so that $\mathbb{E}X \leq n$.

Delete a point from each triangle with area $\leq \varepsilon$.

The expected number of remaining points is $\mathbb{E}[2n - X] \geq n$, and no triangles with area $\leq \varepsilon = c/n^2$.

Thus with positive probability, we end up with $\geq n$ points and no triangle with area $\leq c/n^2$. □

**Algebraic construction.** Here is another construction due to Erdős (in appendix of Roth (1951)) also giving $\Delta(n) \gtrsim n^{-2}$:

Let $p$ be a prime. The set $\{(x, x^2) \in \mathbb{F}_p^2 : x \in \mathbb{F}_p\}$ has no 3 points collinear (a parabola meets every line in $\leq 2$ points). Take the corresponding set of $p$ points in $[p]^2 \subseteq \mathbb{Z}^2$. Then every triangle has area $\geq 1/2$ due to Pick's theorem. Scale back down to a unit square. (If $n$ is not a prime, then use that there is a prime between $n$ and $2n$.)

## 3.3 Markov's inequality

We note an important tool that will be used next.

**Theorem 3.3.1** (Markov's inequality)
Let $X \geq 0$ be random variable. Then for every $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

*Proof.* $\mathbb{E}[X] \geq \mathbb{E}[X 1_{X \geq a}] \geq \mathbb{E}[a 1_{X \geq a}] = a\mathbb{P}(X \geq a)$ □

Take-home message: for r.v. $X \geq 0$, if $\mathbb{E}X$ is *very* small, then *typically* $X$ is small.

## 3.4 High girth and high chromatic number

If a graph has a $k$-clique, then you know that its chromatic number is at least $k$.

Conversely, if a graph has high chromatic number, is it always possible to certify this fact from some "local information"?

Surprisingly, the answer is no. The following ingenious construction shows that a graph can be "locally tree-like" while still having high chromatic number.

The ***girth*** of a graph is the length of its shortest cycle.

**Theorem 3.4.1** (Erdős 1959)
For all $k, \ell$, there exists a graph with girth $> \ell$ and chromatic number $> k$.

*Proof.* Let $G \sim G(n, p)$ with $p = (\log n)^2/n$ (the proof works whenever $\log n/n \ll p \ll n^{-1+1/\ell}$). Here $G(n, p)$ is Erdős–Rényi random graph ($n$ vertices, every edge appearing with probability $p$ independently).

Let $X$ be the number of cycles of length at most $\ell$ in $G$. By linearity of expectations, as there are exactly $\binom{n}{i}(i-1)!/2$ cycles of length $i$ in $K_n$ for each $3 \leq i \leq n$, we have (recall that $\ell$ is a constant)

$$\mathbb{E}X = \sum_{i=3}^{\ell} \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=3}^{\ell} n^i p^i = \ell (\log n)^{2i} = o(n).$$

By Markov's inequality

$$\mathbb{P}(X \geq n/2) \leq \frac{\mathbb{E}X}{n/2} = o(1).$$

(This allows us to get rid of all short cycles.)

How can we lower bound the chromatic number $\chi(\cdot)$? Note that $\chi(G) \geq |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number (the size of the largest independent set).

With $x = (3/p)\log n = 3n/\log n$,

$$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < n^x e^{-px(x-1)/2} = (ne^{-p(x-1)/2})^x = n^{-\Theta(n)} = o(1).$$

Let $n$ be large enough so that $\mathbb{P}(X \geq n/2) < 1/2$ and $\mathbb{P}(\alpha(G) \geq x) < 1/2$. Then there is some $G$ with fewer than $n/2$ cycles of length $\leq \ell$ and with $\alpha(G) \leq 3n/\log n$.

Remove a vertex from each cycle to get $G'$. Then $|V(G')| \geq n/2$, girth $> \ell$, and $\alpha(G') \leq \alpha(G) \leq 3n/\log n$, so

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{3n/\log n} = \frac{\log n}{6} > k$$

if $n$ is sufficiently large. □

***Remark* 3.4.2.** Erdős (1962) also showed that in fact one needs to see at least a linear number of vertices to deduce high chromatic number: for all $k$, there exists $\varepsilon = \varepsilon_k$ such that for all sufficiently large $n$ there exists an $n$-vertex graph with chromatic number $> k$ but every subgraph on $\lfloor \varepsilon n \rfloor$ vertices is 3-colorable. (In fact, one can take $G \sim G(n, C/n)$; see "Probabilistic Lens: Local coloring" in Alon–Spencer)

## 3.5 Random greedy coloring

In Section 1.3, we saw a simple argument showing that every $k$-uniform hypergraph with than $2^{k-1}$ edges is 2-colorable (meaning that we can color the vertices red/blue without no monochromatic edge). Take a moment to remember the proof.

In this section, we improve this result. The next result gives the current best known bound.

**Theorem 3.5.1** (Radhakrishnan and Srinivasan (2000))

There is some constant $c > 0$ so that every $k$-uniform hypergraph with at most $c\sqrt{\frac{k}{\log k}}2^k$ edges is 2-colorable.

Recall from Section 1.3 that there exists a non-2-colorable $k$-uniform hypergraph on $k^2$ vertices and $O(k^2 2^k)$ edges, via a random construction.

Here we present a simpler proof, based on a ***random greedy coloring***, due to Cherkashin and Kozik (2015), following an approach of Pluhaár (2009).

*Proof.* Consider a $k$-graph with $m$ edges.

Let us order the vertices using a uniformly random chosen permutation.

Color vertices greedily from left to right: color a vertex blue unless it would create a monochromatic edge, in which case color it red (i.e., every red vertex is the final vertex in an edge with all earlier $k-1$ vertices have already been colored blue).

The resulting coloring has no blue edges. The greedy coloring succeeds if it does not create a red edge.

Analyzing a greedy coloring is tricky, since the color of a single vertex may depend on the entire history. Instead, we identify a specific feature that necessarily results from a unsuccessful coloring.

If there is a red edge, then there must be two edges $e, f$ so that the last vertex of $e$ is the first vertex of $f$. Call such pair $(e, f)$ ***conflicting*** (note that whether $(e, f)$ is conflicting depends on the random ordering of the vertices, but not on how we assigned colors).

What is the probability of seeing a conflicting pair? Here is the randomness comes from the random ordering of vertices.

Each pair of edges with exactly one vertex in common conflicts with probability $\frac{(k-1)!^2}{(2k-1)!} = \frac{1}{2k-1}\binom{2k-2}{k-1}^{-1} \asymp k^{-1/2}2^{-2k}$. Summing over all $\leq m^2$ pairs of edges that share a unique vertex, we find that the expected number of conflicting pairs is at most $\lesssim m^2 k^{-1/2} 2^{-2k}$, which is $< 1$ for some $m \asymp k^{1/4}2^k$. In this case, there is some ordering of vertices creating no conflicting pairs, in which case the greedy coloring always succeeds.

The above argument, due to Pluhaár (2009), yields $m \lesssim k^{1/4}2^k$. Next we will refine the argument to obtain a better bound of $\sqrt{\frac{k}{\log k}}2^k$ as claimed.

Instead of just considering a random permutation, let us map each vertex to $[0, 1]$ independently and uniformly at random. This map induces an ordering of the vertices, but it comes with further information that we will use.

Write $[0, 1] = L \cup M \cup R$ where ($p$ to be decided)

$$L := \left[0, \frac{1-p}{2}\right), \qquad M := \left[\frac{1-p}{2}, \frac{1+p}{2}\right], \qquad R := \left(\frac{1+p}{2}, 1\right].$$

The probability that a given edge lands entirely in $L$ is $(\frac{1-p}{2})^k$, and likewise with $R$. Taking a union bound over all edges,

$$\mathbb{P}(\text{some edge lies in } L \text{ or } R) \leq 2m\left(\frac{1-p}{2}\right)^k.$$

Suppose that no edge of $H$ lies entirely in $L$ or entirely in $R$. If $(e, f)$ conflicts, then their unique common vertex $x_v \in e \cap f$ must lie in $M$. So the probability that $(e, f)$

conflicts is (here we use $x(1-x) \le 1/4$)

$$\int_{(1-p)/2}^{(1+p)/2} x^{k-1}(1-x)^{k-1}\,dx \le p4^{-k+1}.$$

Taking a union bound over all $\le m^2$ pairs of edges, we find that

$$\mathbb{P}(\text{some conflicting pair has the common vertex in } M) \le m^2 p 4^{-k+1}.$$

Thus

$$\begin{aligned}
&\mathbb{P}(\text{there is a conflicting pair}) \\
&\quad \le \mathbb{P}(\text{some edge lies in } L \text{ or } R) + \mathbb{P}(\text{some conflicting pair has the common vertex in } M) \\
&\quad \le 2m\left(\frac{1-p}{2}\right)^k + m^2 p 4^{-k+1} \\
&\quad < 2^{-k+1} m e^{-pk} + (2^{-k+1}m)^2 p
\end{aligned}$$

set $p = \log(2^{k-1}k/m)/k$ to minimize the right-hand side to get

$$= \frac{m^2}{4^{k-1}k} + \frac{m^2}{4^{k-1}k}\log\left(\frac{2^{k-2}k}{m}\right)$$

which is $< 1$ for $m = c2^k\sqrt{k/\log k}$ with $c > 0$ being a sufficiently small constant (we should assume that is $k$ large enough to ensure $p \in [0,1]$; smaller values of $k$ can be handled in the theorem exceptionally by later reducing the constant $c$). □

*Food for thought*: what is the source of the gain in the the $L \cup M \cup R$ argument? The expected number of conflicting pairs is unchanged. It must be that we are somehow clustering the bad events by considering the event when some edge lies in $L$ or $R$.

It remains an intriguing open problem to improve this bound further.

## Exercises

1. Using the alteration method, prove the Ramsey number bound

   $$R(4,k) \ge c(k/\log k)^2$$

   for some constant $c > 0$.

2. Prove that every 3-uniform hypergraph with $n$ vertices and $m \ge n$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least

$cn^{3/2}/\sqrt{m}$, where $c > 0$ is a constant.

3. Prove that every $k$-uniform hypergraph with $n$ vertices and $m$ edges has a transversal (i.e., a set of vertices intersecting every edge) of size at most $n(\log k)/k + m/k$.

4. *Zarankiewicz problem.* Prove that for every positive integers $n \geq k \geq 2$, there exists an $n \times n$ matrix with $\{0, 1\}$ entries, with at least $\frac{1}{2}n^{2-2/(k+1)}$ 1's, such that there is no $k \times k$ submatrix consisting of all 1's.

5. Fix $k$. Prove that there exists a constant $c_k > 1$ so that for every sufficiently large $n > n_0(k)$, there exists a collection $\mathcal{F}$ of at least $c_k^n$ subsets of $[n]$ such that for every $k$ distinct $F_1, \ldots, F_k \in \mathcal{F}$, all $2^k$ intersections $\bigcap_{i=1}^k G_i$ are nonempty, where each $G_i$ is either $F_i$ or $[n] \setminus F_i$.

6. *Acute sets in* $\mathbb{R}^n$. Prove that, for some constant $c > 0$, for every $n$, there exists a family of at least $c(2/\sqrt{3})^n$ subsets of $[n]$ containing no three distinct members $A, B, C$ satisfying $A \cap B \subseteq C \subseteq A \cup B$.
   Deduce that there exists a set of at least $c(2/\sqrt{3})^n$ points in $\mathbb{R}^n$ so that all angles determined by three points from the set are acute.
   *Remark.* The current best lower and upper bounds for the maximum size of an "acute set" in $\mathbb{R}^n$ (i.e., spanning only acute angles) are $2^{n-1} + 1$ and $2^n - 1$ respectively.

7. ⋆ *Covering complements of sparse graphs by cliques*

   a) Prove that every graph with $n$ vertices and minimum degree $n - d$ can be written as a union of $O(d^2 \log n)$ cliques.

   b) Prove that every bipartite graph with $n$ vertices on each side of the vertex bipartition and minimum degree $n - d$ can be written as a union of $O(d \log n)$ complete bipartite graphs (assume $d \geq 1$).

8. ⋆ Let $G = (V, E)$ be a graph with $n$ vertices and minimum degree $\delta \geq 2$. Prove that there exists $A \subseteq V$ with $|A| = O(n(\log \delta)/\delta)$ so that every vertex in $V \setminus A$ contains at least one neighbor in $A$ and at least one neighbor not in $A$.

9. ⋆ Prove that every graph $G$ without isolated vertices has an induced subgraph $H$ on at least $\alpha(G)/2$ vertices such that all vertices of $H$ have odd degree. Here $\alpha(G)$ is the size of the largest independent set in $G$.

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