MATHEMATICAL IDEAS AND NOTIONS OF QUANTUM FIELD THEORY

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Introduction

Physics has always been a major source of both motivation and applications for several central fields of mathematics, such as analysis, differential equations and probability. However, the development of quantum field theory and string theory in the last four decades has taken interactions between these two disciplines to an unprecedented level, incorporating into physics such traditionally "pure" areas of mathematics as algebraic topology, category theory, differential and algebraic geometry, representation theory, combinatorics, and even number theory. This interaction has been highly fruitful in both directions, and led to a necessity for physicists to know the basics of modern mathematics and for mathematicians to know the basics of modern physics. Physicists have been quick to learn, and nowadays good physicists often understand relevant areas of mathematics as deeply as professional mathematicians. On the other hand, many mathematicians have been dragging their feet, deterred by lack of rigor in physical texts, and, more importantly, by a different manner of presentation. In particular, even the basic setting of quantum field theory, necessary for understanding its more advanced (and more mathematically exciting) parts, is already largely unfamiliar to mathematicians. Nevertheless, many of the basic ideas of quantum field theory can in fact be presented in a rigorous and mathematically natural way. Doing so is the main goal of this text.

Namely, these are slightly expanded lecture notes for a graduate course on basic mathematical structures of quantum field theory that I gave at the MIT Mathematics Department in 2002 and then again in 2023. The reader should not hope to learn quantum field theory from this text - this is impossible, for instance, because I know less about this subject than a beginning physics graduate student. Rather, as mentioned above, its aim is to present the basic setup of quantum field theory in a mathematically motivated manner, highlighting its connections with various fields of mathematics. As such, it could serve to prepare the reader for more advanced texts in this genre, such as [QFS], or for reading a regular QFT textbook or lecture notes, such as [Co, W, IZ, PS] from a mathematician's viewpoint. Note that a lot of important material is contained in exercises, which I strongly recommend the reader to solve while reading the text.

We begin with a general discussion of classical and quantum mechanics and field theory (Chapter 1). Then we proceed to prove the steepest descent and stationary phase formulas in classical asymptotic analysis, which serve as a finite dimensional model for perturbative

computations with path integrals (Chapter 2). Then, in Chapter 3, we develop Feynman calculus, the main combinatorial tool in perturbative quantum field theory. To illustrate Feynman calculus, we give a number of ins applications to enumerative combinatorics (the matrix-tree theorem and its specializations).

In Chapter 4, we extend Feynman calculus to matrix integrals, and show that for such integrals Feynman graphs are replaced by fat graphs (surfaces), so that the coefficients of the asymptotic expansion in $1/N$ (where N is the matrix size) are sums over fat graphs of a given genus. This allows us, in Chapter 5, to prove Harer-Zagier's theorem on the Euler characteristic of the moduli space of curves, and in Chapter 6 to obtain non-trivial counts of planar graphs.

All this material is, however, about quantum field theory in 0 spacetime dimensions, or, as one may jokingly say, in -1-dimensional space. To connect to real physics, we must go up at least one dimension, i.e., consider quantum field theory in $0+1$ spacetime (or 0 space) dimensions, which is quantum mechanics (Chapter 7). We begin with a review of Lagrangian formalism of classical mechanics (Lagrangians, least action principle), and then proceed to quantize this formalism, developing the path integral approach to quantum mechanics. Namely, we describe perturbative expansion of quantum-mechanical path integrals using Feynman diagrams and give several examples. We also explain that quantum mechanical path integrals are related to (rigorously defined) Wiener integrals in the theory of stochastic processes by the Wick rotation of the time, $t \mapsto it$.

In Chapter 8, after reviewing Hamiltonian formalism in classical mechanics, we describe its quantization, which gives a rigorous basis for non-perturbative quantum mechanics. We also prove the Feynman-Kac formula which relates the correlation functions obtained in the Lagrangian and Hamiltonian approaches.

In Chapters 9 and 10, we discuss the super-generalization of the material of the previous chapters, i.e., describe classical and quantum mechanics for fermions. We begin with a review of supergeometry and Berezin's integration theory on supermanifolds and proceed to extend Feynman calculus to the super-case. Then we discuss classical and quantum mechanics for fermions.

In Chapter 11 we finally get to the actual quantum field theory, in $d+1$ spacetime dimension with $d \geq 1$. We start with reviewing Lagrangian classical field theory and then pass to its quantization, in particular describing the theories of free bosons and fermions. We review the classical theory of spinors (in particular, real Bott periodicity for them, modulo 8) and use it to describe the possible kinetic terms

and mass terms in fermionic lagrangians. Then we turn to hamiltonian formalism in both classical and quantum field theory and discuss Wightman axioms. We conclude with describing the quantum theory of a free scalar boson from this point of view.

In Chapter 12, we describe the basics of the perturbative renomalization theory. In particular, we discuss ultraviolet divergences of Feynman amplitudes and regularization of such divergences by introducing counterterms in the Lagrangian depending on the cutoff Λ in the momentum space. We define super-renormalizable, renormalizable and non-renormalizable theories, critical dimensions for various theories and terms in the Lagrangian, and discuss the key examples.

Finally, in Chapter 13 we give a brief introduction to 2-dimensional conformal field theory. After a review of classical field theory of a massless scalar in 1+1 dimensions, we quantize it and construct its Hilbert space from the Fock representation of the infinite dimensional Heisenberg Lie algebra. We show that the partition function of this theory (normalized using the zeta function regularization) is modular invariant, reflecting its conformal symmetry. Then we show that the Hilbert space of the theory carries two commuting projective actions of the Lie algebra W of polynomial vector fields on $\mathbb{C}^\times$, which expresses the infinitesimal conformal symmetry. We explain that this action is truly projective, i.e., both copies of W are replaced by its non-trivial central extension - the Virasoro algebra (conformal anomaly). Then we discuss a circle-valued version of this theory, vertex operators and T-duality. We also briefly discuss the quantum theory of a free fermion in 1+1 dimensions and the Wess-Zumino-Witten model.

1. Generalities on quantum field theory

1.1. **Classical mechanics.** In classical mechanics, we study the motion of a particle (say, of mass 1) in a Euclidean space V . This motion is described by a function of one variable, $q = q(t) \in V$, representing the position of the particle at a time t. This function must satisfy the *Newton equation of motion*,

$$\ddot{q} = -U'(q),$$

where U is the potential energy.

Another way to express this law of motion is to say that $q(t)$ must be a solution of a certain variational problem. Namely, one introduces the *Lagrangian*

$$\mathcal{L}(q) := \frac{\dot{q}^2}{2} - U(q)$$

(the difference of kinetic and potential energy), and the *action* functional

$$S(q) := \int_a^b \mathcal{L}(q)dt$$

(for some fixed $a < b$). Then the law of motion can be expressed as the *least action principle*: $q(t)$ must be a critical point of S on the space of all functions with given $q(a)$ and $q(b)$, i.e., the Newton equation is the Euler-Lagrange equation for a solution of the variational problem defined by S. Indeed, using integration by parts, for $\varepsilon \in C^1[a, b]$ with $\varepsilon(a) = \varepsilon(b) = 0$ we have

$$\tfrac{d}{ds}|_{s=0} \int_a^b \mathcal{L}(q + s\varepsilon)dt = \int_a^b (\tfrac{\partial \mathcal{L}}{\partial q}\varepsilon + \tfrac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon})dt =$$

$$\int_a^b (-U'(q)\varepsilon + \dot{q}\dot{\varepsilon})dt = -\int_a^b (U'(q) + \ddot{q})\varepsilon dt,$$

and this vanishes for all ε iff q satisfies the Newton equation $\ddot{q} = -U'(q)$.

Remark 1.1. The name "least action principle" comes from the fact that in some cases (for example when $U'' \leq 0$) the action is not only extremized but also minimized at the solution $q(t)$. In general, however, this is not the case, and the trajectory of the particle may be not a (local) minimum, but only a critical point of the action. Therefore, the law of motion is better formulated as the "extremal (or stationary) action principle"; this is the way we will think of it in the future.

Exercise 1.2. *(i) Consider the motion of a particle in a Euclidean space V. Show that if the potential is concave ($U''(q) \leq 0$) then for any $\mathbf{a}, \mathbf{b} \in V$ and $a < b \in \mathbb{R}$ there exists at most one solution of the*

Newton equation with $q(a) = \mathbf{a}$ and $q(b) = \mathbf{b}$, and it is the strict global minimum for the action with these boundary conditions (if exists).

(ii) Show that the conclusion of (i) holds if $U''(q) < \frac{\pi^2}{(b-a)^2}$ (prove and use Wirtinger's inequality: if $\varepsilon \in C^1[a,b]$ and $\varepsilon(a) = \varepsilon(b) = 0$ then $\int_a^b \varepsilon'(t)^2 dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b \varepsilon(t)^2 dt$).

(iii) Compute the unique solution in (i) if $U(q) = -\frac{1}{2}B(q,q)$, where B is a nonnegative definite symmetric bilinear form on V.

(iv) Show that the statements of (i) fail for $\dim V = 1$, $U(q) = \frac{1}{2}q^2$ and $b - a \geq \pi$.

(v) Let $\dim V = 1$ and U be a smooth potential on $\mathbb{R}$. Suppose that $\limsup_{|x|\to\infty} \frac{U(x)}{x^2} \leq 0$. Show that a solution in (i) (possibly non-unique) exists for any $a, b, \mathbf{a}, \mathbf{b}$. Give an example of a smooth potential U for which a solution in (i) does not always exist.[1]

Remark 1.3. Physicists often consider solutions of Newton's equation on the whole line rather than on a fixed interval $[a,b]$. In this case, the naive definition of an extremal does not make sense, since the action integral $S(q) = \int_{\mathbb{R}} \mathcal{L}(q)dt$ is improper and in general diverges. Instead, one makes the following "corrected" definition: a function $q(t)$ on $\mathbb{R}$ is an extremal of S if the expression

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} \mathcal{L}(q + s\varepsilon)dt := \int_{\mathbb{R}} (\frac{\partial \mathcal{L}}{\partial q}\varepsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon})dt,$$

where $\varepsilon(t)$ is any compactly supported perturbation, is identically zero. With this definition, the extremals are exactly the solutions of Newton's equation (which, as before, is easily seen by integration by parts).

Remark 1.4. Note that this formalism also describes the motion of a system of n particles, if we combine the vectors representing their positions in a Euclidean space V into a single vector in V^n. More generally, we may consider a particle moving on a Riemannian manifold M. In this case $q(t)$ is a path on M, and the motion is described by the same equation, where $\ddot{q}$ means the covariant derivative $\nabla_{\dot{q}}\dot{q}$ of $\dot{q}$ with respect to the Levi-Civita connection. For example, if $U = 0$, this is the *geodesic flow*, whose trajectories are the geodesics on M. The same applies to a system of n particles on M, in which case $q(t)$ is a path on the configuration space M^n. Finally, a similar analysis applies to more general Lagrangians, which are arbitrary smooth functions of (finitely many) derivatives of q.

[1] One can show using calculus of variations that for any $\dim V$, if $U(q) \leq 0$ for all q then the solution always exists.

1.2. **Classical field theory.** In classical field theory, the situation is similar, but with infinitely many particles. Namely, in this case we should think not of a single particle or a finite system of particles, but rather of a "continuum of particles" (e.g. a string, a membrane, a jet of fluid); so in a $d+1$-dimensional classical field theory the motion is described by a *classical field* – a (vector-valued) function $\phi(x,t)$ depending on both space and time coordinates ($x \in \mathbb{R}^d$, $t \in \mathbb{R}$). Consequently, the equation of motion is a partial differential equation. For example, for a string or a membrane the equation of motion is the *wave equation* $\Box\phi = 0$, where $\Box$ is the *D'Alembertian* $\partial_t^2 - v^2\Delta$ (here Δ is the Laplacian with respect to the space coordinates, and v the velocity of wave propagation, e.g. for the string v^2 is proportional to the string tension).

As in classical mechanics, in classical field theory there is a Lagrangian $\mathcal{L}(\phi)$ (a smooth function of finitely many partial derivatives of ϕ), whose integral

$$S(\phi) = \int_D \mathcal{L}(\phi) dx dt$$

over a compact region D in the *spacetime* $\mathbb{R}^{d+1}$ is called the *action*. The law of motion can be expressed as the condition that the action must be extremized over any such region D with fixed boundary conditions; so the equations of motion (also called the *field equations*) are the Euler-Lagrange equations for this variational problem. For example, in the case of string or membrane, the Lagrangian is

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\phi_t^2 - v^2(\nabla\phi)^2).$$

Remark 1.5. Like in mechanics, in field theory solutions of the equations of motion on the whole space-time (rather than a compact region D) are extremals of the action in the sense that

$$\tfrac{d}{ds}|_{s=0} \int_{\mathbb{R}^{d+1}} \mathcal{L}(u + s\varepsilon) dx dt = 0,$$

where ε is a compactly supported perturbation.

1.3. **Brownian motion.** One of the main differences between classical and quantum mechanics is, roughly speaking, that quantum particles do not have to obey the classical equations of motion, but can randomly deviate from their classical trajectories. Therefore, given the position and velocity of the particle at a given time, we cannot determine its position at a later time, but can only determine the density of probability that at this later time the particle will be found at a given point.

In this sense quantum particles are similar to random (Brownian) particles. Brownian particles are a bit easier to understand conceptually, so let us begin with them.

The motion of a Brownian particle in $\mathbb{R}^k$ in a potential field

$$U : \mathbb{R}^k \to \mathbb{R}$$

is described by a stochastic process $q = q(t)$, $q = (q_1, \dots, q_k) \in \mathbb{R}^k$. That is, for each real t we have a random variable $q(t) \in \mathbb{R}^k$ (the position of the particle at a time t), such that the dependence of t is regular in some sense. Namely, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$ the random dynamics of the particle conditioned to have $q(a) = \mathbf{a}$, $q(b) = \mathbf{b}$ is "defined" as follows:[2] if $y : [a, b] \to \mathbb{R}^k$ is a continuously differentiable function with $y(a) = \mathbf{a}, y(b) = \mathbf{b}$, then the density of probability that $q(t) = y(t)$ for $t \in [a, b]$ is proportional to $e^{-S(y)/\kappa}$, where

$$S(y) := \int_a^b (\tfrac{1}{2} y'^2 + U(y)) dt$$

is the action and $\kappa > 0$ is the diffusion coefficient. Thus, the likeliest $q(t)$ is the one that minimizes S (in particular, solves the classical equations of motion $\ddot{q} = U'(q)$), while the likelihood of the other paths decays exponentially with the deviation of the action of these paths from the minimal possible.

Remark 1.6. 1. This discussion thus assumes that the extremum of S at q is actually a minimum, which we know is not always the case, but is so when U is convex, i.e., $U''(q) \geq 0$ for all q (see Exercise 1.2).

2. The reader must have noticed that compared to the discussion of classical mechanics, the sign in front of the potential U has been changed to the opposite one. This is not a misprint! It has to do with the fundamental fact discussed below that statistical mechanics is related to usual (quantum) mechanics by the *Wick rotation* $t \mapsto it$, where $i = \sqrt{-1}$. In particular, this means that Brownian motion is well defined in the physically important case of convex potential, such as the multidimensional *harmonic oscillator* potential $\frac{1}{2} B(q, q)$ where B is a positive definite bilinear form.

All the information we can hope to get about the stochastic process $q(t)$ is contained in the *correlation functions*

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle,$$

[2]We put the word "defined" in quotation marks because this definition is obviously heuristic and not rigorous; see below for more explanations.

which by definition are the expectation values of the products of random variables $q_{j_1}(t_1), \dots, q_{j_n}(t_n)$, (more specifically, by Kolmogorov's theorem the stochastic process $q(t)$ is completely determined by these functions). So such functions should be regarded as the output, or answer, of the theory of the Brownian particle.

Thus the main question is how to compute the correlation functions. Physicists write down the following "answer" motivated by the above definition: given points $t_1, \dots, t_n \in [a, b]$,

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq, \tag{1.1}$$

where integration is carried out over the space $P_{\mathbf{a},\mathbf{b}}$ of paths

$$q : [a, b] \to \mathbb{R}^n, \ q(a) = \mathbf{a}, \ q(b) = \mathbf{b},$$

and Dq is a Lebesgue measure on this space such that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}} Dq = 1.$$

Alternatively, when they do not want to normalize the Lebesgue measure, they write

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \frac{1}{Z} \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq, \tag{1.2}$$

where

$$Z := \int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}} Dq$$

is the *partition function*. Such an integral is called a *path integral*, since it is an integral over the space of paths.

It is clear, however, that such definition and answer are a priori not satisfactory from the mathematical viewpoint, since the infinite dimensional integration requires justification. In the case of Brownian motion, such a justification is actually possible within the framework of the Lebesgue measure theory, and the corresponding integration theory is called the theory of *Wiener integral*. (To be more precise, one cannot define the measure Dq, but one can define the measure $e^{-\frac{S(q)}{\kappa}} Dq$ for sufficiently nice potentials $U(q)$).

Remark 1.7. As $\kappa \to 0$, the non-optimal trajectories become increasingly less likely relatively to the optimal one, so in the limit we recover the deterministic system:

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n),$$

where $\mathbf{q}(t)$ is the classical trajectory with $\mathbf{q}(a) = \mathbf{a}, \mathbf{q}(b) = \mathbf{b}$ (note that if $U \geq 0$ then this trajectory is unique by Exercise 1.2).

1.4. **Quantum mechanics.** Now let us turn to a quantum particle. Quantum mechanics is notoriously difficult to visualize, and the randomness of the behavior of a quantum particle is less intuitive and more subtle than that of a Brownian particle; nevertheless, it was pointed out by Feynman that the behavior of a quantum particle in a potential field $U(q)$ is correctly described by the same model, with the real positive parameter κ replaced by the imaginary number $-i\hbar$ where $\hbar > 0$ is the *Planck constant*, and the time t is replaced by it. In other words, the dynamics of a quantum particle can be expressed (we will discuss later how) via the *correlation functions*

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{\frac{iS(q)}{\hbar}} Dq, \tag{1.3}$$

where Dq is normalized so that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{\frac{iS(q)}{\hbar}} Dq = 1, \tag{1.4}$$

and $S(q)$ is now given by the same formula as in classical mechanics (and differing by sign from Brownian motion):

$$S(q) = \int_a^b (\tfrac{\dot{q}^2}{2} - U(q)) dt.$$

As before, we have to make sense of this path integral, and now the theory of Wiener integrals unfortunately does not work any more: for instance, the absolute value of the integrand in (1.4) does not decay as the path $q(t)$ deviates from the classical trajectory (in fact, it identically equals to 1!). So we will be able to make sense of (1.3) only partially, and an effective mathematically rigorous approach to quantum mechanics is, in fact, based on different techniques (Hamiltonian formalism); this is discussed in more detail below. Still, formula (1.3) is extremely helpful for motivational purposes and with appropriate care can be used for computation.

Remark 1.8. Similarly to Brownian motion (cf. Remark 1.7), in the limit $\hbar \to 0$ we are supposed to recover the classical system:

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n).$$

However, now this is achieved not because individual non-optimal trajectories become less likely, but rather due to cancellation in the oscillatory integral (1.3) which corresponds to the physical phenomenon of *quantum interference*. We will observe how this cancellation occurs in

finite-dimensional oscillatory integrals when we discuss the stationary phase formula below.

1.5. **Quantum field theory.** The situation is the same in field theory, but with infinitely many particles. Namely, a useful theory of quantum fields (used in the study of interactions of elementary particles) is obtained when one considers correlation functions

(1.5)
$$\langle \phi_{j_1}(x_1,t_1)\dots\phi_{j_n}(x_n,t_n)\rangle = \int \phi_{j_1}(x_1,t_1)\dots\phi_{j_n}(x_n,t_n)e^{\frac{iS(\phi)}{\hbar}}D\phi,$$

where $D\phi$ is normalized so that $\int e^{\frac{iS(\phi)}{\hbar}}D\phi = 1$.

Of course, from the mathematical point of view, this setting is a priori even less satisfactory than the one for a quantum particle, since it involves integration with respect to the complex-valued measure $e^{\frac{iS(q)}{\hbar}}Dq$ on functions of ≥ 2 variables which nobody knows how to define in general (even after the Wick rotation). Nevertheless, physicists imagine that certain integrals of this type exist and come to correct and interesting conclusions (both physical and mathematical). Therefore, making sense of such integrals is an interesting problem for mathematicians, and will be one of our main occupations.[3]

[3]To be more precise, we will make sense of path integrals as power series in $\hbar$.

2. The steepest descent and stationary phase formulas

Now, let us forget for a moment that the integrals (1.1,1.3,1.5) are infinite dimensional and hence problematic to define, and ask ourselves the following question: why should we expect to recover the usual classical mechanics or field theory when the parameter κ or $\hbar$ goes to zero? The answer is that this expectation is based on the *steepest descent* (respectively, *stationary phase*) principle from classical analysis: if $f(x)$ is a function on $\mathbb{R}^d$ then the integrals $\int g(x)e^{-\frac{f(x)}{\kappa}}dx$, $\int g(x)e^{\frac{if(x)}{\hbar}}dx$ "localize" to minima, respectively critical points, of the function f. As this classical fact is of central importance to the whole subject, let us now discuss it in some detail.

2.1. **Gaussian integrals.** We start with auxiliary facts from linear algebra and analysis. Let V be a real vector space of dimension d. Let $\mathbf{M}(V)$ be the set of non-degenerate complex-valued symmetric bilinear forms on V with non-negative definite real part. We have an open dense subset $\mathbf{M}^\circ(V) \subset \mathbf{M}(V)$ of forms with positive definite real part. If $B = P + iQ \in \mathbf{M}^\circ(V)$ where P, Q are the real and imaginary parts of B, then $P^{-1}Q : V \to V$ is a self-adjoint operator with respect to P, which therefore has real eigenvalues and diagonalizes in an orthonormal basis. In this basis $B(x,y) = \sum_{j=1}^d a_j x_j y_j$ where $\mathrm{Re}(a_j) = 1$. Thus $B^{-1} \in \mathbf{M}^\circ(V^*)$. It follows that the map $B \mapsto B^{-1}$ is a homeomorphism $\mathbf{M}(V) \cong \mathbf{M}(V^*)$ which restricts to a homeomorphism $\mathbf{M}^\circ(V) \cong \mathbf{M}^\circ(V^*)$.

Now fix a translation-invariant volume form dx on V. Then for every complex-valued symmetric bilinear form B on V we can define its determinant $\det B$. Thus we can define a continuous function $(\det B)^{-\frac{1}{2}}$ on $\mathbf{M}(V)$ using the branch of the square root which is positive on positive definite forms (it exists and is unique because $\mathbf{M}(V)$ is star-like with respect to any point of $\mathbf{M}^\circ(V)$, hence simply connected). Note that if $B = iQ$ where Q is a real non-degenerate form then $(\det B)^{-\frac{1}{2}} = e^{\frac{\pi i \sigma(Q)}{4}} |\det Q|^{-\frac{1}{2}}$, where σ is the signature of Q. Indeed, it suffices to check the statement for diagonal forms, hence for $d = 1$, in which case it is straighforward.

Let $\mathcal{S}(V)$ be the *Schwartz space* of V, i,.e., the space of smooth functions on V whose all derivatives are rapidly decaying at ∞ (faster than any power of $|x|$). In other words, $\mathcal{S}(V)$ is the space of smooth functions f on V such that $D(V)f \subset L^2(V)$, where $D(V)$ is the algebra of differential operators on V with polynomial coefficients. The Schwartz space has a natural Fréchet topology defined by the seminorms $||Df||_{L^2}$, $D \in D(V)$. The topological dual space $\mathcal{S}'(V)$ is the

space of *tempered distributions* on V. Note that we have natural inclusions $\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V)$. Recall that the Fourier transform is the operator

$$\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$$

given by

$$\mathcal{F}(g)(p) := (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i(p,x)} dx,$$

which defines an isometry $L^2(V) \to L^2(V^*)$ such that $(\mathcal{F}^2 g)(x) = g(-x)$. By duality, it defines an operator

$$\mathcal{F} : \mathcal{S}'(V) \to \mathcal{S}'(V^*)$$

which extends $\mathcal{F}$. For any complex symmetric bilinear form B with $\mathrm{Re}B \geq 0$ the function $e^{-\frac{1}{2}B(x,x)}$ belongs to $\mathcal{S}'(V)$, and moreover to $\mathcal{S}(V)$ iff $B \in \mathbf{M}^\circ(V)$. Furthermore, it depends continuously on B as an element of these spaces. We will call it the *complex Gaussian distribution*.

Lemma 2.1. *(Gaussian integral) For any $B \in \mathbf{M}(V)$ we have*

$$\mathcal{F}(e^{-\frac{1}{2}B(x,x)}) = (\det B)^{-\frac{1}{2}} e^{-\frac{1}{2}B^{-1}(p,p)}.$$

Proof. By continuity, it suffices to prove this when $\mathrm{Re}B > 0$. In this case B is diagonalizable, so the statement reduces to the case $d = 1$. In this case we have to show that for every $a \in \mathbb{C}$ with $\mathrm{Re}a > 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx - \frac{1}{2}ax^2} dx = \frac{1}{\sqrt{a}} e^{-\frac{1}{2a}p^2}.$$

Since both sides are holomorphic in a, it is enough to check the statement when a is real. The integral in question can be written as

$$\frac{e^{-\frac{1}{2}a^{-1}p^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx.$$

But using Cauchy's theorem,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+ia^{-1}p} e^{-\frac{1}{2}ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx.$$

Thus the result follows from the Poisson integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

by rescaling x. □

In the sense of Lemma 2.1 we can say, setting $p = 0$, that

$$(2\pi)^{-\frac{d}{2}} \int_V e^{-\frac{1}{2}B(x,x)} dx = (\det B)^{-\frac{1}{2}}. \tag{2.1}$$

Note that this equality is also true in the sense of absolute convergence if $B \in \mathbf{M}^\circ(V)$ and conditional convergence otherwise (check it!).

2.2. **Gaussian integrals with insertions.** Now let $g \in \mathcal{S}(V)$. Consider the integral

$$I_g(\hbar) := \int_V g(\hbar^{\frac{1}{2}} x) e^{-\frac{1}{2}B(x,x)} dx, \ \hbar \geq 0,$$

where for $\hbar = 0$ we use (2.1), so

$$I_g(0) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0). \tag{2.2}$$

Let $\Delta_B : \mathcal{S}(V) \to \mathcal{S}(V)$ be the Laplace operator corresponding to B: $\Delta_B = \sum_{j=1}^d \partial_{B^{-1}e_j^*} \partial_{e_j}$ for a basis $\{e_i\}$ of V.

Theorem 2.2. *We have*

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar), \ \hbar \geq 0.$$

Thus $I_g \in C^\infty[0, \infty)$. In particular, if g vanishes at the origin to order $2n+1$ then $I_g(0) = ... = I_g^{(n)}(0) = 0$.

The rest of the subsection is occupied by the proof of Theorem 2.2.

Lemma 2.3. *I_g is a continuous function.*

Proof. Only continuity at $\hbar = 0$ requires proof. By Plancherel's theorem and Lemma 2.1,

$$I_g(\hbar) = (g(\hbar^{\frac{1}{2}} x), e^{-\frac{1}{2}B(x,x)}) =$$

$$\hbar^{-\frac{d}{2}} (\det B)^{-\frac{1}{2}} (\widehat{g}(\hbar^{-\frac{1}{2}} p), e^{-\frac{1}{2}B^{-1}(p,p)}) = (\det B)^{-\frac{1}{2}} (\widehat{g}(p), e^{-\frac{\hbar}{2}B^{-1}(p,p)}),$$

where $\widehat{g}$ is the Fourier transform of g. But $e^{-\frac{\hbar}{2}B^{-1}(p,p)} \to 1$ in $\mathcal{S}'(V^*)$ as $\hbar \to 0$ (as the complex Gaussian distribution depends continuously of the bilinear form). Thus

$$\lim_{\hbar \to 0} I_g(\hbar) = (\det B)^{-\frac{1}{2}} (\widehat{g}(p), 1) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0) = I_g(0),$$

as desired. □

Lemma 2.4. *If $\ell \in V^*$ and $f \in \mathcal{S}(V)$ then*

$$I_{\ell f}(\hbar) = \hbar I_{\partial_{B^{-1}\ell} f}(\hbar).$$

Proof. We have

$$I_{\ell f}(\hbar) = \hbar^{\frac{1}{2}}(\ell(x)f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \ell(x)e^{-\frac{1}{2}B(x,x)}) =$$

$$-\hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \partial_{B^{-1}\ell}e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(\partial_{B^{-1}\ell}f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) =$$

$$\hbar((\partial_{B^{-1}\ell}f)(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar I_{\partial_{B^{-1}\ell}f}(\hbar).$$

This proves the lemma. □

Now we prove Theorem 2.2. If $\hbar > 0$ then by direct differentiation we get

$$I'_g(\hbar) = \tfrac{1}{2}\hbar^{-1}I_{Eg}(\hbar),$$

where $E := \sum_{j=1}^d e_j^*\partial_{e_j}$ is the Euler vector field on V. Thus by Lemma 2.4 we have

$$I'_g(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar),\ \hbar > 0. \tag{2.3}$$

So, using Lemma 2.3, it suffices to show that $I_g \in C^1[0,\infty)$ (then smoothness will follow by repeated application of (2.3)). To this end, note that if C is a positive definite form on V then

$$I_{e^{-\frac{1}{2}C(x,x)}}(\hbar) = \int_V e^{-\frac{1}{2}(B+\hbar C)(x,x)}dx = (2\pi)^{\frac{d}{2}}\det(B+\hbar C)^{-\frac{1}{2}},$$

which is analytic, hence continuously differentiable on $[0,\infty)$. So subtracting from g a multiple of such function, it suffices to prove that $I_g \in C^1[0,\infty)$ when $g(0) = 0$. In this case g is well known to be a linear combination of functions of the form ℓf where $f \in \mathcal{S}(V)$ and $\ell \in V^*$. So it suffices to check that $I_g \in C^1[0,\infty)$ for $g = \ell f$. But then by Lemma 2.4 $I'_g(0) = I_{\partial_{B^{-1}\ell}f}(0) = I_{\frac{1}{2}\Delta_B g}(0)$, as

$$\tfrac{1}{2}\Delta_B g(0) = \tfrac{1}{2}\Delta_B(\ell f)(0) = \sum_j \ell(e_j)\partial_{B^{-1}e_j^*}f(0) = \partial_{B^{-1}\ell}f(0).$$

This completes the proof.

Exercise 2.5. *Let $\mathcal{S}_m(V) \subset C^m(V)$ be the subspace of functions whose derivatives of order $\leq m$ are rapidly decaying. Prove that the differentiation formula of Theorem 2.2 holds for $g \in \mathcal{S}_2(V)$. Deduce that if $g \in \mathcal{S}_{2n}(V)$ then $I \in C^n[0,\infty)$, and that if moreover g vanishes at 0 to order $2n+1$ then $I_g(0) = ... = I_g^{(n)}(0) = 0$.*

2.3. **The steepest descent formula.** Let $a < b$ be real numbers and $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are smooth on (a, b).

Theorem 2.6. *(Steepest descent formula) Assume that f attains a global minimum at a unique point $c \in [a, b]$, such that $a < c < b$ and $f''(c) > 0$. Then one has*

$$\int_a^b g(x)e^{-\frac{f(x)}{\hbar}}dx = \hbar^{\frac{1}{2}}e^{-\frac{f(c)}{\hbar}}I(\hbar), \tag{2.4}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = \sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}}.$$

Proof. Without loss of generality we may put $c = 0, f(c) = 0$. Let $f''(c) = M$. Making a change of variable, we may reduce to a situation where $f(x) = \frac{M}{2}x^2$ when x is in some neighborhood U of 0. Let h be a "bump" function - a smooth function supported in U which equals 1 in a smaller neighborhood $0 \in U' \subset U$. Write $g = g_1 + g_2$, where $g_1 = hg$ and $g_2 = (1 - h)g$. Let I be defined by equation (2.4), and I_1, I_2 be defined by the same equation for g replaced by g_1, g_2, so $I = I_1 + I_2$. Since f has a unique global minimum, we see by direct differentiation that for all n, $I_2^{(n)}(\hbar)$ is rapidly decaying as $\hbar \to 0$. Thus for $g = g_2$ the result is obvious, and our job is to prove it for $g = g_1$. In other words, we may assume without loss of generality that $g = g_1$ and $g_2 = 0$. We extend g by zero to the whole real line.

Let us make a change of variables $y := \hbar^{-\frac{1}{2}}x$. Then we get

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}}y)e^{-\frac{M}{2}y^2}dy. \tag{2.5}$$

Thus the result follows from (2.2) and Theorem 2.2. □

Remark 2.7. Theorem 2.6, in fact, provides an explicit formula for the Taylor coefficients of $I(\hbar)$. Namely, as in the proof of Theorem 2.6, assume that $c = 0$ and $f(x) = \frac{1}{2}p(x)^2$ near 0, where

$$p'(0) = \sqrt{f''(0)} > 0.$$

Ignoring limits of integration (which, as we have seen, are irrelevant for the asymptotic expansion of $I(\hbar)$), we have[4]

$$I(\hbar) = \hbar^{-\frac{1}{2}} \int g(x)e^{-\frac{p(x)^2}{2\hbar}}dx \sim \int_{-\infty}^{\infty} \widetilde{g}(\hbar^{\frac{1}{2}}y)e^{-\frac{y^2}{2}}dy$$

[4]Recall that for $I \in C^\infty[0, \varepsilon)$ we write $I(\hbar) \sim \sum_{n=0}^{\infty} a_n\hbar^n$ if for every $N \geq 0$ we have $I(\hbar) = \sum_{n=0}^{N-1} a_n\hbar^n + O(\hbar^N)$ as $\hbar \to 0$.

where

$$\widetilde{g}(z) := g(p^{-1}(z))(p^{-1})'(z) = \frac{g(p^{-1}(z))}{p'(p^{-1}(z))}.$$

By Theorem 2.2, the first $n+1$ terms of the Taylor expansion of this integral are given by the integral

$$I_N(\hbar) := \int_{-\infty}^{\infty} \widetilde{g}_N(\hbar^{\frac{1}{2}} y) e^{-\frac{y^2}{2}} dy$$

where $\widetilde{g}_N$ is the $2N$-th Taylor polynomial of $\widetilde{g}$ at 0. Thus if $\widetilde{g}(z) \sim \sum_{n=0}^{\infty} b_n z^n$ then

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} \hbar^n \int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy.$$

But, setting $u = \frac{y^2}{2}$, we have

(2.6)

$$\int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy = 2^{n+\frac{1}{2}} \int_0^{\infty} u^{n-\frac{1}{2}} e^{-u} du = 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}} (2n-1)!!,$$

where $(2n-1)!! := \prod_{1 \le j \le n} (2j-1)$. Hence

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) \hbar^n.$$

2.4. **Stationary phase formula.** Theorem 2.6 has the following imaginary analog, called the *stationary phase formula*.

Theorem 2.8. *(Stationary phase formula) Let $f, g : [a,b] \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point $c \in [a,b]$, such that $a < c < b$ and $f''(c) \neq 0$, and g has vanishing derivatives of all orders at a and b. Then*

$$\int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar),$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that

$$I(0) = \sqrt{2\pi} e^{\pm \frac{\pi i}{4}} \frac{g(c)}{\sqrt{|f''(c)|}},$$

where $\pm$ is the sign of $f''(c)$.[5]

Remark 2.9. It is important to assume that g has vanishing derivatives of all orders at a and b. Otherwise we will get additional boundary contributions.

[5]This is called the stationary phase formula because the main contribution comes from the point where the phase $\frac{f(x)}{\hbar}$ is stationary.

Proof. The proof is analogous to the proof of the steepest descent formula, but slightly more subtle, as we have to keep track of cancellations. First we need the following very simple but important lemma which allows us to do so.

Lemma 2.10. *(Riemann lemma) (i) Let $f : [a,b] \to \mathbb{R}$ be a smooth function such that $f'(x) > 0$ for all $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ a C^n-function such that*

$$g(a) = ... = g^{(n-1)}(a) = g(b) = ... = g^{(n-1)}(b) = 0.$$

Let

$$I(\hbar) := \int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx.$$

Then $I(\hbar) = O(\hbar^n),\ \hbar \to 0$.

(ii) Suppose g is smooth on $[a,b]$ and all derivatives of g at a and b are zero. Then I extends (by setting $I(0) := 0$) to a smooth function on $[0,\infty)$ whose all derivatives are rapidly decaying as $\hbar \to 0$.

Proof. (i) By making a change of variables we may assume without loss of generality that $f(x) = x$. Then the proof is by induction in n. The base case $n = 0$ is obvious. For $n > 0$ note that

$$\int_a^b g(x) e^{\frac{ix}{\hbar}} dx = i\hbar \int_a^b g'(x) e^{\frac{ix}{\hbar}} dx$$

(integration by parts), which justifies the induction step.

(ii) follows from (i) by repeated differentiation. □

Now we proceed to prove the theorem. As in the proof of the steepest descent formula, we may assume that $c = 0$ and $f = \frac{M}{2}x^2$ near 0 for some $M \neq 0$, and write I as the sum $I_1 + I_2$. Moreover, by Lemma 2.10(ii)

$$I_2(\hbar) = \int_a^b g_2(x) e^{\frac{if(x)}{\hbar}} dx$$

is rapidly decaying with all derivatives, so it suffices to prove the theorem for $g = g_1$.

Again following the proof of the steepest descent formula, we have

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}} y) e^{\frac{iM}{2}y^2} dy, \tag{2.7}$$

so as before the result follows from (2.2) and Theorem 2.2. □

Remark 2.11. Since computation of the asymptotic expansion of $I(\hbar)$ is a purely algebraic procedure, the explicit formula for this expansion

in the imaginary case is the same as in the real case (Remark 2.7) but with $\hbar$ replaced by $i\hbar$:

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2})(i\hbar)^n.$$

2.5. **Non-analyticity of $I(\hbar)$ and Borel summation.** Even though $I(\hbar)$ is smooth at $\hbar = 0$, its Taylor series is usually only an asymptotic expansion which diverges for any $\hbar \neq 0$, so that this function is not analytic at 0. To illustrate this, consider the integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \hbar^{\frac{1}{2}} I(\hbar),$$

where

$$I(\hbar) = \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy. \tag{2.8}$$

Since this integral is divergent for any $\hbar < 0$, we cannot conclude its analyticity at $\hbar = 0$, and it indeed fails to be so. Namely, as in Remark 2.7, the asymptotic expansion of integral (2.8) is obtained by expanding the exponential $e^{-\frac{1}{2}\hbar y^4}$ into a Taylor series and integrating termwise using (2.6):

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n,$$

where

$$a_n = (-1)^n \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{y^{4n}}{2^n n!} dy =$$

$$(-1)^n \frac{2^{n+\frac{1}{2}} \Gamma(2n+\frac{1}{2})}{n!} = (-1)^n \sqrt{2\pi} \frac{(4n-1)!!}{2^n n!}.$$

It is clear that this sequence has super-exponential growth, so the radius of convergence of the series is zero.

Let us now discuss the question: to what extent does the asymptotic expansion of the function $I(\hbar)$ (which we can find using Feynman diagrams as explained below) actually determine this function?

Suppose that

$$\widetilde{I}(\hbar) = \sum_{n \geq 0} a_n \hbar^n$$

is a series with zero radius of convergence. In general, we cannot uniquely determine a function I on $[0, \varepsilon)$ whose expansion is given by such a series: it always exists (check it!) but in general there is

no canonical choice. However, assume that the exponential generating function of a_n

$$g(\hbar) = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!}$$

is convergent in some neighborhood of 0, analytically continues to $[0,\infty)$, and has at most exponential growth as $\hbar \to \infty$. In this case there is a "canonical" way to construct a smooth function I on $[0,\varepsilon)$ with (asymptotic) Taylor expansion $\widetilde{I}$, called the *Borel summation* of $\widetilde{I}$. Namely, the function I is defined by the formula

$$I(\hbar) = \int_0^\infty g(\hbar u)e^{-u}du = \hbar^{-1}\int_0^\infty g(u)e^{-\frac{u}{\hbar}}du,$$

i.e., $I(\hbar) = \hbar^{-1}(\mathcal{L}g)(\hbar^{-1})$, where $\mathcal{L}$ is the Laplace transform (note that since g grows at most exponentially at infinity, this is well defined for small enough $\hbar > 0$). Note that

$$I(\hbar) = \int_{-\infty}^\infty |v|g(\hbar v^2)e^{-v^2}dv = \hbar^{-\frac{1}{2}}\int_{-\infty}^\infty g_*(\hbar^{\frac{1}{2}}v)e^{-v^2}dv,$$

where $g_*(v) = |v|g(v^2)$. Thus Exercise 2.5 implies that to compute the asymptotic expansion of I, we may replace g by its Taylor polynomials at 0. Hence the identity $\int_0^\infty x^n e^{-x}dx = n!$ implies that I has the Taylor expansion $\widetilde{I}$.

For example, consider the divergent series

$$\widetilde{I} := \sum_{n\geq 0}(-1)^n n!\hbar^n.$$

Then

$$g(\hbar) = \sum_{n\geq 0}(-1)^n\hbar^n = \frac{1}{1+\hbar}.$$

Hence, the Borel summation yields

$$I(\hbar) = \int_0^\infty \frac{e^{-u}}{1+\hbar u}du = \hbar^{-1}e^{\hbar^{-1}}E_1(\hbar^{-1})$$

where $E_1(x) := \int_x^\infty \frac{e^{-u}}{u}du$ is the integral exponential.

Physicists expect that in physically interesting situations perturbation expansions in quantum field theory are Borel summable, and the actual answers are obtained from these expansions by Borel summation. The Borel summability of perturbation series has actually been established in a few nontrivial examples of QFT.

Exercise 2.12. *Show that the function given by* (2.8) *equals the Borel sum of its asymptotic expansion.*

Hint. *The function $g(z)$ in this example is a special case of the hypergeometric function ${}_2F_1$ which does not express in elementary functions. But it satisfies a hypergeometric differential equation. Write down this equation and show that the Laplace transform turns it into another second order linear differential equation, and that the function $I(\hbar)$ given by* (2.8) *satisfies this equation.*

2.6. **Application of steepest descent.** Let us give an application of Theorem 2.6. Consider the integral

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt,\ s > 0.$$

By doing a change of variable $t = sx$, we get

$$\frac{\Gamma(s+1)}{s^{s+1}} = \int_0^\infty x^s e^{-sx} dx = \int_0^\infty e^{-s(x-\log x)} dx.$$

Thus, we can apply Theorem 2.6 for $\hbar = \frac{1}{s}$, $f(x) = x - \log x$, $g(x) = 1$ (of course, the interval $[a, b]$ is now infinite, and the function f blows up on the boundary, but one can easily see that the theorem is still applicable, with the same proof). The function $f(x) = x - \log x$ has a unique critical point on $[0, \infty)$, which is $c = 1$, and we have $f''(c) = 1$. Then we get

$$\Gamma(s+1) \sim s^s e^{-s} \sqrt{2\pi s}(1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots). \tag{2.9}$$

This is the celebrated *Stirling formula*.

Moreover, we can compute the coefficients $a_1, a_2, ...$ using Remark 2.7. Namely,

$$p(x) = \sqrt{2(x - \log(1+x))} = x\sqrt{1 - \tfrac{2x}{3} + \tfrac{x^2}{2} - \ldots} = x - \tfrac{x^2}{3} + \tfrac{7x^3}{36} + ...$$

Thus

$$p^{-1}(z) = z + \tfrac{z^2}{3} + \tfrac{z^3}{36} + ...,$$

hence

$$(p^{-1})'(z) = 1 + \tfrac{2z}{3} + \tfrac{z^2}{12} + ...,$$

So for instance by Remark 2.7 $a_1 = b_2 = \frac{1}{12}$.

Remark 2.13. Another way to compute this asymptotic expansion is to use the Euler product formula for the Gamma function. Differentiating the logarithm of this formula twice, we obtain (for $z > 0$):

$$(\log\Gamma)''(z) = \sum_{n=0}^\infty \frac{1}{(z+n)^2} = \sum_{n=0}^\infty \int_0^\infty t e^{-(z+n)t} dt = \int_0^\infty \frac{t e^{-zt}}{1 - e^{-t}} dt.$$

Recall that the *Bernoulli numbers* are defined by the generating function

$$\sum_{n\geq 0} \frac{B_n t^n}{n!} = \frac{t}{1-e^{-t}},$$

e.g. $B_0 = 1, B_1 = \frac{1}{2}$, $B_{2n+1} = 0$ for $n \geq 1$. Thus we get for $z \to \infty$

$$(\log\Gamma)''(z) \sim \sum_{n\geq 0} B_n z^{-n-1}.$$

Integrating, we get

$$(\log\Gamma)'(z) \sim \log z + C_1 - \sum_{n\geq 1} \frac{B_n}{n} z^{-n},$$

so integrating again and adding $\log z$, we get

$$\log\Gamma(z+1) \sim z\log z - z + C_1 z + \frac{1}{2}\log z + C_2 + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}.$$

From Stirling's formula we have $C_1 = 0, C_2 = \frac{1}{2}\log(2\pi)$, so in the end we get

$$(\log\Gamma)'(z) \sim \log z - \sum_{n\geq 1} \frac{B_n}{n} z^{-n}, \tag{2.10}$$

$$\log\Gamma(z+1) \sim z\log z + \frac{1}{2}\log z + \frac{1}{2}\log(2\pi) + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}. \tag{2.11}$$

So

$$1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots = \exp(\sum_{n\geq 2} \tfrac{B_n}{n(n-1)} s^{-n+1}).$$

In particular, since $B_2 = \frac{1}{6}$, we get $a_1 = \frac{1}{12}$.

Exercise 2.14. *Calculate $\int_0^\pi \sin^n x dx$ for nonnegative integers n using integration by parts. Then apply steepest descent to this integral and discover a formula for π (the so called Wallis formula).*

Exercise 2.15. *The Bessel function $I_0(a)$ is defined by the formula*

$$I_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{a\cos\theta} d\theta.$$

It is an even entire function with Taylor expansion

$$I_0(a) = \sum_{n=0}^{\infty} \frac{a^{2n}}{2^{2n} n!^2}.$$

Use the steepest descent/stationary phase formulas to find the asymptotic expansion of $I_0(a)$ as $a \to +\infty$ and $a \to i\infty$. Compute the first two terms of the expansion (cf. Remark 2.22).

2.7. **Multidimensional versions of steepest descent and stationary phase.** Theorems 2.6,2.8 have multidimensional analogs. To formulate them, let V be a real vector space of dimension d with a fixed volume element dx and $D \subset V$ be a compact region with smooth boundary.[6]

Theorem 2.16. *(Multidimensional steepest descent formula) Let $f, g : D \to \mathbb{R}$ be continuous functions which are smooth in the interior of D. Assume that f achieves global minimum on D at a unique point c, such that c is an interior point and $f''(c) > 0$. Then*

$$\int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I(\hbar), \tag{2.12}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} \frac{g(c)}{\sqrt{\det f''(c)}}.$$

Theorem 2.17. *(Multidimensional stationary phase formula) Let $f, g : D \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point c in D, such that c is an interior point and $\det f''(c) \neq 0$, and g has vanishing derivatives of all orders on ∂D. Then*

$$\int_D g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar), \tag{2.13}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} e^{\frac{\pi i \sigma}{4}} \frac{g(c)}{\sqrt{|\det f''(c)|}},$$

where σ is the signature of the symmetric bilinear form $f''(c)$.

2.8. **Morse lemma.** For the proof of these theorems it is convenient to use a fundamental result in multivariable calculus called *the Morse lemma*. This lemma easily follows by induction in dimension from the following theorem.

[6]The condition of smooth boundary is introduced for simplicity of exposition only and is not essential. The same results and proofs apply with trivial modifications to more general regions, e.g. those whose boundary is only piecewise smooth in an appropriate sense.

Theorem 2.18. *(Separation of variables) Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a non-degenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system near 0 (possibly defined in a smaller ball) in which*

$$f(x_1, ..., x_n) = f(x_1, ..., x_{d-1}) \pm x_d^2.$$

Proof. By making a linear change of variables, we can assume that the quadratic part of f has the form $Q(y) \pm u^2$, where $y := (x_1, ..., x_{d-1})$, $u := x_d$. Consider the hypersurface S defined by the equation

$$\partial_u f(y, u) = 0.$$

The linear part of $\partial_u f(y, u)$ is $\pm 2u$, so by the implicit function theorem there is a change of coordinates F near 0 (with $dF(0) = 1$) in which u is replaced by $v := \pm\frac{1}{2}\partial_u f(y, u)$ and y is kept unchanged; so $u = g(y, v)$ for some function g with $(\partial_v g)(0, 0) \neq 0$. Let

$$f_*(y, v) := f(y, u) = f(y, g(y, u)).$$

Then by the chain rule

$$\partial_v f_*(y, v) = \partial_u f_*(y, v)\frac{\partial u}{\partial v} = \partial_u f(y, u)\frac{\partial u}{\partial v} = \pm 2v\partial_v g(y, v).$$

Thus the hypersurface S in the new coordinates is defined by the equation $v = 0$. So we may assume without loss of generality that S is given by the equation $u = 0$ to start with. Then $(\partial_u f)(y, 0) = 0$, so

$$f(y, u) - f(y, 0) = h(y, u)u^2,$$

where h is a smooth function in B with $h(0, 0) = \pm 1$. By replacing u with $\widetilde{u} := \sqrt{|h(y, u)|}u$ and keeping y unchanged, we may assume that $h = \pm 1$. Then

$$f(u, y) = f(0, y) \pm u^2,$$

as claimed. □

Corollary 2.19. *(Morse lemma) Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a non-degenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system $(x_1, ..., x_d)$ near 0 (possibly defined in a smaller ball) in which*

$$f = x_1^2 + ... + x_m^2 - x_{m+1}^2 - ... - x_d^2.$$

In other words, near a non-degenerate critical point a smooth function is equivalent by a change of coordinates to its quadratic part.

Proof. As mentioned above, this follows easily from Theorem 2.18 by induction in dimension. □

Exercise 2.20. *Let f be a smooth function on $\mathbb{R}^2$ which is a cubic polynomial in x:*

$$f(x,y) = a(y) + b(y)x + c(y)x^2 + d(y)x^3.$$

Assume that $a(0) = a'(0) = 0$, $b(0) = b'(0) = 0$, $a''(0) = c(0) = 2$. Find explicitly local coordinates $u = u(x,y), v = v(x,y)$ near 0 in which $f(x,y) = u^2 + v^2$.

2.9. **Proof of the multidimensional steepest descent and stationary phase formulas.** The proofs of the multidimensional steepest descent and stationary phase formulas are parallel to the proofs of their one-dimensional versions, using the Morse lemma. Namely, the Morse lemma allows us to assume without loss of generality that f is quadratic near the critical point. After this, the proof of the steepest descent formula is identical to the 1-variable case. The same applies to the stationary phase formula, using the following multivariable analog of the Riemann lemma.

Lemma 2.21. *Let $f, g : D \to \mathbb{R}$ be smooth functions such that all derivatives of g vanish on ∂D and df does not vanish anywhere on the support of g. Then the function*

$$I(\hbar) := \int_D g(x) e^{\frac{if(x)}{\hbar}} dx$$

extends to a smooth function on $[0, \infty)$ and has rapidly decaying derivatives of all orders as $\hbar \to 0$.

Proof. Since df does not vanish on supp$g$, we can cover supp$g$ by local charts U_i in which $f(x)$ is the last coordinate x_d. By compactness this cover can be chosen finite. By using a partition of unity $\{h_i\}$ on supp$g$ subordinate to this cover and replacing g with $h_i g$, we may assume without loss of generality that g is supported on a single chart. Then changing variables, we may also assume that $f(x) = x_d$. Then integrating out the variables $x_1, ..., x_{d-1}$, we reduce to the 1-dimensional case covered by Lemma 2.10. □

Remark 2.22. It is clear from the proof of the stationary phase formula that it extends to the case when f may have several critical points but all of them are interior and non-degenerate. In this case the asymptotic expansions coming from different critical points are simply added together. The same applies to the steepest descent formula if the global minimum is attained at several points all of which are interior and non-degenerate.

3. Feynman calculus

3.1. **Wick's theorem.** Let V be a real vector space of dimension d with volume element dx. Let $S(x)$ be a smooth function on a compact region $D \subset V$ with smooth boundary which attains its minimum at a unique point $c \in D$ in the interior of D, and let g be any smooth function on D. In the previous section we proved the steepest descent formula which implies that the function

$$I(\hbar) = \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D g(x) e^{-\frac{S(x)}{\hbar}} dx$$

admits an asymptotic power series expansion in $\hbar$:

$$I(\hbar) = a_0 + a_1\hbar + \cdots + a_m\hbar^m + \cdots \tag{3.1}$$

Our main question now will be: how to compute the coefficients a_i?

Our proof of the steepest descent formula shows that although the problem of computing $I(\hbar)$ is transcendental, the problem of computing the coefficients a_i is, in fact, purely algebraic, and involves only differentiation of the functions S and g at the point c. Indeed, recalling the proof of equation (3.1), we see that the calculation of a_i reduces to calculation of integrals of the form

$$\int_V P(x) e^{-\frac{B(x,x)}{2}} dx,$$

where P is a polynomial and B is a positive definite bilinear form (in fact, $B(v,u) = (\partial_v \partial_u S)(c)$). But such integrals can be exactly evaluated. Namely, it is sufficient to consider the case when P is a product of linear functions, in which case the answer is given by the following elementary formula, known to physicists as *Wick's theorem.*

For a positive integer k, consider the set $\{1, \ldots, 2k\}$. By a *matching* σ on this set we will mean its partition into k disjoint two-element subsets (pairs). A matching can be visualized by drawing $2k$ points and connecting two points with an edge if they belong to the same pair (see Fig. 1). This will give k edges which are not connected to each other.

Let us denote the set of matchings on a set T by $\Pi(T)$ and the set $\Pi(\{1, \ldots, 2k\})$ by Π_k. It is clear that $|\Pi_k| = \frac{(2k)!}{2^k \cdot k!} = (2k-1)!!$. For any $\sigma \in \Pi_k$, we can think of σ as a permutation of $\{1, \ldots, 2k\}$, such that $\sigma^2 = 1$ and σ has no fixed points. Namely, σ maps any element i to the second element $\sigma(i)$ of the pair containing i.

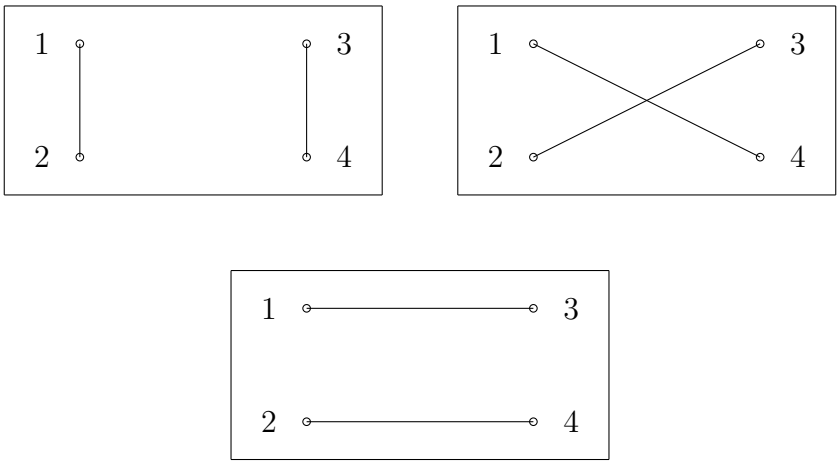


FIGURE 1. Matchings of the set $\{1, 2, 3, 4\}$

Theorem 3.1. *(Wick's theorem) Let B^{-1} denote the inverse form to B on V^*, and $\ell_1, \dots, \ell_N \in V^*$. Then, if N is even, we have*

$$\int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\sigma \in \Pi_{N/2}} \prod_{i \in \{1,\dots,N\}/\sigma} B^{-1}(\ell_i, \ell_{\sigma(i)})$$

If N is odd, the integral is zero.

Proof. If N is odd, the statement is obvious, because the integrand is an odd function. So consider the even case $N = 2k$. Since both sides of the equation are symmetric polylinear forms in $\ell_1, \dots, \ell_N$, it suffices to prove the result when $\ell_1 = \dots = \ell_N = \ell$. Further, it is clear that the formula in question is stable under linear changes of variable, so we can choose a coordinate system in such a way that $B(x, x) = x_1^2 + \dots + x_d^2$, and $\ell(x) = x_1$. Therefore, it is sufficient to assume that $d = 1$ and $\ell(x) = x$. In this case, the theorem says that

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}} dx = (2\pi)^{\frac{1}{2}} (2k-1)!!,$$

which is formula (2.6). □

Example 3.2. We have

$$\int_V \ell_1(x)\ell_2(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} B^{-1}(\ell_1, \ell_2),$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x) e^{-\frac{B(x,x)}{2}} dx =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} (B^{-1}(\ell_1, \ell_2)B^{-1}(\ell_3, \ell_4) + B^{-1}(\ell_1, \ell_3)B^{-1}(\ell_2, \ell_4) + B^{-1}(\ell_1, \ell_4)B^{-1}(\ell_2, \ell_3)).$$

Wick's theorem shows that the problem of computing a_i is of combinatorial nature. In fact, the central role in this computation is played by certain finite graphs, which are called *Feynman diagrams*. They are the main subject of the remainder of this section.

3.2. **Feynman diagrams and Feynman's theorem.** We come back to the problem of computing the coefficients a_i. Since each particular a_i depends only on a finite number of derivatives of g at c, it suffices to assume that g is a polynomial, or, more specifically, a product of linear functions: $g = \ell_1 \dots \ell_N$, $\ell_i \in V^*$. Thus, it suffices to be able to compute the series expansion of the integral

$$\langle \ell_1 \dots \ell_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D \ell_1(x) \dots \ell_N(x) e^{-\frac{S(x)}{\hbar}} dx. \tag{3.2}$$

Without loss of generality we may assume that $c = 0$ and $S(c) = 0$. Then the (asymptotic) Taylor expansion of S at c is

$$S(x) = \frac{B(x,x)}{2} - \sum_{i \geq 3} \frac{B_i(x, \dots, x)}{i!},$$

where $B_i := d^i f(0)$. Therefore, regarding the left hand side of (3.2) as a power series in $\hbar$ and making a change of variable $x \mapsto \hbar^{\frac{1}{2}} x$ (like in the last section), we get

$$\langle \ell_1 \dots \ell_N \rangle = \hbar^{\frac{N}{2}} \int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2} + \sum_{i \geq 3} \hbar^{\frac{i}{2}-1} \frac{B_i(x,\dots,x)}{i!}} dx.$$

Note that this is only an identity of asymptotic expansions in $\hbar$, as we ignored the rapidly decaying error which comes from replacing the region D by the whole space. But it implies in particular that $\langle \ell_1 \dots \ell_N \rangle = O(\hbar^{\lceil \frac{N}{2} \rceil})$ as $\hbar \to 0$ (as the expansion contains only integer powers of $\hbar$).

The theorem below, due to Feynman, gives the value of this integral in terms of Feynman diagrams. This theorem is easy to prove but is central in quantum field theory, and will be one of our main theorems. Before formulating Feynman's theorem, let us introduce some notation.

Let $G_{\geq 3}(N)$ be the set of isomorphism classes of graphs with N 1-valent "external" vertices, labeled by $1, \dots, N$, and a finite number of unlabeled "internal" vertices, of any valency ≥ 3. Note that here and below graphs are allowed to have multiple edges between two vertices and loops from a vertex to itself (see Fig. 2).

For each graph $\Gamma \in G_{\geq 3}(N)$, we define the *Feynman amplitude* of Γ as follows.

1. Put the covector ℓ_j at the j-th external vertex.

2. Put the tensor B_i at each i-valent internal vertex.

3. Take the contraction of the tensors along edges of Γ, using the bilinear form B^{-1}. This will produce a number, called the *(Feynman) amplitude* of Γ and denoted $F_\Gamma(\ell_1, \dots, \ell_N)$.

Remark 3.3. If Γ is not connected, then F_Γ is defined to be the product of numbers obtained from the connected components. Also, the amplitude of the empty diagram is defined to be 1.

Example 3.4. Let

$$B_3 := \sum_i b_i^{13} \otimes b_i^{23} \otimes b_i^{33}, \ B_4 := \sum_j b_j^{14} \otimes b_j^{24} \otimes b_j^{34} \otimes b_j^{44},$$

where $b_i^{jk} \in V^*$. Then for the graph Γ_3 in Fig. 2 the amplitude equals

$$F_{\Gamma_3}(\ell_1, \ell_2) =$$

$$\sum_i B^{-1}(\ell_1, b_i^{13}) B^{-1}(b_i^{23}, b_i^{33}) \cdot \sum_{i,j} B^{-1}(b_i^{13}, b_j^{14}) B^{-1}(b_i^{23}, b_j^{24}) B^{-1}(b_i^{33}, b_j^{34}) B^{-1}(b_j^{44}, \ell_2).$$

Theorem 3.5. *(Feynman) One has*

$$\langle \ell_1 \dots \ell_N \rangle = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N), \tag{3.3}$$

where $b(\Gamma)$ is the number of edges minus the number of internal vertices of Γ.

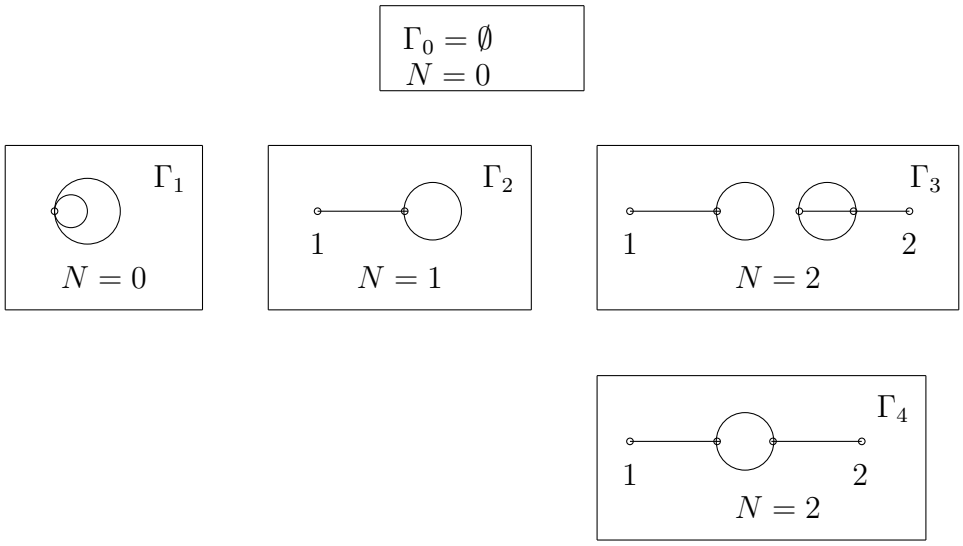


FIGURE 2. Examples of elements of $G_{\geq 3}(N)$.

Here $\mathrm{Aut}(\Gamma)$ denotes the group of automorphisms of Γ, and by an automorphism of Γ we mean a permutation of vertices **and** edges (possibly flipping the self-loops) which fixes each external vertex and preserves the graph structure, see Fig. 3. Thus there can exist nontrivial automorphisms which act trivially on the set of vertices and even ones also acting trivially on the set of edges. For example, there is an automorphism of Γ_4 that flips the upper and lower arc, and an automorphism of Γ_2 that flips the self-loop.

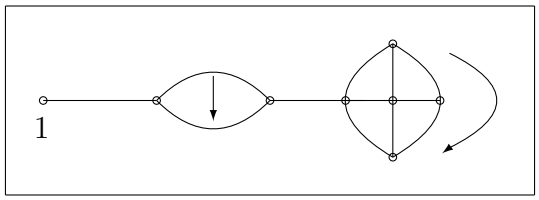


FIGURE 3. An automorphism of a graph

Remark 3.6. 1. Note that this sum is infinite, but $\hbar$-adically convergent.

2. Theorem 3.5 is a generalization of Wick's theorem: the latter is obtained if $S(x) = \frac{B(x,x)}{2}$. Indeed, in this case graphs which give nonzero amplitudes do not have internal vertices, and thus reduce to graphs corresponding to matchings σ.

Let us now make some comments about the terminology. In quantum field theory, the function $\langle \ell_1 \dots \ell_N \rangle$ is called the *N-point correlation function*, and graphs Γ are called *Feynman diagrams*. The form B^{-1} which is put on the edges is called the *propagator.*. The cubic and higher terms $\frac{B_i}{i!}$ in the expansion of the function S are called *interaction terms*, since such terms (in the action functional) describe interaction between particles. The situation in which S is quadratic (i.e., there is no interaction) is called a *free theory*; i.e. for the free theory the correlation functions are determined by Wick's formula.

Remark 3.7. Sometimes it is convenient to consider normalized correlation functions

$$\langle \ell_1 \dots \ell_N \rangle_{\mathrm{norm}} := \frac{\langle \ell_1 \dots \ell_N \rangle}{\langle \emptyset \rangle}$$

where $\langle\emptyset\rangle$ denotes the integral without insertions. Feynman's theorem implies that they are given by the formula

$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\text{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $G^*_{\geq 3}(N)$ is the subset of all graphs in $G_{\geq 3}(N)$ which have no components without external vertices.

3.3. **A weighted version of Feynman's theorem.** Before proving Theorem 3.5, we would like to slightly modify and generalize it. Namely, in quantum field theory it is often useful to consider an interacting theory as a deformation of a free theory. This means that $S(x) = \frac{B(x,x)}{2} + \widetilde{S}(x)$, where $\widetilde{S}(x)$ is a perturbation

$$\widetilde{S}(x) := -\sum_{i \geq 0} g_i \frac{B_i(x, \dots, x)}{i!}$$

in which $g_r, r \geq 0$ are (formal) parameters. One benefit of these parameters is that they will allow us to group the amplitudes of Feynman diagrams in the sum (3.3) by the numbers of vertices of each valency. Namely, consider the *partition function*

$$Z = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx$$

as a series in g_i. Let $\mathbf{n} = (n_0, n_1, n_2, \dots)$ be a sequence of nonnegative integers, almost all zero. Let $G(\mathbf{n})$ denote the set of isomorphism classes of graphs with n_0 0-valent vertices, n_1 1-valent vertices, n_2 2-valent vertices, etc. (thus, now we are considering graphs without external vertices). For $\Gamma \in G(\mathbf{n})$, let F_Γ is the amplitude of Γ defined as before. Thus

$$F_\Gamma = \prod_i g_i^{n_i} \cdot \mathbb{F}_\Gamma,$$

where $\mathbb{F}_\Gamma$ is the Feynman amplitude computed without the factors g_j.

Theorem 3.8. *One has*

$$Z = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\text{Aut}(\Gamma)|} F_\Gamma =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\text{Aut}(\Gamma)|},$$

where $b(\Gamma) = \sum_i n_i(\frac{i}{2} - 1)$ *is the number of edges minus the number of vertices of* Γ.

Note that we may view Z as an element of the algebra

$$\mathbb{C}[g_0\hbar^{-\frac{3}{2}}, g_1\hbar^{-1}, g_2\hbar^{-\frac{1}{2}}; g_j, j \geq 3][[\hbar^{\frac{1}{2}}]],$$

i.e., it can be specialized to numerical values of

$$g_0\hbar^{-\frac{3}{2}}, g_1\hbar^{-1}, g_2\hbar^{-\frac{1}{2}}, g_3, g_4, ...,$$

giving an element of $\mathbb{C}[[\hbar^{\frac{1}{2}}]]$. Also Z can be specialized to $\hbar = 1$, giving an element of $\mathbb{C}[[g_j, j \geq 0]]$, and the theorem is, in fact, equivalent to this specialization. Still we choose to keep $\hbar$ to be able to take the classical limit $\hbar \to 0$.

We will prove Theorem 3.8 in the next subsection. Meanwhile, let us show that Theorem 3.5 is in fact a special case of Theorem 3.8. Indeed, because of symmetry of the correlation functions with respect to $\ell_1, \dots, \ell_N$, it is sufficient to consider the case $\ell_1 = \dots = \ell_N = \ell$. In this case, denote the correlation function $\langle \ell^N \rangle$ (expectation value of ℓ^N). Clearly, to compute $\langle \ell^N \rangle$ for all N, it is sufficient to compute the generating function

$$\langle e^\ell \rangle = \hbar^{-\frac{d}{2}} \int_V e^{\ell(x) - \frac{S(x)}{\hbar}} dx := \sum_{N=0}^{\infty} \frac{\langle \ell^N \rangle}{N!},$$

which up to scaling and multiplication of ℓ by i is the Fourier transform of the Feynman density $e^{-\frac{S(x)}{\hbar}} dx$. But this expectation value is exactly the one given by Theorem 3.8 for $g_i = 1$, $i \geq 3$, $g_0 = g_2 = 0$, $g_1 = \hbar$, $B_1 = \ell$, $B_0 = 0$, $B_2 = 0$. Thus, Theorem 3.8 implies Theorem 3.5 (the factor $N!$ in the denominator is accounted for by the fact that in Theorem 3.8 we consider unlabeled, rather than labeled, 1-valent vertices).

3.4. **Proof of Feynman's theorem.** Now we will prove Theorem 3.8. Let us make a change of variable $y = \hbar^{-\frac{1}{2}} x$. Expanding the exponential in a Taylor series, we obtain

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

where

$$Z_{\mathbf{n}} = \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} (\hbar^{\frac{i}{2}-1} B_i(y, \dots, y))^{n_i} dy.$$

Writing B_i as a sum of products of linear functions, and using Wick's theorem, we find that the value of the integral for each $\mathbf{n}$ can be expressed combinatorially as follows.

1. Attach to each factor B_i a "flower" — a vertex with i outgoing edges (see Fig. 4).

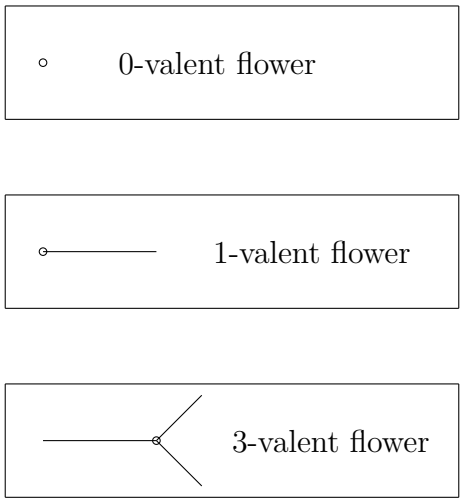


FIGURE 4.

2. Consider the set $T_{\mathbf{n}}$ of ends of these outgoing edges (see Fig. 5), and for any matching σ of this set, consider the corresponding contraction of the tensors B_i using the form B^{-1}. This will produce a scalar $\mathbb{F}(\sigma)$.

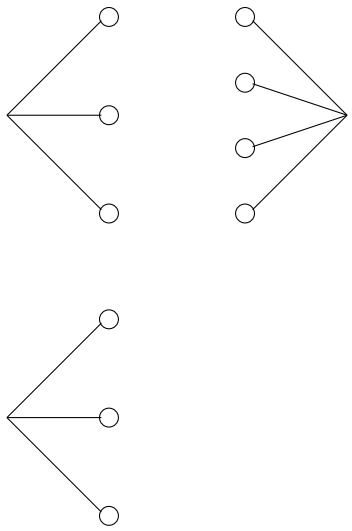


FIGURE 5. The set $T_{\mathbf{n}}$ for $\mathbf{n} = (0, 0, 0, 2, 1, 0, 0, \ldots)$ (the set of white circles)

3. The integral $Z_{\mathbf{n}}$ is given by

$$Z_{\mathbf{n}} = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma). \tag{3.4}$$

Now, recall that matchings on a set can be visualized by drawing its elements as points and connecting them with edges. If we do this with the set $T_{\mathbf{n}}$, all ends of outgoing edges will become connected with each other in some way, i.e. we will obtain a certain (unoriented) graph $\Gamma = \Gamma_\sigma$ (see Fig. 6). Moreover, it is easy to see that the scalar $\mathbb{F}(\sigma)$ is nothing but the amplitude $\mathbb{F}_\Gamma$.

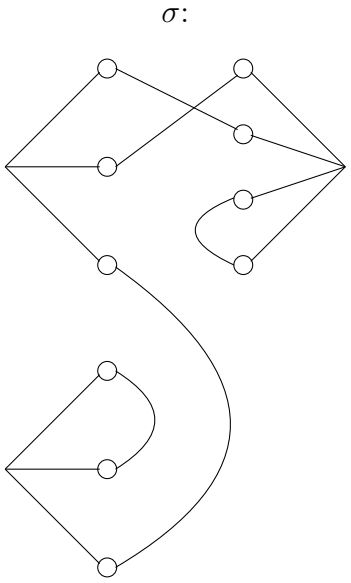


FIGURE 6. A matching σ of $T_{\mathbf{n}}$ and the corresponding graph Γ.

It is clear that any graph Γ with n_i i-valent vertices for each i can be obtained in this way. However, the same graph can be obtained in many different ways, so if we want to collect identical terms in the sum over σ, and turn it into a sum over Γ, we must find the number of σ which yield a given Γ.

For this purpose, we will consider the group $\mathbb{G}_{\mathbf{n}}$ of permutations of $T_{\mathbf{n}}$, which preserves "flowers" (i.e. endpoints of any two edges outgoing from the same flower end up again in the same flower). This group involves

1) permutations of "flowers" with a given valency;

2) permutation of the i edges inside each i-valent "flower".

More precisely, the group $\mathbb{G}_{\mathbf{n}}$ is the semidirect product of symmetric groups

$$\mathbb{G}_{\mathbf{n}} = \prod_i (S_{n_i} \ltimes S_i^{n_i}).$$

Note that $|\mathbb{G}_{\mathbf{n}}| = \prod_i i!^{n_i} n_i!$, which is the product of the numbers in the denominator of formula (3.4).

The group $\mathbb{G}_{\mathbf{n}}$ acts on the set $\Pi(T_{\mathbf{n}})$ of all matchings σ of $T_{\mathbf{n}}$. Moreover, it acts transitively on the set $\Pi_\Gamma(T_{\mathbf{n}})$ of matchings of $T_{\mathbf{n}}$ which yield a given graph Γ. Furthermore, it is easy to see that the stabilizer of a given matching is $\mathrm{Aut}(\Gamma)$. Thus, the number of matchings giving Γ is

$$N_\Gamma = \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|}.$$

Hence,

$$\sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma) = \sum_\Gamma \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|} \mathbb{F}_\Gamma.$$

Finally, note that the exponent of $\hbar$ in equation (3.4) is $\sum_i n_i(\frac{i}{2} - 1)$, which is the number of edges of Γ minus the number of vertices, i.e. $b(\Gamma)$. Substituting this into (3.4), we get the result.

Example 3.9. Let $d = 1$, $V = \mathbb{R}$, $g_i = g$, $B_i = z^i$ for all $i \geq 0$ (where z is a formal variable), $\hbar = 1$. Then we find the asymptotic expansion

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + g e^{zx}} = \sum_{n \geq 0} g^n \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|},$$

where $G(n,k)$ is the set of isomorphism classes of graphs with n vertices and k edges.[7] Expanding the left hand side, we get

$$\sum_k \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{\frac{z^2 n^2}{2}}}{n!},$$

and hence

$$\sum_{\Gamma \in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!}.$$

Exercise 3.10. *Check this by direct combinatorics.*

[7] This integral converges for $g < 0$, $z \in \mathbb{R}$, but this is not important for us here, since we consider the integral formally.

3.5. **Sum over connected diagrams.** Now we will show that the logarithm of the partition function Z is also given by summation over diagrams, but with only connected diagrams taken into account. This significantly simplifies the analysis of Z in the first few orders of perturbation theory, since the number of connected diagrams with a given number of vertices and edges is significantly smaller than the number of all diagrams.

Theorem 3.11. *Let $Z_0 = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}$. Then one has*

$$\log \frac{Z}{Z_0} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_c(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

where $G_c(\mathbf{n})$ is the set of connected graphs in $G(\mathbf{n})$.[8]

Proof. For any graphs Γ_1, Γ_2, let $\Gamma_1\Gamma_2$ stand for the disjoint union of Γ_1 and Γ_2, and for any graph Γ let Γ^n denote the disjoint union of n copies of Γ. Then every graph can be uniquely written as $\Gamma_1^{k_1} \dots \Gamma_l^{k_l}$, where Γ_j are connected non-isomorphic graphs. Moreover, it is clear that $\mathbb{F}_{\Gamma_1\Gamma_2} = \mathbb{F}_{\Gamma_1}\mathbb{F}_{\Gamma_2}$, $b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$, and

$$|\mathrm{Aut}(\Gamma_1^{k_1} \dots \Gamma_l^{k_l})| = \prod_j |\mathrm{Aut}(\Gamma_j)|^{k_j} k_j!.$$

Thus, exponentiating the equation of Theorem 3.11, and using the above facts together with the Taylor series for the function e^x, we arrive at Theorem 3.8. As Theorem 3.8 has been proved, so is Theorem 3.11 □

3.6. **The loop expansion.** Note that since summation in Theorem 3.11 is over connected Feynman diagrams, the number $b(\Gamma)$ is the number of loops in Γ minus 1. In particular, the lowest coefficient in $\hbar$ is that of $\hbar^{-1}$, and it is the sum over all trees; the next coefficient is to $\hbar^0$, and it is the sum over all diagrams with one loop (cycle); the next coefficient to $\hbar$ is the sum over two-loop diagrams, and so on. Therefore, physicists refer to the expansion of Theorem 3.11 as the *loop expansion*.

Let us study the two most singular terms in this expansion (with respect to $\hbar$), i.e. the terms given by the sum over trees and 1-loop graphs.

Let x_0 be the critical point of the function S. It exists and is unique, since g_i are assumed to be formal parameters. Let $G^{(j)}(\mathbf{n})$ denote the

[8]We define a connected graph as a graph with exactly one connected component. So the empty graph, which has zero connected components, is not considered connected.

set of classes of graphs in $G_c(\mathbf{n})$ with j loops. Let

$$\left(\log \frac{Z}{Z_0}\right)_j := \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G^{(j)}(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

so that

$$\log \frac{Z}{Z_0} = \sum_{j=0}^{\infty} \left(\log \frac{Z}{Z_0}\right)_j \hbar^{j-1}.$$

Theorem 3.12.

$$\left(\log \frac{Z}{Z_0}\right)_0 = -S(x_0), \tag{3.5}$$

and

$$\left(\log \frac{Z}{Z_0}\right)_1 = \frac{1}{2} \log \frac{\det B}{\det S''(x_0)}. \tag{3.6}$$

Proof. First note that the statement is purely combinatorial. This means, in particular, that it is sufficient to check that the statement yields the correct asymptotic expansion of the right hand sides of equations (3.5),(3.6) in the case when S is a polynomial with real coefficients of the form $\frac{B(x,x)}{2} - \sum_{i=0}^{N} g_i \frac{B_i(x,\dots,x)}{i!}$ and $\hbar > 0$. To do so, let $Z := \hbar^{-\frac{d}{2}} \int_{\mathbf{B}} e^{-\frac{S(x)}{\hbar}} dx$, where $\mathbf{B}$ is a ball centered at 0. For sufficiently small g_i, the function S has a unique global minimum point x_0 in $\mathbf{B}$, which is non-degenerate. Thus, by the steepest descent formula, we have

$$\frac{Z}{Z_0} = e^{-\frac{S(x_0)}{\hbar}} I(\hbar),$$

where $I(\hbar) \sim \sqrt{\frac{\det B}{\det S''(x_0)}}(1 + a_1\hbar + a_2\hbar^2 + \cdots)$ (asymptotically). Thus,

$$\log \frac{Z}{Z_0} = -S(x_0)\hbar^{-1} + \frac{1}{2} \log \frac{\det B}{\det S''(x_0)} + O(\hbar).$$

This implies the result. □

Physicists call the expression $(\log \frac{Z}{Z_0})_0$ the *classical (or tree) approximation* to the quantum mechanical quantity $\hbar \log \frac{Z}{Z_0}$, and the sum $(\log \frac{Z}{Z_0})_0 + \hbar(\log \frac{Z}{Z_0})_1$ the *one-loop approximation*. Similarly one defines higher loop approximations. Note that the classical approximation is obtained by finding the critical point and value of the classical action $S(x)$, which in the classical mechanics and field theory situation corresponds to solving the classical equations of motion.

3.7. **Nonlinear equations and trees.** As we have noted, Theorem 3.12 does not involve integrals and is purely combinatorial. Therefore, there should exist a purely combinatorial proof of this theorem. Such a proof indeed exists. Here we will give a combinatorial proof of the first statement of the Theorem (formula (3.5)).

Consider the equation $S'(x) = 0$, defining the critical point x_0. This equation can be written as $x = \beta(x)$, where

$$\beta(x) := \sum_{i \geq 1} g_i \frac{B^{-1} B_i(x, \dots, x, -)}{(i-1)!},$$

where $B^{-1} : V^* \to V$ is the operator corresponding to the form B^{-1}.

In the sense of power series norm, β is a contracting mapping. Thus, $x_0 = \lim_{N \to \infty} \beta^N(x)$, for any initial vector, for example $0 \in V$. In other words, we will obtain x_0 if we keep substituting the series $\beta(x)$ into itself. This leads to summation over trees (explain why!). More precisely, we get the following expression for x_0:

$$x_0 = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G^{(0)}(\mathbf{n}, 1)} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

where $G^{(0)}(\mathbf{n}, 1)$ is the set of trees with one external vertex and n_i internal vertices of degree i. Now, since $S(x) = \frac{B(x,x)}{2} - \sum_i g_i \frac{B_i(x,\dots,x)}{i!}$, the expression $-S(x_0)$ equals the sum of expressions $\prod_i g_i^{n_i} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|}$ over all trees (without external vertices). Indeed, the term $\frac{B(x_0,x_0)}{2}$ corresponds to gluing two trees with external vertices (identifying the two external vertices, so that they disappear); so it corresponds to summing over trees with a marked edge, i.e. counting each tree as many times as it has edges. On the other hand, the term $g_i \frac{B_i(x_0,\dots,x_0)}{i!}$ corresponds to gluing i trees with external vertices together at these vertices (making a tree with a marked vertex). So $\sum_i g_i \frac{B_i(x_0,\dots,x_0)}{i!}$ corresponds to summing over trees with a marked vertex, i.e. counting each tree as many times as it has vertices. But the number of vertices of a tree exceeds the number of edges by 1. Thus, the difference $-S(x_0)$ of the above two contributions corresponds to summing over trees, counting each exactly once. This implies formula (3.5).

3.8. **The case $d = 1$.** In the case $d = 1$ we can compute the tree sum $-S(x_0)$ even more explicitly. Namely, let

$$S(x) := \frac{x^2}{2} - gh(x)$$

where $h(x) = \sum_{n\geq 0} c_n x^n$ with $c_1 \neq 0$. Then x_0 is the solution of the equation $x = gh'(x)$, i.e., $x_0 = f(g)$ where $x = f(y)$ is the inverse function to $y = \frac{x}{h'(x)}$. So the tree approximation takes the form $-S(x_0) = F(g)$ where

$$F(g) = -\frac{f(g)^2}{2} + gh(f(g)).$$

Thus

$$F'(g) = -f(g)f'(g) + h(f(g)) + gh'(f(g))f'(g).$$

But $h'(f(g)) = \frac{f(g)}{g}$, so the first and third summands cancel and we get

$$F'(g) = h(f(g)),$$

hence

$$-S(x_0) = \int_0^g h(f(a))da. \tag{3.7}$$

3.9. **Counting trees and Cayley's theorem.** In this section we will apply Theorem 3.12 to tree counting problems, in particular will prove a classical theorem due to Cayley that the number of labeled trees with n vertices is n^{n-2}.

We consider essentially the same situation as we considered above in Example 3.9: $d = 1$, $B_i = 1$, $g_i = g$. Thus, we have $S(x) = \frac{x^2}{2} - ge^x$. By Theorem 3.12, we have

$$\sum_{n\geq 0} g^n \sum_{\Gamma\in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = -S(x_0),$$

where $T(n)$ is the set of isomorphism classes of trees with n vertices, and x_0 is the root of the equation $S'(x) = 0$, i.e. $x = ge^x$.

In other words, let $x = f(y)$ be the function inverse to the function $y = xe^{-x}$ near $x = 0$, then $x_0 = f(g)$. The function $f(y)$ is related to (the principal branch of) the *Lambert function* $W(y)$ by the formula $f(y) = -W(-y)$. By (3.7)

$$-S(x_0) = \int_0^g e^{f(a)}da = \int_0^g \frac{f(a)}{a}da.$$

Thus it remains to find the Taylor expansion of f. This expansion is given by the following classical result.

Proposition 3.13. *One has*

$$f(g) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} g^n.$$

Proof. Let $f(g) = \sum_{n\geq 1} a_n g^n$. Then

$$a_n = \frac{1}{2\pi i}\oint \frac{f(g)}{g^{n+1}}dg = \frac{1}{2\pi i}\oint \frac{x}{(xe^{-x})^{n+1}}d(xe^{-x}) =$$

$$\frac{1}{2\pi i}\oint e^{nx}\frac{1-x}{x^n}dx = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}.$$

□

So we get

$$-S(x_0) = \int_0^g \frac{f(a)}{a}da = \sum_{n\geq 1}\frac{n^{n-2}}{n!}g^n.$$

This shows that

$$\sum_{\Gamma\in T(n)}\frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}.$$

But each isomorphism class of unlabeled trees with n vertices has $\frac{n!}{|\mathrm{Aut}(\Gamma)|}$ nonisomorphic labelings. Thus we obtain

Corollary 3.14. *(A. Cayley) The number of labeled trees with n vertices is n^{n-2}.*

3.10. **Counting trees with conditions.** In a similar way we can count labeled trees with conditions on vertices. For example, let us compute the number of labeled trivalent trees with m vertices (i.e. trees that have only 1-valent and 3-valent vertices). Clearly, $m = 2k$, otherwise there is no such trees. The relevant action functional is

$$S(x) = \tfrac{x^2}{2} - g(x + \tfrac{x^3}{6}).$$

Then the critical point x_0 is obtained from the equation

$$x = g(1 + \tfrac{x^2}{2}),$$

which yields

$$x_0 = \frac{1-\sqrt{1-2g^2}}{g}.$$

Thus, by (3.7) the tree sum equals

$$-S(x_0) = \int_0^g \left(\tfrac{1-\sqrt{1-2a^2}}{a} + \tfrac{(1-\sqrt{1-2a^2})^3}{6a^3}\right)da =$$

$$\frac{2}{3}\int_0^g \tfrac{1-(1+a^2)\sqrt{1-2a^2}}{a^3}da = \frac{(1-2g^2)^{\frac{3}{2}} - (1-3g^2)}{3g^2}.$$

Expanding this in a Taylor series, we find

$$-S(x_0) = \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{(n+1)!}g^{2n}.$$

Hence, we get

Corollary 3.15. *The number N_k of trivalent labeled trees with $2n$ vertices is $(2k-3)!!\frac{(2k)!}{(k+1)!}$.*

For example, $N_1 = 1$ (a single edge), $N_2 = 4$ (a single tree with 4! labelings modulo a group of order 6), $N_3 = 90$ (a single tree with 6! labelings modulo a group of order 8), etc.

3.11. **Counting oriented trees.** Feynman calculus can be used to count not only non-oriented, but also oriented graphs. For example, suppose we want to count labeled oriented trees, whose vertices are either sources or sinks (see Fig. 7). In this case, it is easy to see (check it!) that the relevant integration problem is in two dimensions, with the action $S = xy - be^x - ae^y$ (the form xy is not positive definite, but this is immaterial since our computations are purely formal). So the critical point is found from the equations

$$xe^{-y} = a,\ ye^{-x} = b.$$

Like before, look for a solution $(x, y) = (x_0, y_0)$ in the form

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq} a^p b^q,\ y = b + \sum_{p\geq 1, q\geq 1} d_{pq} a^p b^q.$$

A calculation with residues similar to the one we did for unoriented trees yields

$$c_{pq} = \frac{1}{(2\pi i)^2} \oint\oint \frac{x}{a^{p+1}b^{q+1}} da \wedge db =$$

$$\frac{1}{(2\pi i)^2} \oint\oint \frac{e^{qx+py}}{x^p y^{q+1}} (1 - xy) dx \wedge dy = \frac{q^{p-1}p^{q-1}}{(p-1)!q!}.$$

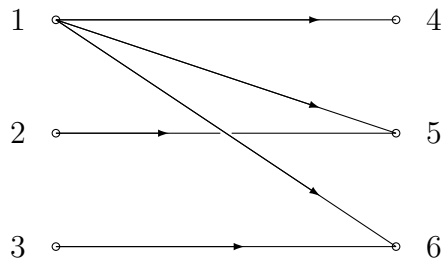


FIGURE 7. A labeled oriented tree with 3 sources and 3 sinks.

Similarly, $d_{pq} = \frac{q^{p-1}p^{q-1}}{p!(q-1)!}$. Now, similarly to the unoriented case, we find that $-a\partial_a S(x,y) = x$, $-b\partial_b S(x,y) = y$, so

$$-S(x,y) = b + \int_0^a \frac{x}{u}du = a + b + \sum_{p,q\geq 1} \frac{p^{q-1}q^{p-1}}{p!q!}a^p b^q$$

This implies that the number of labeled trees with p sources and q sinks is $p^{q-1}q^{p-1}\frac{(p+q)!}{p!q!}$. In particular, if we specify which vertices are sources and which are sinks, the number of labeled trees is $p^{q-1}q^{p-1}$.

Exercise 3.16. *Do this calculation in detail.*

3.12. **The matrix-tree theorem.** These calculations can be generalized to compute the number of *colored* labeled trees. For this we first need to define the *Kirchhoff polynomial* $K_m(\mathbf{u})$. Namely, for a collection of variables $\mathbf{u} := (u_{ik})$, $1 \leq i \neq k \leq m$, $u_{ik} = u_{ki}$ consider the quadratic form

$$U(\mathbf{y}) := \sum_{1\leq i<k\leq m} u_{ik}(y_i - y_k)^2.$$

Generically it has a 1-dimensional kernel spanned by $\mathbf{1} = (1,...,1)$, so it is nondegenerate on the subspace defined by the equation $\sum_i y_i = 0$. This subspace carries a volume form $\omega_0(v_1,...,v_{m-1}) := \omega(v_1,...,v_{m-1},\mathbf{1})$, where ω is the standard volume form on $\mathbb{R}^m$, and with respect to this form we have

$$K_m(\mathbf{u}) := \det U = \det(\delta_{i\ell}\sum_{k\neq\ell} u_{k\ell} - u_{i\ell})_{(j)}$$

for any $1 \leq j \leq m$, where the subscript (j) means that the j-th row and column are removed. The polynomial K_m is called the *Kirchhoff polynomial*. For instance, $K_2 = u_{12}$, $K_3 = u_{12}u_{13} + u_{13}u_{23} + u_{12}u_{23}$, etc.

Now let $\mathbf{p} = (p_1,...,p_m)$ be a m-tuple of positive integers and $\mathbf{r} = (r_{ij}, 1 \leq i \leq j \leq m)$ be a collection of nonnegative integers with $|\mathbf{r}| = |\mathbf{p}| - 1$, where $|\mathbf{r}| := \sum_{i\leq j} r_{ij}$, $|\mathbf{p}| := \sum_k p_k$. Suppose vertices of the tree are given colors $1,...,m$, and we want to compute the number $N(\mathbf{p},\mathbf{r})$ of labeled trees with the first p_1 vertices colored with 1, the next p_2 with 2,..., the last p_m with m, and r_{ij} edges going between vertices of color i and vertices of color j.

It suffices to compute the polynomial

$$Q_{\mathbf{p}}(\mathbf{z}) := \sum_{\mathbf{r}:|\mathbf{r}|=|\mathbf{p}|-1} N(\mathbf{p},\mathbf{r}) \prod_{i\leq j} z_{ij}^{r_{ij}}.$$

Theorem 3.17. *We have*

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1...p_m)^{-1} K(p_k z_{k\ell} p_\ell, k \neq \ell) \prod_\ell (\sum_k p_k z_{k\ell})^{p_\ell - 1}.$$

Note that for $m = 1$ and $\mathbf{z} = 1$ this recovers Cayley's theorem, while for $m = 2$ and $\mathbf{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ it recovers our count of oriented trees.

Proof. We attach to each color j a real variable x_j. Then the corresponding action is

$$S(x, y) = \frac{1}{2} x^T B x - \sum_{j=1}^m a_j e^{x_j},$$

where $B = (b_{ij})$ is inverse to the matrix $\mathbf{z} := (z_{ij})$ with $z_{ij} = z_{ji}$. Then by Theorem 3.12, $Q_{\mathbf{p}}(\mathbf{z})$ is the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$, where x is the critical point of S.

The equation for the critical point of S is

$$\sum_i x_i b_{ij} e^{-x_j} = a_j.$$

Let $X_j := \sum_i x_i b_{ij}$, then $x_i = \sum_j z_{ij} X_j$, $a_i = X_i e^{-x_i}$, and

$$-S(x) = \int X_j \frac{da_j}{a_j}$$

for all j. In other words, the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$ equals the coefficient to the same monomial in $X_j(\mathbf{z}, \mathbf{a})$ divided by p_j. Thus, denoting by $D_T(\mathbf{z})$ the principal minor of $\mathbf{z}$ corresponding to a subset $T \subset \{1, ..., m\}$, we get

$$Q_{\mathbf{p}}(\mathbf{z}) = \frac{p_j^{-1}}{(2\pi i)^m} \oint X_j (\prod_k a_k^{-p_k - 1}) d\mathbf{a} =$$

$$\frac{p_j^{-1}}{(2\pi i)^m} \oint X_j (\prod_k (X_k e^{-x_k})^{-p_k - 1}) d(X_1 e^{-x_1}) \wedge ... \wedge d(X_m e^{-x_m}) =$$

$$\frac{p_j^{-1}}{(2\pi i)^m} \oint \sum_{T \subset \{1,...,m\}} (-1)^{|T|} D_T(\mathbf{z}) X_j (\prod_{\ell \notin T} X_\ell^{-1}) (\prod_\ell X_\ell^{-p_\ell}) e^{\sum_{k,\ell} p_k z_{k\ell} X_\ell} dX_1 \wedge ... \wedge dX_m$$

$$= p_j^{-1} \sum_{T \subset \{1,...,m\}} (-1)^{|T|} \frac{p_j - 1 + \delta_{jT^c}}{\sum_k p_k z_{kj}} D_T(\mathbf{z}) \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - \delta_{\ell T}}}{(p_\ell - \delta_{\ell T})!} =$$

$$p_j^{-1} \left(\frac{p_j - 1}{\sum_k p_k z_{kj}} \det(\delta_{i\ell} \sum_k p_k z_{k\ell} - z_{i\ell} p_\ell) + \det(\delta_{i\ell} \sum_k p_k z_{k\ell} - z_{i\ell} p_\ell)_{(j)} \right) \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell!},$$

where $\delta_{\ell T} = 1$ if $\ell \in T$ and 0 otherwise. The first determinant is zero, so we get

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1...p_m)^{-1} \det(\delta_{i\ell} \sum_k p_k z_{k\ell} p_\ell - p_i z_{i\ell} p_\ell)_{(j)} \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell!}.$$

This implies the theorem. □

Theorem 3.17 is a weighted version of *Kirchhoff's matrix-tree theorem*, which is a generalization of Cayley's theorem. More precisely, take $\mathbf{z} = A_\Gamma$ to be the adjacency matrix of a graph Γ (without self-loops), m the number of vertices of Γ, and $p_i = 1$ for all i. Then $Q_{\mathbf{p}}(\mathbf{z}) = N_\Gamma$ is the number of spanning trees of Γ, and Theorem 3.17 says that

$$N_\Gamma = \det U,$$

where U is the quadratic form

$$U(\mathbf{y}) = \sum_{i<j} (A_\Gamma)_{ij} (y_i - y_j)^2 = (\Delta_\Gamma \mathbf{y}, \mathbf{y}),$$

where $\Delta_\Gamma = D_\Gamma - A_\Gamma$ is the Laplace operator of Γ (D_Γ being the diagonal matrix of vertex degrees). Thus we get

Corollary 3.18. *(The matrix-tree theorem)*

$$N_\Gamma = \frac{1}{m} \lambda_1 ... \lambda_{m-1},$$

where λ_i are the non-zero eigenvalues of Δ_Γ.

Cayley's theorem is obtained from this result when Γ is a complete graph, in which case $\lambda_i = m$ for all i, so we get $N_\Gamma = m^{m-2}$.

3.13. **1-particle irreducible diagrams and the effective action.** Let $Z = Z_S$ be the partition function corresponding to the action S. In the previous subsections we have seen that the "classical" (or "tree") part $(\log \frac{Z_S}{Z_0})_0$ of the quantity $\hbar \log \frac{Z_S}{Z_0}$ is quite elementary to compute – it is just minus the critical value of the action $S(x)$. Thus, if we could find a new "effective" action S_{eff} (a "deformation" of S) such that

$$\hbar^{-1} (\log \tfrac{Z_{S_{\text{eff}}}}{Z_0})_0 = \log \tfrac{Z_S}{Z_0}$$

(i.e. the classical answer for the effective action is the quantum answer for the original one), then we can consider the quantum theory for the action S solved. In other words, the problem of solving the quantum theory attached to S (i.e. finding the corresponding integrals) essentially reduces to the problem of computing the effective action S_{eff}.

We will now give a recipe of computing the effective action in terms of amplitudes of Feynman diagrams, and see that it is computationally easier than computing the sum over connected diagrams.

Definition 3.19. An edge e of a connected graph Γ is said to be a *bridge* if the graph $\Gamma \setminus e$ is disconnected. A connected graph without bridges is called *1-particle irreducible* (1PI).[9]

To compute the effective action, we will need to consider graphs with external edges (but having at least one internal vertex). Such a graph Γ (with N external edges) will be called 1-particle irreducible if so is the corresponding "amputated" graph (i.e. the graph obtained from Γ by removal of the external edges). In particular, a graph with one internal vertex is always 1-particle irreducible, while a single edge graph without internal vertices is defined *not* to be 1-particle irreducible. The notions of a bridge and a 1-particle irreducible graph are illustrated by Fig. 8.

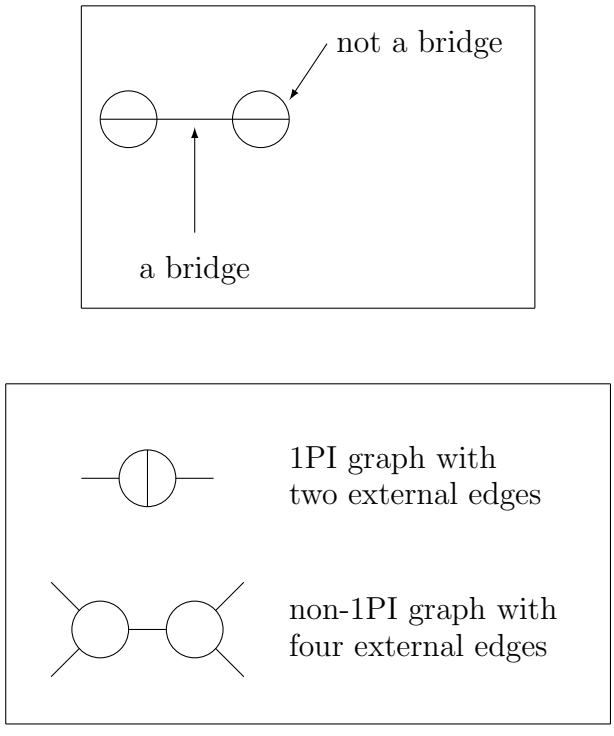


FIGURE 8.

[9]This is the physical terminology. The mathematical term is "2-connected".

Denote by $G_{1\mathrm{PI}}(\mathbf{n}, N)$ the set of isomorphism classes of 1-particle irreducible graphs with N external edges and n_i i-valent internal vertices for each i (where isomorphisms are not allowed to move external edges).

Theorem 3.20. *The effective action S_{eff} is given by the formula*

$$S_{\mathrm{eff}}(x) = \frac{B(x,x)}{2} - \sum_{i\geq 0} \frac{\mathcal{B}_i(x,...,x)}{i!},$$

where

$$\mathcal{B}_N(x,\dots,x) = \hbar \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_{1\mathrm{PI}}(\mathbf{n},N)} \frac{\mathbb{F}_\Gamma(Bx,\dots,Bx)}{|\mathrm{Aut}(\Gamma)|}.$$

Thus, $S_{\mathrm{eff}} = S + \hbar S_1 + \hbar^2 S_2 + ..$ The expressions $\hbar^j S_j$ are called the *j-loop corrections* to the effective action.

This theorem allows physicists to worry only about 1-particle irreducible diagrams, and is the reason why you will rarely see other diagrams in a QFT textbook. As before, it is very useful in doing low order computations, since the number of 1-particle irreducible diagrams with a given number of loops is much smaller than the number of connected diagrams with the same number of loops.

Proof. The proof is based on the following lemma from graph theory.

Lemma 3.21. *Any connected graph Γ can be uniquely represented as a tree whose vertices are 1-particle irreducible subgraphs (with external edges), and edges are the bridges of Γ.*

The lemma is obvious. Namely, let us remove all bridges from Γ. Then Γ will turn into a disjoint union of 1-particle irreducible graphs which should be taken to be the vertices of the said tree.

The tree corresponding to the graph Γ is called the *skeleton* of Γ (see Fig. 9).

It is easy to see that Lemma 3.21 implies Theorem 3.20. Indeed, it implies that the sum over all connected graphs occuring in the expression of $\log \frac{Z_S}{Z_0}$ can be written as a sum over skeleton trees, so that the contribution from each tree is (proportional to) the contraction of tensors $\mathcal{B}_i$ put in its vertices, and $\mathcal{B}_i$ is the (weighted) sum of amplitudes of all 1-particle irreducible graphs with i external edges. □

3.14. **1-particle irreducible diagrams and the Legendre transform.** Recall the notion of *Legendre transform*. Let f be a smooth function on a vector space Y, such that the map $Y \to Y^*$ given by $x \to df(x)$ is a diffeomorphism. Then one can define the Legendre

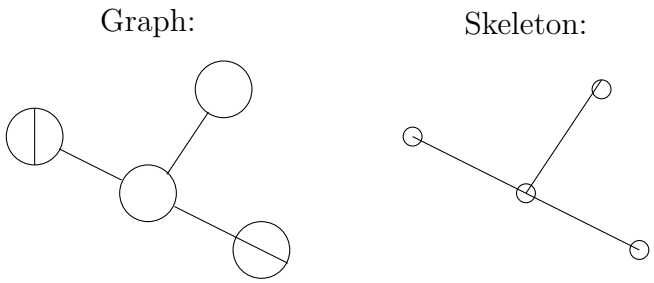


FIGURE 9. The skeleton of a graph.

transform of f as follows. For $p \in Y^*$, let $x_0 = x_0(p)$ be the critical point of the function $(p,x) - f(x)$ (i.e. the unique solution of the equation $df(x) = p$). Then the Legendre transform of f is the function on Y^* defined by

$$L(f)(p) = (p, x_0) - f(x_0).$$

It is easy to see that the differential of $L(f)$ is also a diffeomorphism $Y^* \to Y$ (in fact, inverse to $df(x)$), and that $L^2(f) = f$.

Example 3.22. Let $f(x) = \frac{ax^2}{2}$, $a \neq 0$. Then $px - f = px - \frac{x^2}{2}$ has a critical point at $p = \frac{x}{a}$, and the critical value is $\frac{p^2}{2a}$. Thus $L(\frac{ax^2}{2}) = \frac{p^2}{2a}$. More generally, if $f(x) = \frac{B(x,x)}{2}$ where B is a non-degenerate symmetric form on Y then $L(f)(p) = \frac{B^{-1}(p,p)}{2}$. E.g., the Legendre transform of a Lagrangian $\frac{mv^2}{2} - U(x)$ of a particle of mass m with respect to velocity $v = \dot{x}$ is its Hamiltonian (energy) $\frac{p^2}{2m} + U(x)$, and vice versa. This is, in fact, so in complete generality, which is why Legendre transform plays an important role in classical mechanics and field theory.

Note that the stationary phase formula implies that the Legendre transform is the classical analog of the Fourier transform. Indeed, the leading term of the asymptotics as $\hbar \to 0$ of the logarithm of the (suitably normalized) Fourier transform $\hbar^{-\frac{d}{2}} \int_V e^{\frac{i(-(p,x)+S(x))}{\hbar}} dx$ of the Feynman density $e^{\frac{iS(x)}{\hbar}} dx$ (where the integral is understood in the sense of distributions) is $-\frac{iL(S)(p)}{\hbar}$.

Now let us consider Theorem 3.20 in the situation of Theorem 3.5. Thus, $S(x) = \frac{B(x,x)}{2} + O(x^3)$, and we look at

$$Z(p) = \hbar^{-\frac{d}{2}} \int_V e^{\frac{(p,x)-S(x)}{\hbar}} dx.$$

By Theorem 3.20, one has

$$\log \frac{Z(p)}{Z_0} = -\hbar^{-1} S_{\text{eff}}(x_0, p),$$

where the effective action $S_{\text{eff}}(x, p)$ is the sum over 1-particle irreducible graphs and $x_0 = x_0(p)$ is its critical point.

Now, we must have $S_{\text{eff}}(x, p) = -p \cdot x + S_{\text{eff}}(x)$, since the only 1PI graph which contains 1-valent internal vertices (corresponding to p) is the graph with one edge, connecting an internal vertex with an external one (so it yields the term $-p \cdot x$, and other graphs contain no p-vertices). This shows that $\hbar \log \frac{Z(p)}{Z_0}$ is the critical value of $p \cdot x - S_{\text{eff}}(x)$. Thus we have proved the following.

Proposition 3.23. *We have*

$$S_{\text{eff}}(x) = L(\hbar \log \tfrac{Z(p)}{Z_0}), \ \hbar \log \tfrac{Z(p)}{Z_0} = L(S_{\text{eff}}(x)).$$

Physicists formulate this result as follows: the effective action is the Legendre transform of $\hbar$ times the logarithm of the generating function for quantum correlators (and vice versa).

Exercise 3.24. *Compute the 1-loop contribution to* $\log \frac{Z}{Z_0}$ *for*

$$S(x) = \tfrac{x^2}{2} - g(x + \tfrac{x^3}{6}).$$

Using this, compute the number of labeled n-vertex 1-loop graphs with 1-valent and 3-valent vertices only (be careful with double edges and self-loops!). Check your answer by directly enumerating such graphs with small number of vertices.

Exercise 3.25. *Find the exponential generating function* $\sum_n a_n \frac{z^n}{n!}$ *for the numbers* a_n *of labeled n-vertex trees with 1-valent and 4-valent vertices. You may express the answer via inverse functions to polynomials.*

Exercise 3.26. *Find the one-loop contribution to the effective action for* $S(x) = \frac{x^2}{2} - \frac{gx^3}{6}$. *That is, one has* $S_{\text{eff}} = S + \hbar S_1 + O(\hbar^2)$, *and you need to find* S_1. *Which Feynman diagrams need to be considered?*

Exercise 3.27. *Consider the heat equation* $u_t = \frac{1}{2}\Delta_B u$, *where* Δ_B *is the Laplace operator attached to B defined in Subsection 2.2. It is solved by the heat flow* $u(x,t) = e^{\frac{t\Delta_B}{2}} u(x,0)$. *Show that the effective action* S_{eff} *for the action* $S(x) = \frac{B(x,x)}{2} - \widetilde{S}(x)$ *can be computed as the sum of contributions of 1PI Feynman diagrams without self-loops for the action* $S^\circ(x) := \frac{B(x,x)}{2} - \widetilde{S}^\circ(x)$ *where* $\widetilde{S}^\circ(x) := e^{\frac{\hbar\Delta_B}{2}} \widetilde{S}(x)$ *obtained by transforming* $\widetilde{S}$ *by the heat flow for time* $\hbar$.

4. Matrix integrals

Let $\mathfrak{h}_N$ be the space of Hermitian matrices of size N. The inner product on $\mathfrak{h}_N$ is given by $B(A_1, A_2) = \mathrm{Tr}(A_1 A_2)$. In this section we will consider integrals of the form

$$Z_N := \hbar^{-\frac{N^2}{2}} \int_{\mathfrak{h}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where the Lebesgue measure dA is normalized by the condition

$$\int_{\mathfrak{h}_N} e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = 1$$

(so we don't have to drag around the $\sqrt{2\pi}$-factors), and

$$S(A) := \frac{\mathrm{Tr}(A^2)}{2} - \sum_{m \geq 1} g_m \frac{\mathrm{Tr}(A^m)}{m}$$

is the action functional.[10] We will be interested in the behavior of the coefficients of the expansion of Z_N in g_i for large N. The study of this behavior will lead us to considering not simply Feynman graphs, but actually fat (or ribbon) graphs, which are in fact 2-dimensional surfaces. Thus, before we proceed further, we need to do some 2-dimensional combinatorial topology.

4.1. **Fat graphs.** Recall from the proof of Feynman's theorem that given a finite collection of flowers and a matching σ on the set T of endpoints of their edges, we can obtain a graph Γ_σ by connecting (or gluing) the points which fall into the same pair.

Now, given an i-flower, let us inscribe it in a closed disk D (so that the ends of the edges are on the boundary). Then take its small tubular neighborhood in D. This produces a region with piecewise smooth boundary. We will equip this region and its boundary with the standard orientation, and call it a *fat i-valent flower*. The boundary of a fat i-valent flower has the form $P_1 Q_1 P_2 Q_2 \dots P_i Q_i P_1$, where P_i, Q_i are the angle points, the intervals $P_j Q_j$ are arcs on ∂D, and $Q_j P_{j+1}$ are (smooth) arcs lying inside D (see Fig. 10).

Now, given a collection of usual flowers and a matching σ as above, we can consider the corresponding fat flowers, and glue them, respecting the orientation, along intervals $P_j Q_j$ according to σ. This will produce a compact oriented surface with boundary (the boundary is glued from intervals $Q_j P_{j+1}$). We will denote this surface by $\widetilde{\Gamma}_\sigma$, and

[10] Note that we divide by m and not by $m!$. We will see below why such normalization will be more convenient.

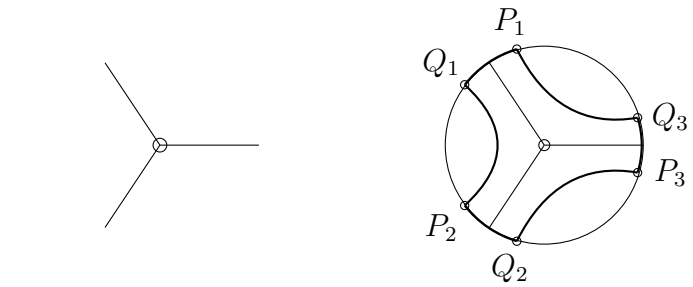


FIGURE 10.

call it the *fattening* of Γ with respect to σ. A fattening of a graph will be called a *fat (or ribbon) graph*.

Thus, a fat graph is not just an oriented surface with boundary, but such a surface together with a partition of this surface into fat flowers.

Note that the same graph Γ can have many fattenings which are non-homeomorphic (albeit homotopy equivalent) surfaces, and in particular the genus g of the fattening is *not* determined by Γ (see Fig. 11).

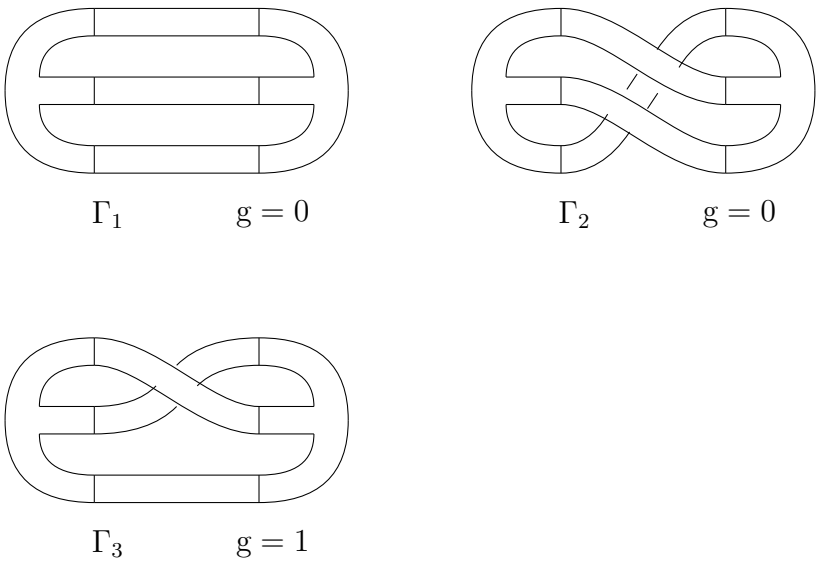


FIGURE 11. Gluing a fat graph from fat flowers

4.2. **Matrix integrals in large N limit, planar graphs, and the genus expansion.** Let us now return to the study of the integral Z_N. We have

$$B_m(A, ..., A) = (m-1)!\mathrm{Tr}(A^m).$$

Thus by Feynman's theorem,

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{(g_i \hbar^{\frac{i}{2}-1})^{n_i}}{i!^{n_i} n_i!} \sum_{\sigma \in \Pi_c(T_{\mathbf{n}})} \mathbb{F}(\sigma),$$

where the summation is taken over the set $\Pi_c(T_{\mathbf{n}})$ of all matchings of $T = T_{\mathbf{n}}$ that produce a connected graph Γ_σ, and $\mathbb{F}(\sigma)$ denotes the contraction of the tensors $(m-1)!\mathrm{Tr}(A^m)$ using σ. So let us compute $\mathbb{F}(\sigma)$.

Let $\{e_i\}$ be the standard basis of $\mathbb{C}^N$, and $\{e_i^*\}$ the dual basis of the dual space. Then the tensor $\mathrm{Tr}(A^m)$ can be written as

$$\mathrm{Tr}(A^m) = \sum_{i_1,\dots,i_m=1}^{N} (e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}).$$

Thus

$$B_m = \sum_{s \in S_{m-1}} \sum_{i_1,\dots,i_m=1}^{N} s(e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*)$$

(sum over all possible cyclic orderings of edges of an m-valent flower). Hence

$$\mathbb{F}(\sigma) = \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma),$$

where $\widetilde{\mathbb{F}}(\sigma)$ is obtained by contracting the tensors

$$\sum_{i_1,\dots,i_m=1}^{N} e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^* \tag{4.1}$$

according to the fat graph $\widetilde{\Gamma}_\sigma$. It follows that

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i!^{n_i} n_i!} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma) =$$

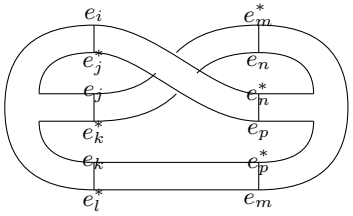
$$\sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i^{n_i} n_i!} \sum_{\sigma} \widetilde{\mathbb{F}}(\sigma)$$

(the product $\prod_i i!^{n_i}$ in the denominator got replaced by $\prod_i i^{n_i}$ since in the sum $\sum_{s,\sigma} \widetilde{\mathbb{F}}(s\sigma)$ every term $\widetilde{\mathbb{F}}(\sigma)$ occurs $|\prod_i S_{i-1}^{n_i}| = \prod_i (i-1)!^{n_i}$ times).

For a surface Σ with boundary, let $\nu(\Sigma)$ denote the number of connected components of the boundary.

Proposition 4.1. $\widetilde{\mathbb{F}}(\sigma) = N^{\nu(\widetilde{\Gamma}_\sigma)}$.

Proof. One can visualize each summand in the sum (4.1) as a labeling of the angle points $P_1, Q_1, \dots, P_m, Q_m$ on the boundary of a fat m-valent flower by $i_1, i_2, i_2, i_3, \dots, i_m, i_1$. Now, the contraction using σ of some set of such monomials is nonzero iff the subscript is constant along each boundary component of $\widetilde{\Gamma}_\sigma$ (see Fig. 12). This implies the result. □



Contraction nonzero iff
$i = r$, $j = p$, $j = m$, $k = r$,
$k = p$, $i = m$,
that is
$i = r = k = p = j = m$.

FIGURE 12. Contraction defined by a fat graph.

Let $\widetilde{G}_c(\mathbf{n})$ be the set of isomorphism classes of connected fat graphs with n_i i-valent vertices for $i \geq 1$. For $\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})$, let $b(\widetilde{\Gamma})$ be the number of edges minus the number of vertices of the underlying usual graph Γ.

Corollary 4.2.

$$\log Z_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|} =$$

$$\sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\widetilde{\Gamma})} \hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Proof. Let $\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}} := \prod_i (S_{n_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{n_i})$. This group acts on $T_{\mathbf{n}}$, so that $\widetilde{\Gamma}_\sigma = \widetilde{\Gamma}_{g\sigma}$, for any $g \in \mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}$. Moreover, the group acts transitively on the set of σ giving a fixed fat graph $\widetilde{\Gamma}_\sigma$, and the stabilizer of any σ is $\mathrm{Aut}(\widetilde{\Gamma}_\sigma)$. This implies the result, as $|\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}| = \prod_i i^{n_i} n_i!$ which cancels the denominators. □

Now for any compact connected surface Σ with boundary, let $\mathrm{g}(\Sigma)$ be the genus of Σ. Then for a connected fat graph $\widetilde{\Gamma}$,

$$b(\widetilde{\Gamma}) = 2\mathrm{g}(\widetilde{\Gamma}) - 2 + \nu(\widetilde{\Gamma})$$

(minus the Euler characteristic). Thus, defining

$$\widehat{Z}_N(\hbar) := Z_N(\tfrac{\hbar}{N}),$$

we obtain

Theorem 4.3.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

This implies the following important result, due to t'Hooft.

Theorem 4.4. *(1) There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c(\mathbf{n})[0]$ *denotes the set of* **planar** *connected fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g} \in \mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected fat graphs of genus* g.

Remark 4.5. Genus zero fat graphs are said to be planar because the underlying usual graphs can be put on the 2-sphere (and hence on the plane) without self-intersections.

Remark 4.6. t'Hooft's theorem may be interpreted in terms of the usual Feynman diagram expansion. Namely, it implies that for large N, the leading contribution to $\log Z_N(\frac{\hbar}{N})$ comes from the terms in the Feynman diagram expansion corresponding to planar graphs (i.e. those that admit an embedding into the 2-sphere).

4.3. **Integration over real symmetric matrices.** One may also consider the matrix integral over the space $\mathfrak{s}_N$ of real symmetric matrices of size N. Namely, one puts

$$Z_N = \hbar^{-\frac{N(N+1)}{4}} \int_{\mathfrak{s}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where S and dA are as above. Let us generalize Theorem 4.4 to this case.

As before, consideration of the large N limit leads to consideration of fat flowers and gluing of them. However, the exact nature of gluing

is now somewhat different. Namely, in the Hermitian case we had $(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il}\delta_{jk}$, which forced us to glue fat flowers preserving orientation. On the other hand, in the real symmetric case $e_i^* = e_i$, and the inner product of the functionals $e_i \otimes e_j$ on the space of symmetric matrices is given by $(e_i \otimes e_j, e_k \otimes e_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. This means that besides the usual (orientation preserving) gluing of fat flowers, we now must allow gluing with a twist of the ribbon by 180°. Fat graphs thus obtained will be called *twisted fat graphs*. That means, a twisted fat graph is a surface with boundary (possibly not orientable), together with a partition into fat flowers, and orientations on each of them (which may or may not match at the cuts, see Fig.13).

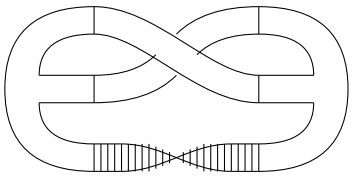


FIGURE 13. Twisted fat graph

Now one can show analogously to the Hermitian case that the $\frac{1}{N}$ expansion of $\log \widehat{Z}_N$ (where $\widehat{Z}_N := Z_N(\frac{2\hbar}{N})$) is given by the same formula as before, but with summation over the set $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})$ of twisted fat graphs:

Theorem 4.7.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Here the genus g of a (possibly non-orientable) surface is defined for closed surfaces by $\mathrm{g} := 1 - \frac{\chi}{2}$, where χ is the Euler characteristic. Thus the genus of $\mathbb{RP}^2$ is $\frac{1}{2}$, the genus of the Klein bottle is 1, and so on.

In particular, we have the following analog of t'Hooft's theorem.

Theorem 4.8. *(1) There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]$ *denotes the set of* **planar** *connected twisted fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g} \in \frac{1}{2}\mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected twisted fat graphs which have genus* g.

Exercise 4.9. *Consider the matrix integral over the space* $\mathfrak{q}_N$ *of quaternionic Hermitian matrices of size* N*. Show that in this case the results are the same as in the real case, except that each twisted fat graph counts with a sign equal to* $(-1)^\nu$*, where* ν *is the number of boundary components. In other words,* $\log Z_N^{\mathrm{quat}}(\hbar)$ *equals* $\log Z_{2N}^{\mathrm{real}}(\hbar)$ *with* N *replaced by* $-N$*.*

Hint: *Use that the quaternionic unitary group* $U(N, \mathbb{H})$ *is a real form of* $Sp(2N)$*, and* $\mathfrak{q}_N$ *is a real form of the representation of* $\Lambda^2 V$*, where* V *is the standard (vector) representation of* $Sp(2N)$*. Compare to the case of real symmetric matrices, where the relevant representation is* $S^2 V$ *for* $O(N)$*, and the case of complex Hermitian matrices, where it is* $V \otimes V^*$ *for* $GL(N)$*.*

4.4. **The number of ways to glue a surface from a polygon and the Wigner semicircle law.** Matrix integrals are so rich that even the simplest possible example reduces to a nontrivial counting problem. Namely, consider the matrix integral Z_N over complex Hermitian matrices with $\hbar = 1$ in the case $S(A) = \frac{\mathrm{Tr}(A^2)}{2} - s\frac{\mathrm{Tr}(A^{2m})}{2m}$, where $s^2 = 0$ (i.e. we work over the ring $\mathbb{C}[s]/(s^2)$). Then from Theorem 4.4 we get

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = P_m(N),$$

where $P_m(N)$ is a polynomial given by the formula

$$P_m(N) = \sum_{\mathrm{g} \geq 0} \varepsilon_{\mathrm{g}}(m) N^{m+1-2\mathrm{g}},$$

and $\varepsilon_{\mathrm{g}}(m)$ is the number of ways to glue a surface of genus g from a $2m$-gon with labeled sides, i.e., to match the sides and then glue the matching ones to each other in an orientation-preserving manner. Indeed, in this case we have only one fat flower of valency $2m$, which has to be glued with itself; so a direct application of our Feynman

rules leads to counting ways to glue a surface of a given genus from a polygon.

The value of this integral is given by the following non-trivial theorem.

Theorem 4.10. *(Harer-Zagier,* [HZ] *1986)*

$$P_m(x) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} \binom{m}{p} 2^p \frac{x(x-1)\dots(x-p)}{(p+1)!}.$$

The theorem is proved in the next subsections.

Looking at the leading coefficient of P_m, we get

Corollary 4.11. *The number of ways to glue a sphere from a $2m$-gon is the Catalan number $C_m = \frac{(2m)!}{m!(m+1)!} = \frac{1}{m+1}\binom{2m}{m}$.*

Corollary 4.11 actually has another (elementary combinatorial) proof, which is as follows. For each matching σ on the set of sides of the $2m$-gon, let us connect the midpoints of the matched sides by straight lines (Fig.14). It is geometrically evident that if these lines don't intersect then the gluing will give a sphere. We claim that the converse is true as well. Indeed, assume the contrary, i.e. that for cyclically ordered edges a, b, c, d, the edge a connects to c and b to d. Then it is easy to see that gluing these two pairs of edges gives a torus with a hole (or without if $m = 2$). But an (open) torus with a hole can't be embedded into a sphere (e.g. it contains a copy of K_5), contradiction.

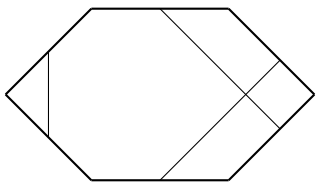


FIGURE 14. Matching of sides of a 6-gon.

Now it remains to count the number of ways to connect midpoints of sides with lines without intersections. Suppose we draw one such line, such that the number of sides on the left of it is $2k$ and on the right is $2l$ (so that $k + l = m - 1$). Then we face the problem of connecting the two sets of $2k$ and $2l$ sides without intersections. This shows that the number of gluings D_m satisfies the recursion

$$D_m = \sum_{k+l=m-1} D_k D_l, \ D_0 = 1.$$

In other words, the generating function

$$h(x) := \sum_m D_m x^m = 1 + x + \cdots$$

satisfies the equation $h(x) - 1 = xh(x)^2$. This implies that

$$h(x) = \frac{1 - \sqrt{1-4x}}{2x},$$

which yields that $D_m = C_m$. We are done.

Corollary 4.11 can be used to derive the following fundamental result from the theory of random matrices, discovered by Wigner in 1955.

Theorem 4.12. *(Wigner's semicircle law) Let f be a continuous function on $\mathbb{R}$ of at most polynomial growth at infinity. Then*

$$\lim_{N\to\infty} \frac{1}{N} \int_{\mathfrak{h}_N} \mathrm{Tr} f(\tfrac{A}{\sqrt{N}}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} = \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4-x^2}dx.$$

This theorem is called the semicircle law because it says that the graph of the density of eigenvalues of a large random Hermitian matrix distributed according to the "Gaussian unitary ensemble" (i.e. with density $e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA$) is a semicircle. In particular, we see that for large N almost all eigenvalues of A belong to the interval $[-2\sqrt{N}, 2\sqrt{N}]$, so the limit does not depend on the values of f outside $[-2, 2]$.

Proof. By Weierstrass' theorem on uniform approximation of a continuous function on an interval by polynomials, we may assume that f is a polynomial. (Exercise: Justify this step). Thus, it suffices to check the result if $f(x) = x^{2m}$. In this case, by Corollary 4.11, the left hand side is C_m. On the other hand, an elementary computation yields

$$\frac{1}{2\pi} \int_{-2}^{2} x^{2m}\sqrt{4-x^2}dx = C_m,$$

which implies the theorem. □

4.5. **Hermite polynomials.** The proof[11] of Theorem 4.10 given below uses Hermite polynomials. So let us recall their properties.

Hermite polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

So the leading term of $H_n(x)$ is $(2x)^n$.

We collect the standard properties of $H_n(x)$ in the following theorem.

[11] I adopted this proof from D.Jackson's notes.

Theorem 4.13. *(i) The exponential generating function of $H_n(x)$ is*

$$f(x,t) = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

(ii) $H_n(x)$ satisfy the differential equation $f'' - 2xf' + 2nf = 0$. In other words, $H_n(x)e^{-x^2/2}$ are eigenfunctions of the operator $L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$ (Hamiltonian of the quantum harmonic oscillator) with eigenvalues $n + \frac{1}{2}$.

(iii) $H_n(x)$ are orthogonal:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\delta_{mn}.$$

Moreover, the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal basis of $L^2(\mathbb{R})$.

(iv) One has

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x)dx = \frac{(2m)!}{(m-k)!}2^{2(k-m)}$$

(if $k > m$, the answer is zero).

(v) One has

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^{r} \frac{r!}{2^k k!^2 (r-k)!} H_{2k}(x).$$

Proof. (sketch) (i) Follows immediately from the fact that the operator $\sum_{n\geq 0}(-1)^n \frac{t^n}{n!}\frac{d^n}{dx^n}$ maps a function $g(x)$ to $g(x-t)$.

(ii) Follows from (i) and the fact that the function $f(x,t)$ satisfies the PDE

$$f_{xx} - 2xf_x + 2tf_t = 0.$$

(iii) The orthogonality follows from (i) by direct integration:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)e^{-x^2}dx = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2ut-(x-u-t)^2}dx = e^{2ut}.$$

Thus the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal system in $L^2(\mathbb{R})$.

To show that these functions are complete, denote by $E \subset L^2(\mathbb{R})$ the closure of their span $\mathbb{C}[x]e^{-\frac{x^2}{2}}$. By approximating the function e^{ipx} by its Taylor polynomials, it is easy to see that $e^{ipx-\frac{x^2}{2}} \in E$ for any $p \in \mathbb{R}$. Thus for any compactly supported smooth $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\phi(x)e^{-\frac{x^2}{2}} = \int_{\mathbb{R}} \widehat{\phi}(p)e^{ipx-\frac{x^2}{2}}dp \in E.$$

where $\widehat{\phi}$ is the (suitably normalized) Fourier transform of ϕ. In other words, $C_0^\infty(\mathbb{R})$ is dense in E. But $C_0^\infty(\mathbb{R})$ is clearly dense in $L^2(\mathbb{R})$, so $E = L^2(\mathbb{R})$, as claimed.

(iv) By (i), one should calculate $\int_{\mathbb{R}} x^{2m} e^{2xt-t^2} e^{-x^2} dx$. This integral equals

$$\int_{\mathbb{R}} x^{2m} e^{-(x-t)^2} dx = \int_{\mathbb{R}} (y+t)^{2m} e^{-y^2} dy == \sqrt{\pi} \sum_p \binom{2m}{2p} \frac{(2m-2p)!}{2^{m-p}(m-p)!} t^{2p}.$$

The result is now obtained by extracting individual coefficients.

(v) By (iii), it suffices to show that

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_r^2(x) H_{2k}(x) e^{-x^2} dx = \frac{2^{r+k} r!^2 (2k)!}{k!^2 (r-k)!}$$

To prove this identity, let us integrate the product of three generating functions. By (i), we have

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x,t) f(x,u) f(x,v) e^{-x^2} dx =$$

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2(ut+uv+tv)-(x-u-t-v)^2} dx = e^{2(ut+tv+uv)}.$$

Extracting the coefficient of $t^r u^r v^{2k}$, we get the result. □

4.6. **Proof of Theorem 4.10.** We need to compute the integral

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA.$$

To do this, we note that the integrand is invariant with respect to conjugation by unitary matrices. Therefore, the integral can be reduced to an integral over the eigenvalues $\lambda_1, \dots, \lambda_N$ of A.

More precisely, consider the spectrum map $\sigma : \mathfrak{h}_N \to \mathbb{R}^N/S_N$. It is well known (due to H.Weyl) that the direct image $\sigma_* dA$ is given by the formula $\sigma_* dA = Ce^{-\sum_i \frac{\lambda_i^2}{2}} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda$, where $C > 0$ is a normalization constant that will not be relevant to us. Thus, we have

$$P_m(N) = \frac{NJ_m}{J_0}, \ J_m := \int_{\mathbb{R}^N} (\frac{1}{N} \sum_i \lambda_i^{2m}) e^{-\sum_i \frac{\lambda_i^2}{2}} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda.$$

To calculate J_m, we will use Hermite polynomials. Observe that since $H_n(x)$ are polynomials of degree n with highest coefficient 2^n, we have

$$\prod_{i<j} (\lambda_i - \lambda_j) = 2^{-\frac{N(N-1)}{2}} \det(H_k(\lambda_\ell)),$$

where k runs through the set $0, 1, \ldots, N-1$ and ℓ through $1, ..., N$. Thus, we find

(4.2)
$$J_m = 2^{m+\frac{N^2}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \det(H_k(\lambda_j))^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}} \sum_{\sigma,\tau \in S_N} (-1)^\sigma (-1)^\tau \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma(i)}(\lambda_i) H_{\tau(i)}(\lambda_i) d\lambda.$$

(Here $(-1)^\sigma$ denotes the sign of σ).

Since Hermite polynomials are orthogonal, the only terms in this sum which are nonzero are the terms with $\sigma(i) = \tau(i)$ for $i = 2, \ldots, N$. That is, the nonzero terms have $\sigma = \tau$. Thus, we have

(4.3)
$$J_m = 2^{m-\frac{N(N-2)}{2}} \sum_{\sigma \in S_N} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma i}(\lambda_i)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}} (N-1)! \gamma_0 \ldots \gamma_{N-1} \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx,$$

where $\gamma_i := \int_{-\infty}^{\infty} H_i(x)^2 e^{-x^2} dx$ are the squared norms of the Hermite polynomials. Applying this for $m = 0$ and dividing NJ_m by J_0, we find

$$P_m(N) = 2^m \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx.$$

Using Theorem 4.13 (iii) and (v), we find that $\gamma_i = 2^i i! \sqrt{\pi}$, and hence

$$P_m(N) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^m x^{2m} H_{2k}(x)}{2^k k!^2 (j-k)!} e^{-x^2} dx.$$

Now, using Theorem 4.13 (iv), we get

$$P_m(N) = \frac{(2m)!}{2^m} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^k j!}{(m-k)! k!^2 (j-k)!} =$$

$$\frac{(2m)!}{2^m m!} \sum_{j=0}^{N-1} \sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k}.$$

The sum over k can be represented as the constant term of a polynomial:

$$\sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k} = C.T.((1+z)^m(1+2z^{-1})^j).$$

Therefore, summation over j (using the formula for the sum of the geometric progression) yields

$$P_m(N) = \frac{(2m)!}{2^m m!} C.T. \left((1+z)^m \frac{(1+2z^{-1})^N - 1}{2z^{-1}} \right) =$$

$$\frac{(2m)!}{2^m m!} \sum_{p=0}^{m} 2^p \binom{m}{p} \binom{N}{p+1}.$$

We are done.

Exercise 4.14. *Find the number of ways to glue an orientable surface of genus* $\mathrm{g} \geq 1$ *from a* $4\mathrm{g}$*-gon (the gluing must preserve orientation), and prove your answer.*

Answer: $\frac{(4\mathrm{g}-1)!!}{2\mathrm{g}+1}$.

Exercise 4.15. *Consider a random Hermitian matrix* $A \in \mathfrak{h}_N$*, distributed with Gaussian density* $e^{-\mathrm{Tr}(A^2)}dA$*. Show that the most likely eigenvalues of* A *are the roots of the* N*-th Hermite polynomial* H_N*.*

Hint. *1) Write down the system of algebraic equations for the maximum of the density on eigenvalues.*

2) Introduce the polynomial $P(z) = \prod_i (z - \lambda_i)$*, where* λ_i *are the most likely eigenvalues. Let* $f = P'/P$*. Compute* $f' + f^2$ *(look at the poles).*

3) Reduce the obtained Riccati equation for f *to a second order linear differential equation for* P*. Show that this equation is the Hermite's equation, and deduce that* $P = \frac{H_N}{2^N}$.

5. The Euler characteristic of the moduli space of curves

Matrix integrals (in particular, the computation of the polynomial $P_m(x)$) can be used to calculate the orbifold Euler characteristic of the moduli space of curves. This was done by Harer and Zagier in 1986. Here we will give a review of this result (with some omissions).

5.1. **Euler characteristics of groups.** We start with recalling some basic notions from algebraic topology.

Let Γ be a discrete group, and Y be a contractible finite dimensional CW complex, on which Γ acts cellularly. This means that Γ acts by homeomorphisms of Y that map each cell homeomorphically to another cell. We will assume that the stabilizer of each cell is a finite group (i.e. Y is a proper Γ-complex).

Suppose first that the action of Γ is free (i.e. the stabilizers of cells are trivial). This is equivalent to saying that Γ is torsion free (i.e. has no nontrivial finite subgroups), since a finite group cannot act without fixed points on a contractible finite dimensional cell complex (as it has infinite cohomological dimension).

In this case we can define a cell complex Y/Γ (a classifying space for Γ), and we have $H^i(Y/\Gamma, A) = H^i(\Gamma, A)$ for any coefficient group A. In particular, if Y/Γ is finite then Γ has finite cohomological dimension, and the Euler characteristic $\chi(\Gamma) := \sum_i (-1)^i \dim H^i(\Gamma, \mathbb{Q})$ is equal to $\sum_i (-1)^i n_i(Y/\Gamma)$, where $n_i(Y/\Gamma)$ denotes the number of cells in Y/Γ of dimension i.

This setting, however, is very restrictive, since it allows only groups of finite cohomological dimension, and in particular excludes all non-trivial finite groups. So let us consider a more general setting: assume that some finite index subgroup $\Gamma' \subset \Gamma$, rather than Γ itself, satisfies the above conditions. In this case, on may define the Euler characteristic of Γ in the sense of Wall, which is the rational number $[\Gamma : \Gamma']^{-1}\chi(\Gamma')$.

It is easy to check that the Euler characteristic in the sense of Wall can be computed using the following *Quillen's formula*

$$\chi(\Gamma) = \sum_{\sigma \in \mathrm{cells(Y)}/\Gamma} \frac{(-1)^{\dim \sigma}}{|\mathrm{Stab}\sigma|}.$$

In particular, this number is independent of Γ' (which is also easy to check directly).

Example 5.1. If G is a finite group then $\chi(G) = |G|^{-1}$ (one takes the trivial group as the subgroup of finite index).

Example 5.2. $G = SL_2(\mathbb{Z})$. This group contains a subgroup F of index 12, which is free in two generators (check it!). The group F has Euler characteristic -1, since its classifying space Y/F is figure "eight" (i.e., Y is the universal cover of figure "eight"). Thus, the Euler characteristic of $SL_2(\mathbb{Z})$ is $-\frac{1}{12}$.

The Euler characteristic in the sense of Wall has a geometric interpretation in terms of orbifolds. Namely, suppose that Γ is as above (i.e. $\chi(\Gamma)$ is a well defined rational number), and M is a contractible manifold, on which Γ acts freely and properly discontinuously. In this case, stabilizers of points are finite, and thus M/Γ is an orbifold. This means, in particular, that to every point $x \in M/\Gamma$ is attached a finite group $\mathrm{Aut}(x)$, of size $\leq [\Gamma : \Gamma']$. Let X_m be the subset of M/Γ, consisting of points x such that $\mathrm{Aut}(x)$ has order m. It often happens that X_m has the homotopy type of a finite cell complex. In this case, the *orbifold Euler characteristic* of M/Γ is defined to be

$$\chi_{\mathrm{orb}}(M/\Gamma) = \sum_m \frac{\chi(X_m)}{m}.$$

Now, we claim that $\chi_{\mathrm{orb}}(M/\Gamma) = \chi(\Gamma)$. Indeed, looking at the projection $M/\Gamma' \to M/\Gamma$, it is easy to see that $\chi_{\mathrm{orb}}(M/\Gamma) = \frac{1}{[\Gamma:\Gamma']}\chi(M/\Gamma')$. But M/Γ' is a classifying space for Γ', so $\chi(M/\Gamma') = \chi(\Gamma')$, which implies the claim.

Example 5.3. Consider the group $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half plane H. Then H/Γ is the moduli space of elliptic curves. So as a topological space it is $\mathbb{C}$, where all points have automorphism group $\mathbb{Z}/2$, except the point i having automorphism group $\mathbb{Z}/4$, and $\rho = \frac{-1+i\sqrt{3}}{2}$ which has automorphism group $\mathbb{Z}/6$. Thus, the orbifold Euler characteristic of H/Γ is $(-1)\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$. This is not surprising since we proved that $\chi_{\mathrm{orb}}(H/\Gamma) = \chi(\Gamma)$, which was computed to be $-\frac{1}{12}$.

5.2. **The mapping class group.** Now let $\mathrm{g} \geq 1$ be an integer, and Σ be a closed oriented surface of genus g. Let $p \in \Sigma$, and let Γ^1_{g} be the group of isotopy classes of diffeomorphisms of Σ which preserve p. We will recall without proof some standard facts about this group, following the paper of Harer and Zagier, [HZ].

The group Γ^1_{g} is not torsion free, but it has a torsion free subgroup of finite index. Namely, consider the homomorphism $\Gamma^1_{\mathrm{g}} \to \mathrm{Sp}(2\mathrm{g}, \mathbb{Z}/n\mathbb{Z})$ given by the action of Γ^1_{g} on $H_1(\Sigma, \mathbb{Z}/n\mathbb{Z})$. Then for large enough n (in fact, $n \geq 3$), the kernel K_n of this map is torsion free.

It turns out that there exists a contractible finite dimensional cell complex Y_{g}, to be constructed below, on which Γ^1_{g} acts cellularly with finitely many cell orbits. Thus, the Euler characteristic of Γ^1_{g} in the sense of Wall is well defined.

5.3. **The Harer-Zagier theorem.** The Euler characteristic of Γ^1_{g} is given by the following theorem.

Theorem 5.4. *(Harer-Zagier) One has*

$$\chi(\Gamma^1_{\mathrm{g}}) = -\frac{B_{2\mathrm{g}}}{2\mathrm{g}},$$

where B_n are the Bernoulli numbers.

Remark 5.5. The group Γ^1_{g} acts on the Teichmüller space $\mathcal{T}^1_{\mathrm{g}}$, which is, by definition, the space of pairs $((R,z),f)$, where (R,z) is a complex Riemann surface with a marked point z, and f is an isotopy class of diffeomorphisms $R \to \Sigma$ that map z to p. One may show that $\mathcal{T}^1_{\mathrm{g}}$ is a contractible manifold of dimension $6\mathrm{g}-4$, and that the action of Γ^1_{g} on $\mathcal{T}^1_{\mathrm{g}}$ is properly discontinuous. In particular, we may define an orbifold $M^1_{\mathrm{g}} = \mathcal{T}^1_{\mathrm{g}}/\Gamma^1_{\mathrm{g}}$. This orbifold parametrizes pairs (R,z) as above; therefore, it is called the moduli space of Riemann surfaces (=smooth complex projective algebraic curves) of genus g with one marked point. Thus, Theorem 5.4 gives the orbifold Euler characteristic of the moduli space of curves of genus g with one marked point.

Remark 5.6. If $\mathrm{g} > 1$, one may define the analogs of the above objects without marked points, namely the mapping class group Γ_{g}, the Teichmüller space $\mathcal{T}_{\mathrm{g}}$, and the moduli space of curves $M_{\mathrm{g}} = \mathcal{T}_g/\Gamma_g$ (one can do it for $\mathrm{g}=1$ as well, but in this case there is no difference with the case of one marked point, since the translation group allows one to identify any two points on Σ). It is easy to see that for $\mathrm{g}>1$ we have an exact sequence $1 \to \pi_1(\Sigma) \to \Gamma^1_{\mathrm{g}} \to \Gamma_{\mathrm{g}} \to 1$, which implies that $\chi(\Gamma_{\mathrm{g}}) = \chi(\Gamma^1_{\mathrm{g}})/\chi(\Sigma)$. Thus, the Harer-Zagier theorem implies that $\chi(\Gamma_{\mathrm{g}}) = \chi_{\mathrm{orb}}(M_{\mathrm{g}}) = \frac{B_{2\mathrm{g}}}{4\mathrm{g}(\mathrm{g}-1)}$.

5.4. **Construction of the complex Y_{g}.** We begin the proof of Theorem 5.4 with the construction of the complex Y_{g}, following [HZ]. We will first construct a simplicial complex with a Γ^1_{g} action, and then use it to construct Y_{g}.

Let $(\alpha_1,...,\alpha_n)$ be a collection of closed simple unoriented curves on Σ, which begin and end at p, and do not intersect other than at p. Such a collection is called an *arc system* if two conditions are satisfied:

(A) none of the curves is contractible to a point;

(B) none of the curves is contractible to another.

Define a simplicial complex A, whose $n-1$-simplices are isotopy classes of arc systems consisting of $n \geq 1$ arcs, and the boundary of a simplex corresponding to $(\alpha_1, ...\alpha_n)$ is the union of simplices corresponding to the arc system $(\alpha_1, ..., \widehat{\alpha}_i, ..., \alpha_n)$ (α_i is omitted).

It is clear that the group Γ^1_{g} acts simplicially on A.

Example 5.7. Let g $= 1$, i.e. $\Sigma = S^1 \times S^1$. Then $\Gamma^1_{\mathrm{g}} = SL_2(\mathbb{Z})$. Up to its action, there are only three arc systems (Fig. 15). Namely, viewing S^1 as the unit circle in the complex plane, and representing arcs parametrically, we may write these three systems as follows:

$$B_0 = \{(e^{i\theta}, 1)\}; B_1 = \{(e^{i\theta}, 1), (1, e^{i\theta})\}; B_2 = \{(e^{i\theta}, 1), (1, e^{i\theta}), (e^{i\theta}, e^{i\theta})\}$$

From this it is easy to find the simplicial complex A. Namely, let T be the tree with root t_0 connected to three vertices t_1, t_2, t_3, with each t_i connected to two vertices t_{i1}, t_{i2}, each t_{ij} connected to t_{ij1}, t_{ij2}, etc. (Fig.16). Put at every vertex of T a triangle, with sides transversal to the three edges going out of this vertex, and glue the triangles along the sides. This yields the complex A, Fig.17 (check it!). The action of $SL_2(\mathbb{Z})$ (or rather $PSL_2(\mathbb{Z})$) on this complex is easy to describe. Namely, recall that $PSL_2(\mathbb{Z})$ is generated by S, U with defining relations $S^2 = U^3 = 1$. The action of S, U on T is defined as follows: S is the reflection with flip with respect to a side of the triangle Δ_0 centered at t_0 (Fig.18), and U is the rotation by $2\pi/3$ around t_0.

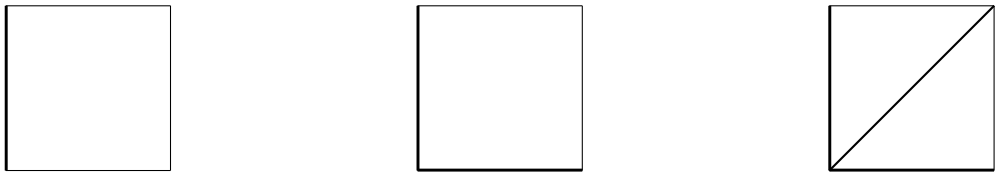


FIGURE 15. Three arc systems.

This example shows that the action of Γ^1_{g} on A is not properly discontinuous, as some simplices have infinite stabilizers (in the example, it is the 0-dimensional simplices). Thus, we would like to throw away the "bad" simplices. To do so, let us say that an arc system $(\alpha_1, .., \alpha_n)$ *fills up* Σ if it cuts Σ into a union of regions diffeomorphic to the open disk. Let A_∞ be the union of the simplices in A corresponding to arc systems that do not fill up Σ. This is a closed subset, since the property

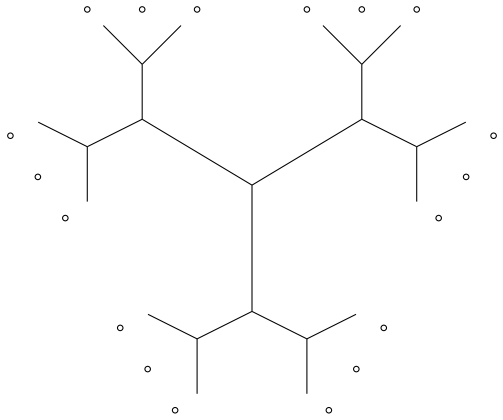


FIGURE 16. The tree T

of not filling up Σ is obviously stable under taking an arc subsystem. Thus, $A \setminus A_\infty$ is an open subset of A. In the example above, it is the complex A with 0-dimensional simplices removed.

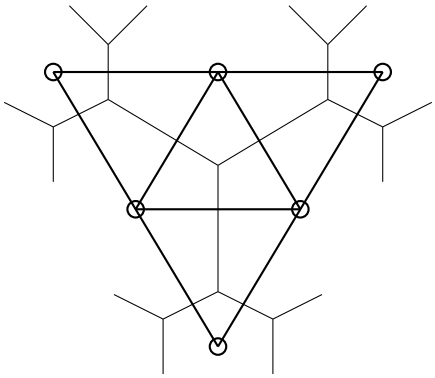


FIGURE 17. The complex A

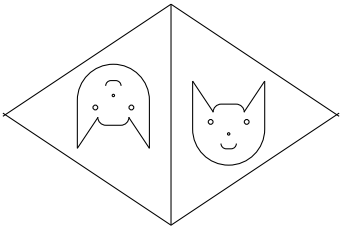


FIGURE 18. Reflection with a flip.

The following theorem shows that $A \setminus A_\infty$ is in fact a combinatorial model for the Teichmüller space $\mathcal{T}^1_{\mathrm{g}}$, with the action of Γ^1_{g}.

Theorem 5.8. *(Mumford) (a) The action of Γ^1_{g} on $A \setminus A_\infty$ is properly discontinuous.*

(b) $A \setminus A_\infty$ is topologically a manifold, which is Γ^1_{g}-equivariantly homeomorphic to the Teichmüller space $\mathcal{T}^1_{\mathrm{g}}$; in particular, it is contractible.

Remark 5.9. Theorem 5.8 exhibits the significance of conditions (A) and (B). Indeed, if either of these conditions were dropped, then one could consider arc systems $(\alpha_1, ..., \alpha_n)$ with arbitrarily large n, while with conditions (A),(B), as seen from Theorem 5.8, the largest value of n is $6\mathrm{g} - 3$.

Remark 5.10. If $\mathrm{g} = 1$, Theorem 5.8 is clear from the explicit description of A (convince yourself of this!).

Theorem 5.8 is rather deep, and we will not give its proof, which is beyond the scope of this text. Rather, we will use it to define the "Poincaré dual" CW complex Y_{g} of $A \setminus A_\infty$. Namely, to each filling arc system $(\alpha_1, ..., \alpha_n)$ we will assign a $6\mathrm{g} - 3 - n$-dimensional cell, and the boundary relation is opposite to the one before. The existence of this CW complex follows from the fact that $A \setminus A_\infty$ is a manifold. For instance, in the case $\mathrm{g} = 1$ the complex Y_{g} is the tree T.

Now, the complex Y_{g} is contractible (since so is $A \setminus A_\infty$), and admits a cellular action of Γ^1_{g} with finitely many cell orbits and finite stabilizers. This means that the Euler characteristic of Γ^1_{g} is given by Quillen's formula.

$$\chi(\Gamma^1_{\mathrm{g}}) = \sum_{\sigma \in \mathrm{cells}(Y_{\mathrm{g}})/\Gamma^1_{\mathrm{g}}} (-1)^{\dim \sigma} \frac{1}{|\mathrm{Stab}\sigma|}.$$

Example 5.11. In the $\mathrm{g} = 1$ case, T has one orbit of 0-cells and one orbit of 1-cells. The stabilizer of a 0-cell in $SL_2(\mathbb{Z})$ is $\mathbb{Z}/6$, and of a 1-cell is $\mathbb{Z}/4$. Hence, $\chi(SL_2(\mathbb{Z})) = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$, which was already computed before by other methods.

5.5. **Enumeration of cells in $Y_{\mathrm{g}}/\Gamma^1_{\mathrm{g}}$.** Now it remains to count the cells in $Y_{\mathrm{g}}/\Gamma^1_{\mathrm{g}}$, i.e. to enumerate arc systems which fill Σ (taking into account signs and stabilizers) To do this, we note that by definition of "filling", any filling arc system S defines a cellular decomposition of Σ. Thus, let S^* be the Poincare dual of this cellular decomposition. Since S has a unique zero cell, S^* has a unique 2-cell. Let n be the number of 1-cells in S (or S^*). Then (Σ, S^*) is obtained by gluing a $2n$-gon

(=the unique 2-cell) according to a matching of its sides preserving orientation. (Note that S can be reconstructed as $(S^*)^*$).

This allows us to link the problem of enumerating filling arc systems with the problem of counting such gluings, which was solved using matrix integrals. Namely, the problem of enumerating filling arc systems is essentially solved modulo one complication: because of conditions (A) and (B) on an arc system, the gluings we will get will be not arbitrary gluings, but gluings which also must satisfy some conditions. Namely, we have

Lemma 5.12. *Let $(\alpha_1, ..., \alpha_n)$ be a system of curves, satisfying the axioms of a filling arc system, except maybe conditions (A) and (B). Then*

(i) $(\alpha_1, ..., \alpha_n)$ satisfies condition (A) iff no edge in the corresponding gluing is glued to a neighboring edge.

(ii) $(\alpha_1, ..., \alpha_n)$ satisfies condition (B) iff no two consequtive edges are glued to another pair of consequtive edges in the opposite order.

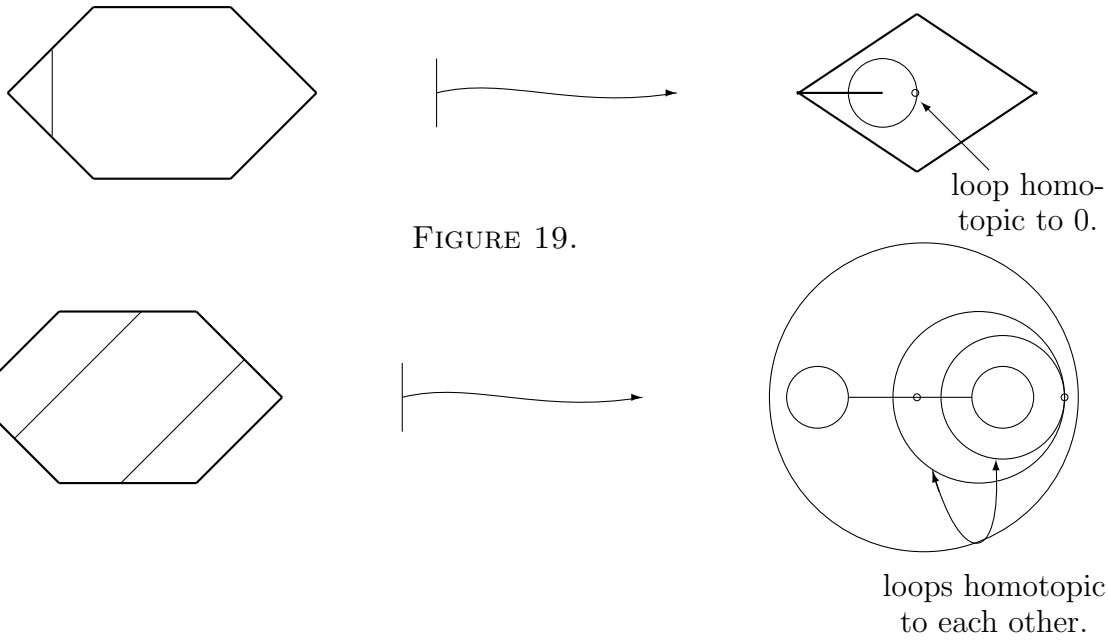


FIGURE 19.

FIGURE 20.

The lemma is geometrically evident, and its proof is obtained by drawing a picture (Fig.19 for (i), Fig.20 for (ii)). Motivated by the lemma, we will refer to the conditions on a gluing in (i) and (ii) also as conditions (A) and (B).

Denote by $\varepsilon_{\mathrm{g}}(n), \mu_{\mathrm{g}}(n), \lambda_{\mathrm{g}}(n)$ the numbers of gluings of a (labeled) $2n$-gon into a surface of genus g, with no conditions, condition (A), and conditions (A),(B), respectively (so $\varepsilon_{\mathrm{g}}(n)$ is the quantity we already studied).

Proposition 5.13. *One has*[12]

$$\chi(\Gamma^1_{\mathrm{g}}) = \sum_n (-1)^{n-1} \frac{\lambda_{\mathrm{g}}(n)}{2n}.$$

Proof. Each filling arc system σ arises from $2n/|\mathrm{Stab}(\sigma)|$ gluings (since the labeling of the polygon does not matter for the resulting surface with an arc system). Thus, the result follows from Quillen's formula. □

5.6. **Computation of $\sum_n (-1)^{n-1} \frac{\lambda_{\mathrm{g}}(n)}{2n}$.** Now it remains to compute the sum on the right hand side. To do this, we will need to link $\lambda_{\mathrm{g}}(n)$ with $\varepsilon_{\mathrm{g}}(n)$, which has already been computed. This is accomplished by the following lemma.

Lemma 5.14. *(i) One has*

$$\varepsilon_{\mathrm{g}}(n) = \sum_i \binom{2n}{i} \mu_{\mathrm{g}}(n-i).$$

(ii) One has

$$\mu_{\mathrm{g}}(n) = \sum_i \binom{n}{i} \lambda_{\mathrm{g}}(n-i).$$

Proof. (i) Let σ be a matching of the sides of a $2n$-gon Δ with labeled vertices. If there is a pair of consecutive edges that are matched, we can glue them to each other to obtain a $2n-2$-gon. Proceeding like this as long as we can, we will arrive at a $2n-2i$-gon Δ_σ, with a matching σ' of its sides which satisfies condition (A). Note that Δ_σ and σ' do not depend on the order in which neighboring edges were glued to each other, and Δ_σ has a canonical labeling by $1, ..., 2n-2i$, in the increasing order of the "old" labels. Now, we claim that each (Δ_σ, σ') is obtained in exactly $\binom{2n}{i}$ ways; this implies the required statement.

Indeed, let us consider the vertices of Δ that ended up in the interior of Δ_σ. They have mapped to i points in the interior (each gluing of a pair of edges produces a new point). Let us call these points $w_1, ..., w_i$, and let ν_j be the smallest label of a vertex of Δ that goes to w_j (where we label the vertices so that the k-th edge connects vertex k with vertex

[12]Note that $\lambda_{\mathrm{g}}(n) = 0$ for almost all n, so this sum is finite.

$k+1$). Then $\nu_1, ..., \nu_i$ is a subset of $\{1, ..., 2n\}$. This subset completely determines the matching σ if (Δ_σ, σ') are given: namely, we should choose a ν_j such that $\nu_j + 1 \neq \nu_k$ for any k, and glue the two edges adjacent to ν_j; then relabel by $1, ..., 2n-2$ the remaining vertices (in increasing order of "old" labels), and continue the step again, and so on. From this it is also seen that any set of ν_j may arise. This proves (i).

(ii) Let σ be a matching of Δ (with labeled edges) which satisfies condition (A) but not necessarily (B). If a_1, a_2 are consecutive edges that are glued to consecutive edges b_2, b_1 in the opposite order, then we may unite a_1, a_2 into a single edge a, and b_2, b_1 into b, and obtain a $2n-2$-gon with a matching. Continuing so as long as we can, we will arrive at a $2n-2i$-gon Δ_σ with a new matching σ', which satisfies conditions (A) and (B). In Δ_σ, each (j-th) pair of edges is obtained for $m_j + 1$ pairs of edges in Δ. Thus, $\sum_{j=1}^{n-i} m_j = i$. Furthermore, for any (Δ_σ, σ') the collection of numbers $m_1, ..., m_{n-i}$ defines (Δ, σ) uniquely, up to deciding which of the $m_1 + 1$ edges constituting the first edge of Δ_σ should be labeled by 1. Thus, each (Δ_σ, σ') arises in the number of ways given by the formula

$$\sum_{m_1, ..., m_{n-i} : \sum_{j=1}^{n-i} m_j = i} (m_1 + 1).$$

It is easy to show (check!) that this number is equal to $\binom{n}{i}$. This proves (ii). □

The completion of the proof of Theorem 5.4 depends now on the following computational lemma.

Lemma 5.15. *Let $\varepsilon(n), \mu(n), \lambda(n)$, $n \geq 0$, be sequences satisfying the equations*

$$\varepsilon(n) = \sum_i \binom{2n}{i} \mu(n-i);$$

$$\mu(n) = \sum_i \binom{n}{i} \lambda(n-i).$$

Assume also that $\varepsilon(n) = \binom{2n}{n} f(n)$, where f is a polynomial such that $f(0) = 0$. Then $\lambda(0) = 0$, $\lambda(n)$ has finitely many nonzero values, and

$$\sum_{n \geq 1} (-1)^{n-1} \frac{\lambda(n)}{2n} = f'(0).$$

Proof. Let us first consider any sequences $\varepsilon(n)$, $\mu(n)$, and $\lambda(n)$ linked by the equations of the lemma. Let $E(z)$, $M(z)$, and $L(z)$ be their generating functions (i.e. $E(z) = \sum_{n\geq 0} \varepsilon(n) z^n$ etc.). We claim that

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} L\left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right).$$

To see this, it suffices to consider the case $\lambda_i = \delta_{ki}$ for some k. In this case,

$$E(z) = \sum_{i,n} \binom{2n}{i} \binom{n-i}{k} z^n = \sum_{p,q\geq 0} \binom{2p+2q}{p} \binom{q}{k} z^{p+q}.$$

But the function

$$F_r(z) := \sum_{p\geq 0} \binom{2p+r}{p} z^p$$

equals

$$F_r(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z}\right)^r,$$

as may be easily seen by induction from the recursion

$$F_r = z^{-1}(F_{r-1} - F_{r-2}),$$

$r \geq 2$. Substituting this in the formula for $E(z)$, one gets (after trivial simplifications)

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} \left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right)^k,$$

as desired.

Now assume that $\varepsilon(n)$ satisfies the polynomiality condition. This means that $E(z) = P(z\partial)|_{z=0} \frac{1}{\sqrt{1-4z}}$, where P is a polynomial with vanishing constant term. To prove our claim, it suffices to consider the case $P(z) = (1+a)^z - 1$, where a is a formal parameter (so $P'(0) = \log(1+a)$); indeed, the coefficients of this formal series are $\binom{z}{j}$, $j \geq 1$, which form a basis in the space of polynomials of z with vanishing constant term. In this case we get

$$E(z) = \frac{1}{\sqrt{1-4(1+a)z}} - \frac{1}{\sqrt{1-4z}}.$$

Hence,

$$L(u) = \frac{1}{1+u}\left(\frac{1}{\sqrt{1-4au(1+u)}} - 1\right).$$

Therefore,

$$\sum_k (-1)^{k-1}\frac{\lambda_k}{2k} = \frac{1}{2}\int_{-1}^{0} L(u)\frac{du}{u} = \frac{1}{2}\sum_{p\geq 1}\binom{2p}{p}(-1)^{p-1}a^p\int_0^1 x^{p-1}(1-x)^{p-1}dx.$$

But $\int_0^1 x^{p-1}(1-x)^{p-1}dx$ is an Euler Beta integral, and it equals $\frac{(p-1)!^2}{(2p-1)!}$. Thus,

$$\sum_k (-1)^{k-1}\frac{\lambda_k}{2k} = \sum_{p\geq 1}(-1)^{p-1}\frac{a^p}{p} = \log(1+a),$$

as desired. □

5.7. **End of proof of Theorem 5.4.** Now we finish the proof of the Harer-Zagier theorem. Recall that using matrix integrals we have proved the formula

$$P_n(x) := \sum_{\mathrm{g}} \varepsilon_{\mathrm{g}}(n)x^{n+1-2\mathrm{g}} = \frac{(2n)!}{2^n n!}\sum_{p\geq 0}\binom{n}{p}2^p\binom{x}{p+1}. \tag{5.1}$$

Let us set $q := n-p$. Then expression (5.1) takes the form

$$P_n(x) = \binom{2n}{n}\sum_{q\geq 0}2^{-q}\binom{n}{q}\frac{n!}{(n-q+1)!}x(x-1)...(x-n+q). \tag{5.2}$$

We claim now that the coefficient of $x^{-2\mathrm{g}}$ ($\mathrm{g}\geq 1$) in the polynomial $\frac{P_n(x)}{x^{n+1}}$ is of the form $\binom{2n}{n}f_{\mathrm{g}}(n)$, where f_{g} is a polynomial. Indeed, contributions to the coefficient of $x^{-2\mathrm{g}}$ come from terms with $q\leq 2\mathrm{g}$ only, so it suffices to check that each of these contributions is as stated. This reduces to checking that the coefficients of the Laurent polynomial $Q(x,n) = (1-\frac{1}{x})...(1-\frac{n}{x})$ are polynomials in n, which vanish at -1 (except, of course, the leading coefficient). To see this, let $Q(x,a) = \frac{\Gamma(x)}{\Gamma(x-a)x^a}$ (this equals to $Q(x,n)$ if $a=n$). This function has an asymptotic Taylor expansion in $\frac{1}{x}$ as $x\to+\infty$ which is obtained from the Stirling asymptotic expansion of $\Gamma(x)$ given by (2.9), and it is easy to show that the coefficients are polynomials in a. Moreover, $Q(x,-1)=1$, which implies the required statement.

Furthermore, we claim that $f_{\mathrm{g}}(0)=0$: again, this follows from the fact that the non-leading coefficients of the expansion of $Q(x,a)$ vanish at $a=0$. But this is clear, since $Q(x,0)=1$.

Thus, we are in a situation where Lemma 5.15 can be applied. So it remains to compute $\sum_{\mathrm{g}\geq 1} f'_{\mathrm{g}}(0)x^{-2\mathrm{g}}$. To do this, observe that the terms with $q>1$ do not contribute to $f'_{\mathrm{g}}(0)$, as they are given by polynomials

of n that are divisible by n^2. So we only need to consider $q = 0$ and $q = 1$. For $q = 1$, the contribution is the value of

$$\frac{1}{2x}(1 - \frac{1}{x})...(1 - \frac{n}{x})$$

at $n = 0$, i.e. it is $\frac{1}{2x}$. For $q = 0$, the contribution is the derivative at 0 with respect to n of $\frac{1}{n+1}(1 - \frac{1}{x})...(1 - \frac{n}{x})$, i.e. it is

$$\frac{d}{da}|_{a=0}\frac{Q(x,a)}{a+1} = -1 + \frac{d}{da}|_{a=0}Q(x, a).$$

Thus, we have (asymptotically)

$$\sum_{\mathrm{g}\geq 1} f'_{\mathrm{g}}(0)x^{-2\mathrm{g}} = \frac{1}{2x} + \frac{d}{da}|_{a=0}Q(x, a) = \frac{1}{2x} + \frac{\Gamma'(x)}{\Gamma(x)} - \log x$$

Now, the asymptotic expansion for Γ'/Γ given by (2.10) implies that $f'_{\mathrm{g}}(0) = -\frac{B_{2\mathrm{g}}}{2\mathrm{g}}$. This completes the proof.

Exercise 5.16. *Prove Theorem 5.8 for* $\mathrm{g} = 1$.

Exercise 5.17. *Let* $\Gamma(N)$ *be the congruence subgroup of* $SL_2(\mathbb{Z})$ *which consists of matrices equal to* 1 *modulo* N.

(a) Show that $\Gamma(N)$ *is free for* $N \geq 3$. *(Hint: consider the action of* $\Gamma(N)$ *on the upper half-plane). Show that* $\Gamma(2)$ *is the direct product of a free group* $\Gamma_+(2)$ *on two generators with* $\mathbb{Z}/2\mathbb{Z}$.

(b) Find the number of generators of $\Gamma(N)$, $N \geq 3$ *which generate it without relations. (Hint: compute* $\chi(\Gamma(N))$*).*

Exercise 5.18. *Let* Γ *be the group defined by the generators* a, b, c *with defining relation* $ab = ba$. *Find the Euler characteristic of* Γ.

Exercise 5.19. *Consider a triangle* Δ *in the hyperbolic plane* $H = \mathbb{C}_+$ *with angles* $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{7}$, *and let* Γ *be the subgroup of* $PSL_2(\mathbb{R})$ *generated by rotations* a, b, c *around the vertices of* Δ *by angles* $2\alpha, 2\beta, 2\gamma$ *respectively.*

(i) Show that H/Γ *is naturally homeomorphic to a sphere glued out of two copies of* Δ, *which can be viewed as an orbifold with three points with nontrivial stabilizers (orders 2,3,7).*

(ii) Compute the Euler characteristic $\chi(\Gamma)$.

(iii) Show that the defining relations for Γ *are*

$$a^2 = 1,\ b^3 = 1,\ c^7 = 1,\ abc = 1$$

(use an orbifold version of van Kampen's theorem).

(iv) Construct a surjective homomorphism $\phi : \Gamma \to PSL_2(\mathbb{F}_7)$.

(v) Show that $\mathrm{Ker}\phi$ *is torsion free and* $H/\mathrm{Ker}\phi$ *is a compact Riemann surface* X *of genus* 3 *with an action of* $PSL_2(\mathbb{F}_7)$. *Identify* X *with the Klein quartic* $x^3y + y^3z + z^3x = 0$ *in* $\mathbb{CP}^2$.

6. Matrix integrals and counting planar diagrams

6.1. **The number of planar gluings.** Let us return to the setting of Section 4. Thus, we have a potential

$$U(x) = \frac{x^2}{2} - \sum_{j\geq 1} g_j \frac{x^j}{j}$$

(with g_j being formal parameters), and consider the matrix integral

$$Z_N(\hbar) = \hbar^{-\frac{N^2}{2}} \int_{\mathfrak{h}_N} e^{-\mathrm{Tr}U(A)} dA.$$

Let $\widehat{Z}_N(\hbar) = Z_N(\hbar/N)$. We have seen that

$$\lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2} = W_\infty,$$

where W_∞ is given by summation over planar fat graphs:

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

In particular, the coefficient of $\prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i}$ is the number of (orientation preserving) gluings of a fat graph of genus zero out of a collection of fat flowers containing n_i i-valent flowers for each i, divided by $\prod_i i^{n_i} n_i!$.

On the other hand, one can compute W_∞ explicitly as a function of g_i by reducing the matrix integral to an integral over eigenvalues, and then using a fundamental fact from the theory of random matrices: the existence of an asymptotic distribution of eigenvalues in the limit $N \to \infty$. This approach allows one to obtain simple closed formulas for the numbers of planar gluings, which are quite nontrivial and for which direct combinatorial proofs were discovered much later.

To illustrate this method, we will restrict ourselves to the case of the potential $U(x) = \frac{x^2}{2} + gx^4$ (so $g_4 = -4g$ and other $g_i = 0$), and set $\hbar = 1$. Then

$$W_\infty = \sum_{n\geq 1} c_n \frac{(-1)^n g^n}{n!},$$

where c_n is a number of connected planar gluings of a set of n 4-valent flowers. In other words, c_n is the number of ways (up to isotopy) to connect n "crosses" in the 2-sphere so that all crosses are connected with each other, all the arms are used, and the connecting lines do not intersect.

Exercise 6.1. *Check by drawing pictures that* $c_1 = 2$, $c_2 = 36$.

Theorem 6.2. *(Brézin, Itzykson, Parisi, Zuber,* [BIPZ]*, 1978). One has*

$$c_n = (12)^n \frac{(2n-1)!}{(n+2)!}.$$

The proof of this theorem (with some omissions) is given in the next subsection.

6.2. **Proof of Theorem 6.2.** We follow the paper [BIPZ]. We will assume that g is a positive real number, and compute the function $W_\infty(g)$ explicitly. The relevant matrix integral has the form

$$\widehat{Z}_N = \int_{\mathfrak{h}_N} e^{-N\mathrm{Tr}(\frac{1}{2}A^2 + gA^4)} dA.$$

Passing to eigenvalues, we get

$$\widehat{Z}_N = \frac{J_N(g)}{J_N(0)},$$

where

$$J_N(g) = \int_{\mathbb{R}^N} e^{-N(\frac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4)} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda. \tag{6.1}$$

Thus, $W_\infty(g) = E(g) - E(0)$, where $E(g) = \lim_{N\to\infty} N^{-2} \log J_N(g)$.

Proposition 6.3. *(Steepest descent principle) $E(g)$ equals the leading coefficient of the asymptotics as $N \to \infty$ of the maximal value of the logarithm of the integrand in* (6.1).

The proposition says, essentially, that the integrand has a sufficiently sharp maximum, so that the leading behavior of the integral can be computed by the steepest descent formula. We note that we cannot apply the steepest descent formula without explanations, since the integral is over a space whose dimension grows as the perturbation parameter $1/N$ goes to 0. In other words, it is necessary to do some estimates which we will omit. We will just mention that for $g = 0$, this result can be derived from the explicit evaluation of the integral using Hermite polynomials (see §4). For the general case, we refer the reader to the book [De].

The logarithm of the integrand

$$K(\lambda_1, ..., \lambda_N) := -N(\tfrac{1}{2}\sum_i \lambda_i^2 + g\sum_i \lambda_i^4) + 2\sum_{i<j} \log|\lambda_i - \lambda_j|$$

has a unique maximum, because it is concave (check it!). This maximum is found by equating the partial derivatives to zero. This yields

$$\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}=N(\tfrac{1}{2}\lambda_i+2g\lambda_i^3). \tag{6.2}$$

Let $\lambda_1<\lambda_2<...<\lambda_N$ be the unique (up to permutations) solution of this system of equations.

Proposition 6.4. *The normalized counting measures $\frac{1}{N}\sum_i\delta(x-\lambda_i)$ converge weakly to a measure $\mu(x)=f(x,g)dx$, where $f(x,g)$ is a continuous function supported on a finite interval $[-2a,2a]$ and differentiable on the interior of this interval.*

For the proof we again refer the reader to [De] (p.132 and later). We note that for $g=0$, by Wigner's semicircle law, $a=1$ and $f(x,0)=\frac{1}{2\pi}\sqrt{4-x^2}$; so $f(x,g)=\frac{1}{2\pi}\sqrt{4-x^2}+O(g)$.

Now our job will be to find the function $f(x,g)$. Passing to the limit in equation (6.2) (which requires justification that we will omit), we get

$$\int_{-2a}^{2a}\frac{1}{y-x}f(x,g)dx=\frac{1}{2}y+2gy^3,\ |y|\leq 2a$$

where the integral is understood in the sense of principal value.

This is a linear integral equation on $f(x,g)$, which can be solved in a standard way. Namely, one considers the analytic function

$$F(y)=\int_{-2a}^{2a}\frac{1}{y-x}f(x,g)dx$$

for y in the complex plane but outside of the interval $[-2a,2a]$. For $y\in[-2a,2a]$, let $F_+(y)$, $F_-(y)$ denote the limits of $F(y)$ from above and below. Then by the Plemelj formula, the integral equation implies

$$\frac{1}{2}(F_+(y)+F_-(y))=\frac{1}{2}y+2gy^3.$$

On the other hand, $F_-(y)=\overline{F_+(y)}$. Hence,

$$\mathrm{Re}F_+(y)=\mathrm{Re}F_-(y)=\frac{1}{2}y+2gy^3.$$

Now set $y:=a(z+z^{-1})$. Then, as y runs through the exterior of $[-2a,2a]$, z runs through the exterior of the unit circle. So the function $G(z):=F(y)$ is analytic on the outside of the unit circle, with decay at infinity, and

$$\mathrm{Re}G(z)=\frac{1}{2}a(z+z^{-1})+2ga^3(z+z^{-1})^3$$

when $|z| = 1$. This implies that $G(z)$ is twice the sum of all negative degree terms of this Laurent polynomial. In other words, we have

$$G(z) = 4ga^3 z^{-3} + (a + 12ga^3)z^{-1}.$$

This yields

$$F(y) = \frac{1}{2}y + 2gy^3 - \left(\frac{1}{2} + 4ga^2 + 2gy^2\right)\sqrt{y^2 - 4a^2}.$$

Now $f(y, g)$ is found as the jump of F:

$$f(y, g) = \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2 + 2gy^2\right)\sqrt{4a^2 - y^2}.$$

It remains to find a in terms of g. We have $yF(y) \to 1$, $y \to \infty$ (as $\int f(x, g)dx = 1$), hence $zG(z) \to 1/a$, $z \to \infty$. This yields

$$\frac{1}{a} = a + 12ga^3,$$

or

$$12ga^4 + a^2 - 1 = 0.$$

This allows one to determine a uniquely:

$$a = \left(\frac{(1 + 48g)^{1/2} - 1}{24g}\right)^{1/2}.$$

Now let us calculate $E(g)$. It follows from the above that

$$E(g) = \int_{-2a}^{2a}\int_{-2a}^{2a} \log|x-y| f(x, g)f(y, g)dxdy - \int_{-2a}^{2a} (\tfrac{1}{2}x^2 + gx^4)f(x, g)dx.$$

On the other hand, let us integrate the integral equation defining $f(x, g)$ with respect to y (from 0 to u). Then we get

$$2\int_{-2a}^{2a} (\log|x - u| - \log|x|)f(x, g)dx = \tfrac{1}{2}u^2 + gu^4.$$

Substituting this into the expression for $E(g)$, we get

$$E(g) = \int_{-2a}^{2a} (\log|u| - \tfrac{1}{4}u^2 - \tfrac{1}{2}gu^4)f(u, g)du.$$

Since $f(u, g)$ is known, this integral can be computed. In fact, can be expressed via elementary functions, and after calculations we get

$$E(g) - E(0) = \log a - \frac{1}{24}(a^2 - 1)(9 - a^2).$$

Substituting here the expression for a, after a calculation one finally gets:

$$E(g) - E(0) = \sum_{k=1}^{\infty} (-12g)^k \frac{(2k-1)!}{k!(k-2)!}.$$

This implies the required formula for c_n.

7. Quantum mechanics

So far we have considered quantum field theory with 0-dimensional spacetime (to make a joke, one may say that the dimension of the space is -1). In this section, we will move closer to actual physics: we will consider 1-dimensional spacetime, i.e. the dimension of the space is 0. This does not mean that we will study motion in a 0-dimensional space (which would be really a pity) but just means that we will consider only point-like quantum objects (particles) and not extended quantum objects (fields). In other words, we will be in the realm of quantum mechanics.

7.1. **The path integral in quantum mechanics.** Let $U(q)$ be a smooth function on the real line (the potential). We will assume that $U(0) = 0$, $U'(0) = 0$, and $U''(0) = m^2$, where $m > 0$.

Remark 7.1. In quantum field theory the parameter m in the potential is called the *mass* parameter. To be more precise, in classical mechanics it has the meaning of frequency ω of oscillations. However, in quantum theory thanks to Einstein frequency is identified with energy ($E = \hbar\omega/2\pi$), while in relativisitic theory energy is identified with mass (again thanks to Einstein, $E = mc^2$).

We want to construct the theory of a quantum particle moving in the potential field $U(q)$. According to what we discussed before, this means that we want to give sense to and to evaluate the normalized correlation functions

$$\langle q(t_1) \dots q(t_n) \rangle := \frac{\int q(t_1) \dots q(t_n) e^{\frac{iS(q)}{\hbar}} Dq}{\int e^{\frac{iS(q)}{\hbar}} Dq},$$

where $S(q) = \int \mathcal{L}(q) dt$, and $\mathcal{L}(q) = \frac{\dot{q}^2}{2} - U(q)$.

As we discussed, such integrals cannot be handled rigorously by means of measure theory if $\hbar$ is a positive number; so we will only define these path integrals "in perturbation theory", i.e. as formal series in $\hbar$.

Before giving this (fully rigorous) definition, we will explain the motivation behind it. We warn the reader that this explanation is heuristic and involves steps which are mathematically non-rigorous (or "formal" in the language of physicists).

7.2. **Wick rotation.** In Section 1 we discussed path integrals with imaginary exponential (quantum mechanics), as well as real exponential (Brownian motion). If $\hbar$ is a number, then the integrals with imaginary exponential cannot be defined measure-theoretically. Therefore,

people study integrals with real exponential (which can be rigorously defined), and then perform a special analytic continuation procedure called the *Wick rotation*.

In our formal setting ($\hbar$ is a formal parameter), one can actually define the integrals in both the real and the imaginary case. Still, the real case is a bit easier, and thus the Wick rotation is still useful. Besides, the Wick rotation is very important conceptually. Therefore, while it is not technically necessary, we start with introducing the Wick rotation here.

Namely, let us denote $\langle q(t_1)...q(t_n)\rangle$ by $\mathcal{G}_n^M(t_1,...,t_n)$, and "formally" make a change of variable $\tau = it$ in the formula for $\mathcal{G}_n^M(t_1,...,t_n)$. Let $q(t) := q_*(\tau)$. Then, taking into account that $d\tau = idt$, $\frac{dq}{dt} = i\frac{dq_*}{d\tau}$, we get

$$\mathcal{G}_n^M(t_1,...,t_n) = \frac{\int q_*(\tau_1)\dots q_*(\tau_n)e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}{\int e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}.$$

This shows that

$$\mathcal{G}_n^M(t_1,...,t_n) = \mathcal{G}_n^E(it_1,...,it_n),$$

where

$$\mathcal{G}_n^E(t_1,...,t_n) := \frac{\int q(t_1)\dots q(t_n)e^{-\frac{S_E(q)}{\hbar}}Dq}{\int e^{-\frac{S_E(q)}{\hbar}}Dq}.$$

with $S_E(q) = \int \mathcal{L}_E(q)d\tau$, and $\mathcal{L}_E(q) = \frac{\dot{q}^2}{2} + U(q)$ (i.e. $\mathcal{L}_E$ is obtained from $\mathcal{L}$ by replacing U with $-U$).

This manipulation certainly does not make rigorous sense, but it motivates the following definition.

Definition 7.2. The function $\mathcal{G}_n^M(t_1,...,t_n)$ ($t_i \in \mathbb{R}$) is the analytic continuation of the function $\mathcal{G}_n^E(\tau_1,...,\tau_n)$ from the point $(t_1,...,t_n)$ to the point $(it_1,...,it_n)$ along the path $\theta \mapsto e^{i\theta}(t_1,...,t_n)$, $0 \le \theta \le \pi/2$.

Of course, this definition will only make sense if we define the function $\mathcal{G}_n^E(t_1,...,t_n)$ and show that it admits the required analytic continuation. This will be done below.

Remark 7.3. (On the terminology.) The function $\mathcal{G}_n^M(t_1,...,t_n)$ is called the *Minkowskian* (time ordered) correlation function, while the function $\mathcal{G}_n^E(t_1,...,t_n)$ is called the *Euclidean* correlation function (hence the notation). This terminology will be explained later, when we consider relativistic field theory.

From now on, we will mostly deal with Euclidean correlation functions, and therefore will omit the superscript E when there is no danger of confusion.

7.3. **Definition of Euclidean correlation functions.** Now our job is to define the Euclidean correlation functions $\mathcal{G}_n(t_1, ..., t_n)$. Our strategy (which will also be used in field theory) will be as follows. Recall that if our integrals were finite dimensional then by Feynman's theorem the expansion of the correlation functions in $\hbar$ would be given by a sum of amplitudes of Feynman diagrams. So, in the infinite dimensional case, we will use the sum over Feynman diagrams as a *definition* of correlation functions.

More specifically, because of the conditions on U we have an action functional without constant and linear terms in q, so that the correlation function $\mathcal{G}_n(t_1, ..., t_n)$ should be given by the sum

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\Gamma \in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_n), \tag{7.1}$$

where $G^*_{\geq 3}(n)$ is defined in Remark 3.7. Thus, we should make sense of (=define) the amplitudes F_Γ in our situation. For this purpose, we need to define the following objects.

1. The space V.
2. The form B on V which defines B^{-1} on V^*.
3. The tensors corresponding to non-quadratic terms in the action.
4. The covectors ℓ_i.

It is clear how to define these objects naturally. Namely, V should be a space of functions on $\mathbb{R}$ with some decay conditions. There are many choices for V, which do not affect the final result; for instance, a good choice (which we will make) is the space $C_0^\infty(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$. Thus V^* is the space of generalized functions on $\mathbb{R}$. Note that V is equipped with the inner product $(f, g) = \int_{\mathbb{R}} f(x)g(x)dx$.

The form B, by analogy with the finite dimensional case, should be twice the quadratic part of the action. In other words,

$$B(q, q) = \int (\dot{q}^2 + m^2 q^2)dt = (Aq, q),$$

where A is the operator

$$A = -\frac{d^2}{dt^2} + m^2.$$

This means that $B^{-1}(f, f) = (A^{-1}f, f)$.

The operator A^{-1} is an integral operator, with kernel

$$K(x, y) = G(x - y),$$

where $G(x)$ is the Green's function of A, i.e. the fundamental (decaying at infinity) solution of the differential equation

$$(AG)(x) = \delta(x).$$

It is straightforward to find that

$$G(x) = \frac{e^{-m|x|}}{2m}.$$

(thus B^{-1} is actually defined not on the whole V^* but on a dense subspace of V^*).

Remark 7.4. Here we already see the usefulness of the Wick rotation. Namely, the spectrum of A (interpreted as usual as a self-adjoint unbounded operator on $L^2(\mathbb{R})$) is $[m^2, +\infty)$, so it is invertible and the inverse is bounded. However, if we did not make a Wick rotation, we would deal with the operator $A' = -\frac{d^2}{dt^2} - m^2$, whose spectrum is $[-m^2, +\infty)$, i.e. contains 0, so this operator does not have a bounded inverse.

To make sense of the cubic and higher terms in the action as tensors, consider the decomposition of U in the (asymptotic) Taylor series at $x = 0$:

$$U(x) = \frac{m^2x^2}{2} - \sum_{n\geq 3} \frac{g_n x^n}{n!}.$$

This shows that cubic and higher terms in the action have the form

$$B_r(q, q, , ..., q) = \int q^r(t)dt.$$

Thus $B_r(q_1, ..., q_r)$ is an element of $(S^rV)^*$ given by the generalized function $\delta_{t_1=...=t_r}$ (the delta function of the diagonal).

Finally, the functionals ℓ_i are given by $\ell_i(q) = q(t_i)$, so $\ell_i = \delta(t - t_i)$.

This leads to the following Feynman rules of defining the amplitude of a diagram Γ.

1. To the i-th external vertex of Γ assign the number t_i.
2. To each internal vertex j of Γ, assign a variable s_j.
3. On each internal edge connecting vertices j and j', write the Green's function $G(s_j - s_{j'})$.
4. On each external edge connecting i and j write $G(t_i - s_j)$.
5. On each external edge connecting i and i' write $G(t_i - t_{i'})$.
6. Let $G_\Gamma(\mathbf{t}, \mathbf{s})$ be the product of all these functions.
7. Let $F_\Gamma(\ell_1, ..., \ell_n) := \prod_j g_{v(j)} \int G_\Gamma(\mathbf{t}, \mathbf{s})d\mathbf{s}$, where $v(j)$ is the valency of j.

We are finally able to give the following definition.

Definition 7.5. The function $\mathcal{G}_n(t_1, ..., t_n)$ is defined by formula (7.1).

Remark 7.6. Note that the integrals defining F_Γ are convergent since the integrand always decays exponentially at infinity. It is, however, crucial that we consider only graphs without components having no external vertices; for example, if Γ has a single 4-valent vertex connected to itself by two loops (Fig.21) then the amplitude integral involves $\int_{\mathbb{R}} G(0)^2 ds$, which is obviously divergent.

With this definition, the function $\mathcal{G}_n(t_1, ..., t_n)$ is a Laurent series in $\hbar$ whose coefficients are symmetric functions of $t_1, ..., t_n$ given by linear combinations of explicit (and convergent) finite dimensional integrals. Furthermore, it is easy to see that these integrals are in fact computable in elementary functions, i.e. are (in the region $t_1 \geq ... \geq t_n$) linear combinations of products of functions of the form $t_i^r e^{at_i}$. This implies the existence of the analytic continuation required in the Wick rotation procedure.

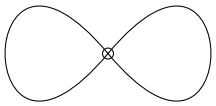


FIGURE 21.

Remark 7.7. As in the finite dimensional case, an alternative setting for making this definition is to assume that g_i are formal parameters. In this case, $\hbar$ can be given a numerical value, e.g. $\hbar = 1$, and the function $\mathcal{G}_n$ will be a well defined power series in $g_3, g_4, ...$

As an example consider a free massive theory, i.e., a *harmonic oscillator*: $U(q) = \frac{m^2 q^2}{2}$. In this case, there are no internal vertices, hence we get

Proposition 7.8. *(Wick's theorem) One has $\mathcal{G}_n(t_1, ..., t_n) = 0$ if n is odd, and*

$$\mathcal{G}_{2k}(t_1, ..., t_{2k}) = \hbar^k \sum_{\sigma \in \Pi_k} \prod_{i \in \{1,...,2k\}/\sigma} G(t_i - t_{\sigma(i)}).$$

In particular, $\mathcal{G}_2(t_1, t_2) = \hbar G(t_1 - t_2)$. In other words, $\mathcal{G}_2(t_1, t_2)$ is (proportional to) the Green's function. Motivated by this, physicists often refer to all correlation functions of a quantum field theory as *Green's functions*.

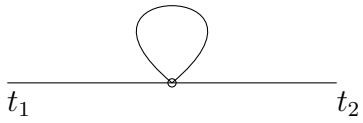


FIGURE 22.

Example 7.9. Consider the potential $U(q) = \frac{m^2q^2}{2} - \frac{gq^4}{24}$, and set $\hbar = 1$. In this case, let us calculate the 2-point correlation function modulo g^2. In other words, we have to compute the coefficient of g in this function. Thus we have to consider Feynman diagrams with two external edges and one internal vertex. Such a diagram Γ is unique: it consists of one edge with a loop attached in the middle (Fig. 22). This diagram has automorphism group $\mathbb{Z}/2$. The amplitude of this diagram is

$$F_\Gamma = g\int_{\mathbb{R}} G(s,t_1)G(s,t_2)G(s,s)ds = \frac{g}{8m^3}\int_{\mathbb{R}} e^{-m(|s-t_1|+|s-t_2|)}ds.$$

Because of symmetry in t_1 and t_2, we may assume that $t_1 \geq t_2$. Splitting the integral in a sum of three integrals, over $(-\infty, t_2], [t_2, t_1]$, and $[t_1, \infty)$, respectively we get:

$$F_\Gamma = \frac{g}{8m^3}\left(2\int_0^\infty e^{-m(2s+|t_1-t_2|)}ds + |t_1-t_2|e^{-m|t_1-t_2|}\right) = \frac{g}{8m^4}e^{-m|t_1-t_2|}(1+m|t_1-t_2|).$$

Thus

$$\mathcal{G}_2(t_1,t_2) = \widetilde{G}(t_1-t_2),$$

where

$$\widetilde{G}(t) := \tfrac{1}{2m}e^{-m|t|} + \tfrac{g}{16m^4}e^{-m|t|}(1+m|t|) + O(g^2).$$

This expression is called the *1-loop approximation* to the 2-point function, because it comes from 0-loop and 1-loop Feynman diagrams.

Remark 7.10. Here we are considering quantum mechanics of a single 1-dimensional particle. However, everything generalizes without difficulty to the case of an n-dimensional particle or system of particles (i.e., to path integrals over the space of vector-valued, rather than scalar, functions of one variable). Indeed, if q takes values in a Euclidean space V then the quadratic part of the Lagrangian is of the form $\frac{1}{2}(\dot{q}^2 - M(q))$, where M is a positive definite quadratic form on V. Diagonalizing M, we may assume that the quadratic part of the

Lagrangian looks like $\frac{1}{2}\sum_i(\dot{q_i}^2 - m_i^2 q_i^2)$, which corresponds to a system of independent harmonic oscillators. Thus in quantum theory the propagator will be the diagonal matrix with diagonal entries $\frac{e^{-m_i|t-s|}}{2m_i}$, and the correlation functions can be defined by the usual Feynman diagram procedure.

7.4. **Connected correlation functions.** Let $\mathcal{G}_n^c(t_1, ..., t_n)$ be the *connected correlation (or Green) functions*, defined by the sum of the same amplitudes as $\mathcal{G}_n(t_1, ..., t_n)$ but taken over connected Feynman diagrams only. It is clear that

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\{1,...,n\}=S_1\sqcup...\sqcup S_k} \prod \mathcal{G}^c_{|S_i|}(t_j; j \in S_i).$$

For example, $\mathcal{G}_2(t_1, t_2) = \mathcal{G}_2^c(t_1, t_2) + \mathcal{G}_1^c(t_1)\mathcal{G}_1^c(t_2)$, etc. Thus, to know the correlation functions, it is sufficient to know the connected correlation functions.

Example 7.11. In a free theory ($U = \frac{m^2q^2}{2}$, the harmonic oscillator), all connected Green's functions except $\mathcal{G}_2$ vanish.

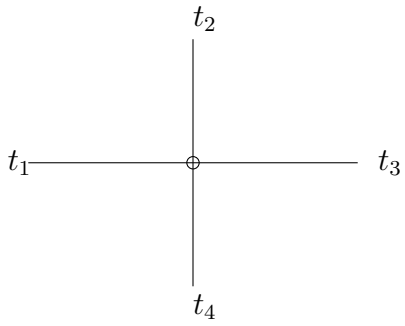


FIGURE 23.

Example 7.12. Let us compute the connected 4-point function in the theory associated to the quartic potential $U = \frac{m^2q^2}{2} - \frac{gq^4}{4}$ as above, modulo g^2. This means, we should compute the contribution of connected Feynman diagrams with one internal vertex and 4 external edges. Such a diagram Γ is unique – it is the cross (with one internal vertex), Fig. 23. This diagram has no nontrivial automorphisms. Thus,

$$\mathcal{G}_4^c(t_1, t_2, t_3, t_4) = g\int_{\mathbb{R}} G(t_1 - s)G(t_2 - s)G(t_3 - s)G(t_4 - s)ds + O(g^2).$$

It is elementary to compute this integral; we leave it as an exercise.

7.5. **The clustering property.** Note that the Green's function $G(t)$ goes to zero at infinity. This implies the following *clustering property* of the correlation functions of the free theory:

$$\lim_{z\to\infty} \mathcal{G}_n(t_1,...,t_r,t_{r+1}+z,...,t_n+z) = \mathcal{G}_r(t_1,...,t_r)\mathcal{G}_{n-r}(t_{r+1}...t_n).$$

Moreover, it is easy to show that the same is true in the interacting theory (i.e. with potential) in each degree with respect to $\hbar$ (check it!). The clustering property can be more simply expressed by the equation

$$\lim_{z\to\infty} \mathcal{G}_n^c(t_1,...,t_r,t_{r+1}+z,...,t_n+z) = 0.$$

This property has a physical interpretation: processes distant from each other are almost statistically independent. Thus it can be viewed as a necessary condition of a quantum field theory to be "physically meaningful".

Remark 7.13. Nevertheless, there exist theories (e.g. so called *topological quantum field theories*) which do not satisfy the clustering property but are interesting both form a physical and mathematical point of view (see Subsection 10.2 below).

7.6. **The partition function.** Let $J(t)dt$ be a compactly supported measure on the real line. Consider the "partition function with external current J", which is the formal expression

$$Z(J) = \int e^{\frac{-S_E(q)+(J,q)}{\hbar}} Dq.$$

Then we have a formal equality

$$\frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int_{\mathbb{R}^n} \mathcal{G}_n(t_1,...,t_n)J(t_1)...J(t_n)dt_1...dt_n,$$

which, as before, we will use as the definition of $Z(J)/Z(0)$. So the knowledge of $Z(J)/Z(0)$ is equivalent to the knowledge of all the Green's functions (in other words, $Z(J)/Z(0)$ is their generating function). Furthermore, as in the finite dimensional case, we have

Proposition 7.14. *One has*

$$W(J) := \log \frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^c(t_1,...,t_n)J(t_1)...J(t_n)dt_1...dt_n$$

(i.e. W is the generating function of connected Green's functions)

The proof of this proposition is the same as in the finite dimensional case.

Remark 7.15. The statement of the proposition is equivalent to the relation between usual and connected Green's functions given in the previous subsection.

Remark 7.16. The fact that we can only define amplitudes of graphs whose all components have at least one 1-valent vertex (see above) means that we actually cannot define either $Z(0)$ or $Z(J)$ but can only define their ratio $Z(J)/Z(0)$.

Like in the finite dimensional case, we have an expansion

$$W(J) = \hbar^{-1}W_0(J) + W_1(J) + \hbar W_2(J) + ...,$$

where W_j are the j-loop contributions (in particular, W_0 is given by a sum over trees). Furthermore, we have explicit formulas for W_0 and W_1, analogously to the finite dimensional case.

Proposition 7.17. *One has*

$$W_0(J) = -S_E(q_J) + (q_J, J),$$

where q_J is the extremal of the functional $S_E^J(q) := S_E(q) - (q, J)$ which decays at infinity. Furthermore,

$$W_1(J) = -\frac{1}{2}\log\det L_J,$$

where L_J is the linear operator on V such that

$$d^2S_E^J(q_J)(f_1, f_2) = d^2S_E^0(0)(L_Jf_1, f_2).$$

The proof of this proposition, in particular, involves showing that q_J is well defined and that $\det L_J$ exists. It is analogous to the proof of the same result in the finite dimensional case which is given in Subsection 3.7 (to be precise, we gave a proof only in the 0-loop case; but in the 1-loop case, the proof is similar). Therefore we will not give this proof; rather, we will illustrate the statement by an example.

Example 7.18. Let U be the above quartic potential $\frac{m^2q^2}{2} + \frac{gq^4}{2}$ (in which for convenience we change the sign and normalization of the quartic term) and $J(t) = a\delta(t)$. In this case,

$$S_E^J(q) = \int (\tfrac{\dot{q}^2}{2} + U(q))dt - aq(0).$$

The Euler-Lagrange equation has the form

$$\ddot{q} = m^2q + 2gq^3 - a\delta(t).$$

Thus, the function q_J is continuously glued from two solutions q_+, q_- of the nonlinear differential equation

$$\ddot{q} = m^2q + 2gq^3$$

on $(-\infty, 0]$ and $[0, \infty)$, with jump of derivative at 0 equal to $-a$.

The solutions q_+, q_- are required to decay at infinity, so they must be solutions of zero energy:

$$E = \frac{\dot{q_\pm}^2}{2} - U(q_\pm) = 0.$$

Thus, by the standard formula for solutions of Newton's equation, they are defined by the equality

$$t - t_\pm = \int \frac{dq}{\sqrt{2U(q)}} = \int \frac{dq}{mq\sqrt{1+\frac{gq^2}{m^2}}} = \frac{1}{2m} \log \frac{\sqrt{1+\frac{gq^2}{m^2}} - 1}{\sqrt{1+\frac{gq^2}{m^2}} + 1}.$$

After a calculation one gets

$$q_J(t) = \frac{2mg^{-\frac{1}{2}}}{C^{-1}e^{m|t|} - Ce^{-m|t|}},$$

where C is the solution of the equation

$$\frac{C(1+C^2)}{(1-C^2)^2} = \frac{ag^{\frac{1}{2}}}{4m^2}$$

which is given by a power series in a with zero constant term. From this it is elementary (but somewhat lengthy) to compute $W_0 = -S_E^J(q_J)$.

Now, the operator L_J is given by the formula

$$L_J = 1 + \frac{gA^{-1} \circ q_J(t)^2}{2},$$

where $A = -\frac{d^2}{dt^2} + m^2$. Thus $\det L_J$ makes sense. Indeed, the operator $A^{-1} \circ q_J(t)^2$ is an integral operator given by the kernel

$$K_J(x,y) := \frac{e^{-m|x-y|} q_J(y)^2}{2m},$$

which decays exponentially at infinity; hence the determinant of the operator $1 + \frac{gA^{-1} \circ q_J(t)^2}{2}$ is well defined.

Remark 7.19. In these computations, g, a were formal variables, but the above computations in fact make sense for real numerical values of these variables as long as $ga^2 + m^2 > 0$.

7.7. **1-particle irreducible Green's functions.** Let $\mathcal{G}_n^{1PI}(t_1, ..., t_n)$ denote *1-particle irreducible Green's functions*, i.e. those defined by the sum of the same amplitudes as the usual Green's functions, but taken only over 1-particle irreducible Feynman graphs. Define also the *amputated* 1-particle irreducible Green's function: $\mathcal{G}_n^{1PIa} := A^{\otimes n}\mathcal{G}_n^{1PI}$

(it is defined by the same sum of amplitudes, except that instead of $G(t_i - s_j)$ for external edges, we write $\delta(t_i - s_j)$).

Let $S_{\text{eff}}(q)$ be the generating function of $\mathcal{G}_n^{1PIa}$ i.e.,

$$S_{\text{eff}}(q) = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^{1PIa}(t_1, ..., t_n) q(t_1) ... q(t_n) dt_1 ... dt_n.$$

Proposition 7.20. *The function $W(J) = \log(Z(J)/Z(0))$ is the Legendre transform of $S_{\text{eff}}(q)$, i.e. it equals $-S_{\text{eff}}(\widetilde{q}_J) + (J, \widetilde{q}_J)$, where $\widetilde{q}_J$ is the extremal of $-S_{\text{eff}}(q) + (J, q)$ decaying at infinity.*

The proof of this proposition is the same as in the finite dimensional case. The proposition shows that in order to know the Green's functions, it "suffices" to know amputated 1-particle irreducible Green's functions (the generating function of usual Green's functions can be reconstructed from that for 1PI Green's functions by taking the Legendre transform and exponentiation). Which is a good news, since there are a lot fewer 1PI diagrams than general connected diagrams.

7.8. **Momentum space integration.** We saw that the amplitude of a Feynman diagram is given by an integral over the space of dimension equal to the number of internal vertices. This is sometimes inconvenient, since even for tree diagrams such integrals can be rather complicated. However, it turns out that if one passes to Fourier transforms then Feynman integrals simplify and in particular the number of integrations for a connected diagram becomes equal to the number of loops (so for tree diagrams we have no integrations at all).

Namely, we will proceed as follows. Instead of the time variable t we will consider the dual energy variable E. A function $q(t)$ with compact support will be replaced by its Fourier transform $\widehat{q}(E)$. Then, by Plancherel's theorem, for real functions q_1, q_2, we have

$$(q_1, q_2) = \int_{\mathbb{R}} q_1(t) q_2(t) dt = \int_{\mathbb{R}} \widehat{q_1}(E) \overline{\widehat{q_2}(E)} dE = \int_{\mathbb{R}} \widehat{q_1}(E) \widehat{q_2}(-E) dE.$$

This implies that the propagator is given by

$$B^{-1}(f, f) = \int_{\mathbb{R}} \frac{1}{E^2 + m^2} \widehat{f}(E) \widehat{f}(-E) dE.$$

The vertex tensors standing at k-valent vertices were $\delta_{s_1 = ... = s_k}$, so they will be replaced by $\delta_{Q_1 + ... + Q_k = 0}$, where Q_i are dual variables to s_i.

Remark 7.21. (On terminology) Physicists refer to the time variables t_i, s_j as *position variables*, and to energy variables E_i, Q_k as *momentum variables*, since in relativistic mechanics (which is the setting we will deal with when we study field theory) there is no distinction between

time and position and between energy and momentum (due to the action of the Lorentz group).

This shows that the Feynman rules "in momentum space" for a given connected Feynman diagram Γ with n external vertices are as follows.

1. Orient the diagram Γ, so that all external edges are oriented inwards.

2. Assign variables E_i to external edges, and variables Q_j to internal ones. These variables are subject to the linear equations of "the first Kirchhoff law": at every internal vertex, the sum of the variables corresponding to the incoming edges equals the sum of those corresponding to the outgoing edges. Let $Y(\mathbf{E})$ be the space of solutions $\mathbf{Q}$ of these equations (it depends on Γ, but we will not write the dependence explicitly). It is easy to show that this space is nonempty only if $\sum_i E_i = 0$, and in that case $\dim Y(\mathbf{E})$ equals the number of loops of Γ (show this!).

3. On each external edge, write $\frac{1}{E_i^2+m^2}$, and on each internal edge, write $\frac{1}{Q_k^2+m^2}$. Let $\phi_\Gamma(\mathbf{E},\mathbf{Q})$ be the product of all these functions.

4. Define the *momentum space amplitude* of Γ to be the distribution $\widehat{F}_\Gamma(\mathbf{E})$:

$$\widehat{F}_\Gamma(E_1,...,E_n) = \prod_j g_{v(j)} \int_{Y(\mathbf{E})} \phi_\Gamma(\mathbf{E},\mathbf{Q}) d\mathbf{Q} \cdot \delta(E_1+...+E_n) d\mathbf{E},$$

supported on the hyperplane $\sum_i E_i = 0$. It is clear that this distribution is independent on the orientation of Γ.

Remark 7.22. Here we must specify the normalization of the (translation-invariant) Lebesgue measure $d\mathbf{Q}$ on the space $Y(\mathbf{E})$. It is defined in such a way that the volume of $Y(\mathbf{E})/Y_{\mathbb{Z}}(0)$ is 1, where $Y_{\mathbb{Z}}(0)$ is the set of integer elements in $Y(0)$. So if $T \subset \Gamma$ is a spanning tree then in the coordinates $\{Q_e, e \notin T\}$ on $Y(\mathbf{E})$, we have $d\mathbf{Q} = \prod_{e\notin T} dQ_e$.

Now we have

Proposition 7.23. *The Fourier transform of the function $F_\Gamma(\delta_{t_1},...,\delta_{t_n})$ is $\widehat{F}_\Gamma(E_1,...,E_n)$. Hence, the Fourier transform of the connected Green's function is*

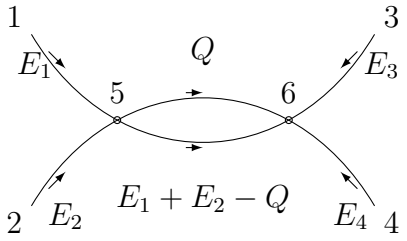
$$\widehat{\mathcal{G}}_n^c(E_1,...,E_n) = \sum_{\Gamma\in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \widehat{F}_\Gamma(E_1,\dots,E_n). \tag{7.2}$$

The proof of the proposition is straightforward.

To illustrate the proposition, consider an example.

Example 7.24. The connected 4-point function for the quartic potential modulo g^2 in momentum space looks like:

$$\widehat{\mathcal{G}}_4^c(E_1,E_2,E_3,E_4) = g\prod_{i=1}^4 \frac{1}{E_i^2+m^2}\delta(\sum_i E_i)d\mathbf{E} + O(g^2).$$



Γ.

FIGURE 24.

Example 7.25. Let us compute the 1PI 4-point function in the same problem, but now modulo g^3. Thus, in addition to the above, we need to compute the g^2 coefficient, which comes from 1-loop diagrams. There are three such diagrams, differing by permutation of external edges. One of these diagrams is as follows: it has external vertices $1,2,3,4$ and internal ones $5,6$ such that $1,2$ are connected to 5, $3,4$ to 6, and 5 and 6 are connected by two edges (Fig.24). This diagram has the symmetry group $\mathbb{Z}/2$, so its contribution is

$$\frac{g^2}{2}\left(\int_{\mathbb{R}} \frac{dQ}{(Q^2+m^2)((E_1+E_2-Q)^2+m^2)}\right)\prod_{i=1}^4 \frac{1}{E_i^2+m^2}\delta(\sum_i E_i)d\mathbf{E}.$$

The integral inside is easy to compute, for example, by residues. This yields

$$\widehat{\mathcal{G}}_4^c(E_1,E_2,E_3,E_4) =$$
$$g\prod_{i=1}^4 \frac{1}{E_i^2+m^2}\left(1+\frac{\pi g}{m}\sum_{i=2}^4 \frac{1}{(E_1+E_i)^2+4m^2}\right)\delta(\sum_i E_i)d\mathbf{E} + O(g^3)$$

(this is symmetric in the E_1,E_2,E_3,E_4 since when $\sum_i E_i = 0$ then for distinct i,j,k,ℓ one has $(E_i+E_j)^2 = (E_k+E_\ell)^2$).

7.9. **The Wick rotation in momentum space.** To obtain the correlation functions of quantum mechanics, we should, after computing them in the Euclidean setting, Wick rotate them back to the Minkowski setting. Let us do it at the level of Feynman integrals in momentum space. (We could do it in position space as well, but it is instructive for the future to do it in momentum space, since in higher dimensional field

theory which we will discuss later, the momentum space representation is more convenient).

Consider the Euclidean propagator

$$\frac{1}{E^2+m^2} = \int_{\mathbb{R}} G(t)e^{iEt}dt,$$

where G is the Green's function. When we do analytic continuation back to the Minkowski setting, we must replace in the correlation functions the time variable t with $e^{i\theta}t$, where θ varies from 0 to $\frac{\pi}{2}$. In particular, the Green's function $G(t)$ must be replaced by $G(e^{i\theta}t)$. So we must consider

$$\int_{\mathbb{R}} G(e^{i\theta}t)e^{iEt}dt = e^{-i\theta}\int_{\mathbb{R}} G(t)e^{ie^{-i\theta}Et}dt = \frac{e^{-i\theta}}{e^{-2i\theta}E^2+m^2}.$$

As $\theta \to \frac{\pi}{2}$, this function tends (as a distribution) to the function $\lim_{\varepsilon\to 0+}\frac{i}{E^2-m^2+i\varepsilon}$. For brevity the limit sign is usually dropped and this distribution is written as $\frac{i}{E^2-m^2+i\varepsilon}$.

We see that in order to compute the correlation functions in momentum space in the Minkowski setting, we should use the same Feynman rules as in the Euclidean setting except that the propagator put on the edges should be

$$\frac{i}{E^2-m^2+i\varepsilon}.$$

For instance, the contribution of the diagram in Fig.24 is

$$-\frac{g^2}{2}\left(\int_{\mathbb{R}}\frac{dQ}{(Q^2-m^2+i\varepsilon)((E_1+E_2-Q)^2-m^2+i\varepsilon)}\right)\prod_{j=1}^{4}\frac{1}{E_j^2-m^2+i\varepsilon}\delta(\sum_i E_j)d\mathbf{E}.$$

7.10. **Quantum mechanics on the circle.** It is reasonable (at least mathematically) to consider Euclidean quantum mechanical path integrals in the case when the time axis has been replaced with a circle of length L, i.e. $t \in \mathbb{R}/L\mathbb{Z}$ (this corresponds to a Brownian particle in a potential field conditioned to return to the original position in a certain time L). In this case, the theory is the same, except the Green's function $G(t)$ is replaced by the periodic solution $G_L(t)$ of the equation $(-\frac{d^2}{dt^2}+m^2)f = \delta(t)$ on the circle. This solution has the form

$$G_L(t) = \sum_{k\in\mathbb{Z}} G(t-kL) = \frac{e^{-m(t-\frac{L}{2})}+e^{-m(\frac{L}{2}-t)}}{2m(e^{\frac{mL}{2}}-e^{-\frac{mL}{2}})},\ 0\le t\le L. \tag{7.3}$$

We note that in the case of a circle, there is no problem with graphs without external edges (as integral over the circle of a constant function

is convergent), and hence one may define not only correlation functions (i.e. $Z(J)/Z(0)$), but also $Z(0)$ itself. Namely, let

$$U(q) = \frac{m^2 q^2}{2} + \sum_{n \geq 3} \frac{g_n q^n}{n!},$$

and let $m^2 = m_0^2 + g_2$ (where g_i are formal parameters). Then we can make sense of the ratio $Z_{m_0,\mathbf{g},L}(0)/Z_{m_0,0,L}(0)$ (where $Z_{m,\mathbf{g},L}(0)$ denotes the partition function for the specified values of parameters; from now on the argument 0 will be dropped). Indeed, this ratio is defined by the formula

$$\frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = \sum_{\Gamma \in G_{\geq 2}(0)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma$$

(where $G_{\geq 2}(0)$ is the set of Feynman graphs without external vertices and all vertices of valency ≥ 2), which is a well-defined expression.

It is instructive to compute this expression in the case

$$g_2 = a, \ g_3 = g_4 = ... = 0.$$

In this case, we have only 2-valent vertices, so the only connected Feynman diagrams are N-gons, which are 1-loop. Hence,

$$\log \frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = W_1 = -\frac{1}{2} \log \det M,$$

where

$$M = 1 + a(-\tfrac{d^2}{dt^2} + m_0^2)^{-1}.$$

This determinant may be computed by looking at the eigenvalues. Namely, the eigenfunctions of $-\frac{d^2}{dt^2} + m_0^2$ in the space $C^\infty(\mathbb{R}/L\mathbb{Z})$ are $e^{\frac{2\pi i n t}{L}}$, with eigenvalues $\frac{4\pi^2 n^2}{L^2} + m_0^2$. So,

$$\det M = \prod_{n \in \mathbb{Z}} \left(1 + \frac{a}{\frac{4\pi^2 n^2}{L^2} + m_0^2}\right).$$

Hence, using the Euler product formula

$$\sinh(z) = z \prod_{n \geq 1} \left(1 + \frac{z^2}{\pi^2 n^2}\right),$$

we get

$$\frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = \frac{\sinh(\frac{m_0 L}{2})}{\sinh(\frac{mL}{2})}.$$

(Double-check this using summation over Feynman diagrams!)

Remark 7.26. More informally speaking, we see that the partition function Z for the theory with $U = \frac{m^2q^2}{2}$ has the form $\frac{C}{\sinh(\frac{mL}{2})}$, where C is a constant of our choice. Our choice from now on will be $C = \frac{1}{2}$; we will see later (in Example 8.25) why such a choice is preferable.

7.11. **The massless case.** Consider now the massless case, $m = 0$. In this case the propagator should be obtained by inverting the operator $-\frac{d^2}{dt^2}$, i.e. it should be the integral operator with kernel $G(t-s)$, where $G(t)$ is an even function satisfying the differential equation

$$-G''(t) = \delta(t).$$

There is a 1-parameter family of such solutions,

$$G(t) = -\frac{1}{2}|t| + C.$$

Using this function (for any choice of C), one may define the correlation functions of the free theory by the Wick formula.

Note that the function G does not decay at infinity. Therefore, this theory will not satisfy the clustering property (i.e. is not "physically meaningful").

We will also have difficulties in defining the corresponding interacting theory (i.e. one with a non-quadratic potential), as the integrals defining the amplitudes of Feynman diagrams will diverge. Such divergences are called *infrared divergences*, since they are caused by the failure of the integrand to decay at large times (or, in momentum space, its failure to be regular at low frequencies).

7.12. **Circle-valued quantum mechanics.** Consider now the theory with the same Lagrangian in which $q(t)$ takes values in the circle of radius r, $\mathbb{R}/2\pi r\mathbb{Z}$ (the "sigma-model"). We can do this at least classically, since the Lagrangian $\frac{\dot{q}^2}{2}$ makes sense in this case.

Let us define the corresponding quantum theory. The main difference from the line-valued case is that since $q(t)$ is circle-valued, we should consider not the usual correlators $\langle q(t_1)...q(t_n)\rangle$, but rather correlation functions of exponentials $\langle e^{\frac{ip_1q(t_1)}{r}}...e^{\frac{ip_nq(t_n)}{r}}\rangle$, where p_j are integers. They should be defined by the path integral

$$\int e^{\frac{ip_1q(t_1)}{r}}...e^{\frac{ip_nq(t_n)}{r}}e^{-\frac{S(q)}{\hbar}}Dq, \tag{7.4}$$

where $S(q) := \frac{1}{2}\int \dot{q}^2dt$ and $\int e^{-\frac{S(q)}{\hbar}}Dq$ is agreed to be 1. Note that it suffices to consider only the case $\sum_j p_j = 0$, otherwise the group of translations along the circle acts nontrivially on the integrand, hence under any reasonable definition the integral should be zero.

Now let us define the integral (7.4). Since the integral is invariant under shifts along the target circle, we may as well imagine that we are integrating over $q : \mathbb{R} \to \mathbb{R}$ with $q(0) = 0$. Now let us use the finite-dimensional analogy. Following this analogy, by completing the square we would get

$$\int e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} e^{-\frac{S(q)}{\hbar}} Dq = e^{-\frac{\hbar}{2r^2} B^{-1}(\sum_j p_j q(t_j), \sum_j p_j q(t_j))} =$$

$$e^{-\frac{\hbar}{2r^2} \sum_{j,\ell} p_\ell p_j G(t_\ell - t_j)} = e^{\frac{\hbar}{2r^2} \sum_{\ell<j} p_\ell p_j |t_\ell - t_j|},$$

where $B(q,q) := \int \dot{q}^2 dt$. Thus, it is natural to define the correlators by the formula

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_k)}{r}} \rangle = e^{\frac{\hbar}{2r^2} \sum_{\ell<j} p_\ell p_j |t_l - t_j|}.$$

We note that this theory, unlike the line-valued one, *does* satisfy the clustering property. Indeed, if $\sum p_j = 0$ (as we assumed), then (assuming $t_1 \geq t_2 \geq ... \geq t_n$), we have

$$\sum_{\ell<j} p_\ell p_j (t_\ell - t_j) = \sum_{j=1}^{n-1} (t_j - t_{j+1})(p_{j+1} + ... + p_n)(p_1 + ... + p_j) =$$

$$-\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2,$$

so the clustering property follows from the fact that $(p_1 + ... + p_j)^2 \geq 0$.

7.13. **Massless quantum mechanics on the circle.** Consider now the theory with Lagrangian $\frac{\dot{q}^2}{2}$, where q is a function on the circle of length L. In this case, according to the Feynman yoga, we must invert the operator $-\frac{d^2}{dt^2}$ on the circle $\mathbb{R}/L\mathbb{Z}$, or equivalently solve the differential equation $-G''(t) = \delta(t)$. Here we run into trouble: the operator $-\frac{d^2}{dt^2}$ is not invertible, since it has an eigenfunction 1 with eigenvalue 0; correspondingly, the differential equation in question has no solutions, as $\int G'' dt$ must be zero, so $-G''(t)$ cannot equal $\delta(t)$ (one may say that the quadratic form in the exponential is degenerate, and therefore the Gaussian integral turns out to be meaningless). This problem can be resolved by the following technique of "killing the zero mode". Namely, let us invert the operator $-\frac{d^2}{dt^2}$ on the space $\{q \in C^\infty(\mathbb{R}/L\mathbb{Z}) : \int q dt = 0\}$ (this may be interpreted as integration over this codimension one subspace, on which the quadratic form is non-degenerate). This means that we must find the solution of the

differential equation $-G''(t) = \delta(t) - \frac{1}{L}$, such that $\int G dt = 0$. Such solution is indeed unique, and it equals

$$G(t) = \frac{(t - \frac{L}{2})^2}{2L} - \frac{L}{24}, \tag{7.5}$$

$t \in [0, L]$. Thus, for example $\langle q(0)^2 \rangle = \frac{L}{12}$.

Higher correlation functions are defined in the usual way. Moreover, one can define the theory with an arbitrary potential using the standard procedure with Feynman diagrams.

7.14. **Circle-valued quantum mechanics on the circle.** Finally, let us consider the circle-valued version of the same theory. Thus, our integration variable is a map $q : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}/2\pi r\mathbb{Z}$. So we have a new feature - there are different homotopy classes of maps labeled by degree. Let us first consider integration over degree zero maps. Then we should argue in the same way as in the case $t \in \mathbb{R}$, and make the definition

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle_0 := e^{-\frac{\hbar}{2r^2} \sum_{\ell,j} p_\ell p_j G(t_\ell - t_j)},$$

where $\sum_j p_j = 0$. (Here subscript 0 stands for degree zero maps). Assuming that $0 \leq t_1, ..., t_n \leq L$, we find after a short calculation using (7.5):

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle_0 = e^{\frac{\hbar}{2r^2} (\sum_{\ell<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum_j p_j t_j)^2}{L})}$$

(the second summand disappears as $L \to \infty$, and we recover the answer on the line).

It is, however, more natural (as we will see later) to integrate over all maps q, not only degree zero. Namely, let N be an integer. Then all maps of degree N have the form $q(t) + \frac{2\pi r N t}{L}$, where q is a map of degree zero. Thus, if we want to integrate over maps of degree N, we should compute the same integral as in degree zero, but with shift $q \mapsto q + \frac{2\pi r N t}{L}$. But it is easy to see that this shift results simply in rescaling of the integrand by the factor $e^{\frac{2\pi i N}{L} \sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}$. Thus, the integral over all maps should be defined by the formula

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle =$$

$$e^{\frac{\hbar}{2r^2} (\sum_{l<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum p_j t_j)^2}{L})} \frac{\sum_{N \in \mathbb{Z}} e^{\frac{2\pi i N}{L} \sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}}{\sum_{N \in \mathbb{Z}} e^{-\frac{2\pi^2 r^2 N^2}{\hbar L}}}. \tag{7.6}$$

Introduce the elliptic theta-function

$$\theta(u, T) := \sum_{N \in \mathbb{Z}} e^{2\pi i u N - \pi T N^2}.$$

Then for $L \geq t_1 \geq ... \geq t_n \geq 0$ formula (7.6) can be rewritten in the form

(7.7)

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle = e^{\frac{\hbar}{2r^2}(\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2 + \frac{(\sum_j p_j t_j)^2}{L})} \frac{\theta(\frac{\sum_j p_j t_j}{L}, \frac{2\pi r^2}{\hbar L})}{\theta(0, \frac{2\pi r^2}{\hbar L})}.$$

Exercise 7.27. *Calculate the 1-particle irreducible 2-point function for a quantum particle with potential* $U(q) := \frac{m^2 q^2}{2} - \frac{gq^4}{4!}$ *modulo* g^3 *in momentum space, for* $\hbar = 1$. *(We have done this modulo* g^2 *in position space).*

Exercise 7.28. *Let* $U(q) := \frac{m^2 q^2}{2} - \frac{gq^3}{3}$.

(i) Calculate the leading term of the 1-point function $\mathcal{G}_1(t)$ *(with respect to* g*).*

(ii) Calculate the connected 2-point function modulo g^3.

Exercise 7.29. *Consider the potential* $U(x) := \frac{m^2 \sinh^2(gx)}{2g^2}$. *Find a formula for* $W_0(J)$ *(the tree part of* $\log(Z(J)/Z(0))$*) as explicitly as you can, when* $J(t) = a\delta(t)$.

8. Operator approach to quantum mechanics

In mechanics and field theory (both classical and quantum), there are two main languages – Lagrangian and Hamiltonian. In the classical setting, the Lagrangian language is the language of variational calculus (i.e. one studies extremals of the action functional), while the Hamiltonian language is that of symplectic geometry and Hamilton equations. Correspondingly, in the quantum setting, the Lagrangian language is the language of path integrals, while the Hamiltonian language is the language of operators and Schrödinger equation. We have now studied the first one (at least in perturbation expansion) and are passing to the second one.

8.1. **Hamilton's equations in classical mechanics.** We start with recalling the Hamiltonian formalism of classical mechanics. For more details, we refer the reader to the excellent book [A].

Recall first the Lagrangian description of the motion of a classical particle or system of particles. The position of a particle is described by a point q of the configuration space X, which we will assume to be a manifold. The Lagrangian of the system is a (smooth) function $\mathcal{L} : TX \to \mathbb{R}$ on the total space of the tangent bundle of X. Then the action functional is $S(q) = \int \mathcal{L}(q, \dot{q})dt$. The trajectories of the particle are the extremals of S. The condition for $q(t)$ to be an extremal of S is equivalent to the Euler-Lagrange equation (=the equation of motion), which in local coordinates has the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}.$$

For example, if X is a Riemannian manifold and $\mathcal{L}(q, v) = \frac{v^2}{2} - U(q)$ where $U : X \to \mathbb{R}$ is a potential function, then the Euler-Lagrange equation is the Newton equation

$$\ddot{q} = -U'(q),$$

where $\ddot{q} = \nabla_{\dot{q}}\dot{q}$ is the covariant derivative with respect to the Levi-Civita connection.

Consider now a system with Lagrangian $\mathcal{L}(q, v)$, whose differential with respect to v (for fixed q) is a diffeomorphism $T_qX \to T_q^*X$. This is definitely true in the above special case of Riemannian X.

Definition 8.1. The *Hamiltonian (or energy function)* of the system with Lagrangian $\mathcal{L}$ is the function $H : T^*X \to \mathbb{R}$, which is the Legendre transform of $\mathcal{L}$ along fibers; that is, $H(q, p) = pv_0 - \mathcal{L}(q, v_0)$, where v_0 is the (unique) critical point of $pv - \mathcal{L}(q, v)$. The manifold T^*X is

called the *phase space (or space of states)*. The variable p is called the *momentum variable*.

For example, if $\mathcal{L} = \frac{v^2}{2} - U(q)$, then $H(q,p) = \frac{p^2}{2} + U(q)$.

Remark 8.2. Since Legendre transform is involutive, we also have that the Lagrangian is the fiberwise Legendre transform of the Hamiltonian.

Let q_i be local coordinates on X. This coordinate system defines a coordinate system (q_i, p_i) on T^*X. We obtain

Proposition 8.3. *The equations of motion are equivalent to the Hamilton equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

in the sense that they are obtained from Hamilton's equations by elimination of p_i.

It is useful to write Hamilton's equations in terms of Poisson brackets. Recall that the manifold T^*X has a canonical symplectic structure $\omega = d\alpha$, where α is the canonical 1-form on T^*M (called the *Liouville form*) constructed as follows: for any $z \in T_{(q,p)}(T^*X)$,

$$\alpha(z) = (p, d\pi(q,p)z),$$

where $\pi : T^*X \to X$ is the projection. In local coordinates, we have

$$\alpha = \sum_i p_i dq_i,\ \omega = \sum_i dp_i \wedge dq_i.$$

Now let (M, ω) be a symplectic manifold (in our case $M = T^*X$). Since ω is non-degenerate, one can define the Poisson bivector ω^{-1}, which is a section of the bundle $\wedge^2 TM$. Now, given any two smooth functions f, g on M, one can define a third function – their *Poisson bracket*

$$\{f, g\} = (df \otimes dg, \omega^{-1}).$$

This operation is skew-symmetric and satisfies the Jacobi identity, i.e. it is a Lie bracket on $C^\infty(M)$. For $M = T^*X$, in local coordinates we have

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right).$$

This shows that Hamilton's equations can be written in the following manner in terms of Poisson brackets:

$$\frac{d}{dt} f(q(t), p(t)) = \{f, H\}(q(t), p(t)). \tag{8.1}$$

for any smooth function ("classical observable") $f \in C^\infty(T^*X)$, or, for shorthand

$$\frac{df}{dt} = \{f, H\}.$$

In other words, Hamilton's equations say that the rate of change of the observed value of f equals the observed value of $\{f, H\}$.

8.2. **Unbounded self-adjoint operators.** The rigorous mathematical treatment of quantum mechanics in the Hamiltonian setting is based on von Neumann's theory of unbounded self-adjoint operators in a Hilbert space. Let us recall the basics of this theory.

8.2.1. *Spectral theorem for bounded self-adjoint operators.* Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\ ,\ \rangle$ (antilinear in the first argument, as is traditional in quantum physics). We first recall the *spectral theorem* for bounded self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$, which generalizes the diagonalization theorem for a Hermitian matrix.

Theorem 8.4. *(von Neumann) Let A be a bounded self-adjoint operator. There exists a measure space (X, μ), an essentially bounded measurable function $h : X \to \mathbb{R}$, and an isometry $\mathcal{H} \to L^2(X, \mu)$ under which A maps to the operator of multiplication by h. Moreover, the spectrum $\sigma(A)$ is the set of $\lambda \in \mathbb{R}$ for which $h^{-1}(\lambda-\varepsilon, \lambda+\varepsilon)$ is positive for each $\varepsilon > 0$, and the eigenvalues of A (if they exist) are $\lambda \in \mathbb{R}$ such that $\mu(h^{-1}(\lambda)) > 0$, with eigenfunctions being indicator functions of subsets of $h^{-1}(\lambda)$ of positive measure.*

8.2.2. *Closable and closed operators.* Now we pass to not necessarily bounded operators. Let $\mathcal{H}'$ be another separable Hilbert space. A *densely defined linear operator* on $\mathcal{H}$ is a pair (A, V) where $V \subset \mathcal{H}$ is a dense subspace and A is a (possibly unbounded) linear operator $V \to \mathcal{H}'$. The space V is called the *domain* of A; in the notation, we will often suppress it and denote the operator just by A. Such an operator A has a graph $\Gamma_A \subset V \times \mathcal{H}' \subset \mathcal{H} \times \mathcal{H}'$. Let $\overline{\Gamma}_A$ be the closure of Γ_A in $\mathcal{H} \times \mathcal{H}'$. The operator A is said to be *closable* if $(0, u) \in \overline{\Gamma}_A$ for $u \in \mathcal{H}'$ implies $u = 0$, i.e., if the first projection $p_1 : \overline{\Gamma}_A \to \mathcal{H}$ is injective. In this case, setting $\overline{V} := p_1(\overline{\Gamma}_A) \subset \mathcal{H}$, we have $V \subset \overline{V}$ and obtain an extension of the operator $A : V \to \mathcal{H}'$ to a densely defined operator $\overline{A} : \overline{V} \to \mathcal{H}'$ which is called the *closure* of A. If A is closable and $\overline{A} = A$, we will say that A is *closed*; in other words, A is closed iff it has closed graph in $\mathcal{H} \times \mathcal{H}'$. Obviously, the closure $\overline{A}$ is closed for any closable A. Also, if A is bounded then it is closable, $\overline{V} = \mathcal{H}$, and $\overline{A} : \mathcal{H} \to \mathcal{H}'$ is just the continuous (=bounded) extension of A.

In general, however, a densely defined operator need not be closable. For example, if $\mathcal{H}'$ is finite dimensional and $A : V \to \mathcal{H}'$ is unbounded then there exists a sequence $v_n \in V$ such that $v_n \to 0$ but $||Av_n|| \geq 1$. Then the sequence $w_n := \frac{v_n}{||Av_n||}$ goes to 0, while $||Aw_n|| = 1$, so, as the unit sphere in $\mathcal{H}'$ is compact, passing to a subsequence if needed, we may assume that $Aw_n \to u$ for some $u \in \mathcal{H}'$ with $||u|| = 1$. Then $(0, u) \in \overline{\Gamma}_A$ and A is not closable. So we see that A is closable iff it is bounded.

On the other hand, if $\mathcal{H}'$ is infinite dimensional, then there are important classes of unbounded closable operators. For example, consider the case $\mathcal{H} = \mathcal{H}'$. Let us say that an operator $A : V \to \mathcal{H}$ is *symmetric* if $\langle v, Aw \rangle = \langle Av, w \rangle$ for all $v, w \in V$. We claim that every symmetric operator is closable and its closure is symmetric. Indeed, suppose $(v_n, Av_n) \to (0, u)$ for $u \in \mathcal{H}$. Fix a sequence $u_k \in V$ such that $u_k \to u$. Then

$$\langle Au_k, v_n \rangle = \langle u_k, Av_n \rangle \to \langle u_k, u \rangle, \ n \to \infty.$$

But the leftmost expression goes to zero, so $\langle u_k, u \rangle = 0$ for all k, hence $||u||^2 = 0$ which gives $u = 0$, i.e., A is closable. Moreover, given $v, w \in \overline{V}$, there exist sequences $v_n \to v, w_n \to w$ in V such that $Av_n \to \overline{A}v$, $Aw_n \to \overline{A}w$, thus

$$\langle v, \overline{A}w \rangle = \lim_{n\to\infty} \langle v_n, Aw_n \rangle = \lim_{n\to\infty} \langle Av_n, w_n \rangle = \langle \overline{A}v, w \rangle,$$

so $\overline{A}$ is symmetric.

8.2.3. *Adjoint operator.* Closed symmetric operators by themselves are not sufficient for quantum mechanics, however, since such operators cannot, in general, be diagonalized. Instead we need *self-adjoint* operators, which are closed symmetric operators satisfying an important additional property. To formulate this property, we first need to define the notion of an adjoint operator.

Let (A, V) be a closed symmetric operator. Denote by $V^\vee$ the space of $u \in \mathcal{H}$ such that the linear functional $v \mapsto \langle u, Av \rangle$ is bounded on V. In this case by the Riesz representation theorem there exists a unique vector $w \in \mathcal{H}$ such that $\langle u, Av \rangle = \langle w, v \rangle$, which depends linearly on u. Thus we obtain an operator $A^\dagger : V^\vee \to \mathcal{H}$. Note that $V^\vee \supset V$ and $A^\dagger$ is an extension of A to $V^\vee$, so $(A^\dagger, V^\vee)$ is a densely defined operator called the *adjoint operator* of (A, V). Furthermore, this operator is closed: if $(u_n, A^\dagger u_n) \to (u, w)$ then for $v \in V$,

$$\langle A^\dagger u_n, v \rangle = \langle u_n, Av \rangle \to \langle u, Av \rangle, \ n \to \infty,$$

and at the same time $\langle A^\dagger u_n, v \rangle \to \langle w, v \rangle$, so $\langle u, Av \rangle = \langle w, v \rangle$, hence $u \in V^\vee$ and $w = A^\dagger u$.

However, we will see that the operator $A^\dagger$ fails to be symmetric, in general. So we may consider the skew-Hermitian form

$$B(v,w) := (A^\dagger v, w) - (v, A^\dagger w)$$

on $V^\vee$ that measures its failure to be symmetric, called the *boundary form* (it is called this way because in examples it corresponds to boundary terms arising from integration by parts). By definition, $V \subset \mathrm{Ker}B$ (in fact, one can show that $V = \mathrm{Ker}B$, but we don't need this fact). It is easy to see that closed symmetric extensions of A correspond to isotropic closed subspaces $V \subset L \subset V^\vee$ with respect to the form B; namely, the extension of A to L is defined to be the restriction of $A^\dagger|_L$. Moreover, the adjoint operator to such an extension $(A^\dagger, L)$ is $(A^\dagger, L^\perp)$, where $L^\perp$ is the orthogonal complement of L in $V^\vee$ with respect to B.

8.2.4. *Self-adjoint operators.* Let us say that a closed symmetric operator (A, V) is *self-adjoint* if $V^\vee = V$, i.e., $A^\dagger = A$. We see that *self-adjoint extensions* of A correspond to *Lagrangian* subspaces L, i.e., those for which $L = L^\perp$. Note that such extensions/subspaces may or may not exist: the necessary and sufficient condition for existence of self-adjoint extensions (or Lagrangian subspaces) is that the signature (n_+, n_-) of the Hermitian form iB satisfies the equation $n_+ = n_-$ (i.e., the so-called *deficiency indices* $n_\pm \in \mathbb{Z}_{\geq 0} \cup \infty$ of A are equal). However, in quantum mechanical models they usually exist and correspond to various spatial boundary conditions.

We say that a symmetric operator (A, V) is *essentially self-adjoint* if the closure $(\overline{A}, \overline{V})$ is self-adjoint. Thus an essentially self-adjoint operator has a unique self-adjoint extension, so having such an operator is basically as good as having a self-adjoint one. This notion is convenient, for instance, when we do not want to describe explicitly the space $\overline{V}$.

The importance of unbounded self-adjoint operators consists in the fact that von Neumann's spectral theorem extends naturally to them. Namely, define the *spectrum* $\sigma(A, V)$ of a self-adjoint operator (A, V) to be the subset of $\lambda \in \mathbb{C}$ for which the operator $A - \lambda : V \to \mathcal{H}$ fails to be surjective. Then we have

Theorem 8.5. *Theorem 8.4 except for the statement that h is essentially bounded holds for not necessarily bounded self-adjoint operators. Moreover, the domain V of A in its spectral theorem realization is the space of $g \in L^2(X, \mu)$ such that $hg \in L^2(X, \mu)$.*

If the measure μ is concentrated on a countable set (i.e., we may take $X = \mathbb{N}$ with $\mu(j) = 1$ for $j \in \mathbb{N}$) then $\mathcal{H}$ has a basis consisting of

eigenfunctions, and vice versa; in this case one says that the spectrum of A is *purely point spectrum*. This happens, for example, when A is compact (the Hilbert-Schmidt theorem). The other extreme is *purely continuous spectrum*, when there are no eigenvalues (i.e., in the spectral theorem realization, all points of X have zero measure). The spectral theorem implies that any self-adjoint operator can be uniquely written as an orthogonal direct sum of two self-adjoint operators with purely point and purely continuous spectrum, respectively.

The spectral theorem also implies the following corollary.

Corollary 8.6. *Let (A, V) be a self-adjoint operator. Then there exists a unique 1-parameter group of unitary operators $U(t) = e^{iAt} : \mathcal{H} \to \mathcal{H}$ strongly continuous in t which preserve V and commute with A, such that for all $v \in V$ the function $t \mapsto U(t)v$ is differentiable and*

$$\frac{d}{dt}(U(t)v) = iAU(t)v.$$

Proof. Using the spectral theorem realization where A is the operator of multiplication by $h : X \to \mathbb{R}$, we may define $U(t)$ as the operator of multiplication by e^{ith}. □

In fact, the converse also holds: every strongly continuous 1-parameter group $U(t)$ (i.e., a unitary representation of the Lie group $\mathbb{R}$ on $\mathcal{H}$) arises uniquely (up to isometry) from a self-adjoint operator.

Remark 8.7. The spectral theorem implies that if (A, V) is a self-adjoint operator and $Av = \lambda v$ for some nonzero $v \in V$ then $\lambda \in \mathbb{R}$. On the contrary, if (A, V) is only symmetric and not self-adjoint, von Neumann showed that the set of eigenvalues of A on V is either the (open) upper-half plane $\mathbb{C}_+$ (if $n_+ > 0, n_- = 0$), or the lower half-plane $\mathbb{C}_-$, (if $n_- > 0, n_+ = 0$), or contains both (if $n_+, n_- > 0$).

8.2.5. *Examples.*

Example 8.8. Consider the symmetric operator $P := -i\frac{d}{dx}$ on $\mathcal{H} = L^2(S)$, where $S := \mathbb{R}/2\pi\mathbb{Z}$ (the momentum operator on the circle). This operator is symmetric on the space $V := C^\infty(S)$, and one can show that it is moreover essentially self-adjoint on this space (check it!). The corresponding space $\overline{V}$ is the *Sobolev space* $H^1(S)$ of functions $f \in L^2(S)$ with $f' \in L^2(S)$ in the sense of distributions (note that such functions are continuous). The spectrum of the corresponding self-adjoint operator is purely point and equals $\mathbb{Z}$, with eigenfunctions e^{inx}, i.e., $Pe^{inx} = ne^{inx}$. Thus the spectral realization of A is on $\ell_2(\mathbb{Z})$ with counting measure on which P acts by multiplication by the function n

(i.e., this realization reduces simply to the Fourier expansion of functions on S). Similarly, the energy operator $P^2 = -\frac{d^2}{dx^2}$ is essentially self-adjoint on the same space but with smaller domain of the closure - the Sobolev space $H^2(S)$ of functions $f \in L^2(S)$ such that $f'' \in L^2(S)$. Its spectrum in $\mathbb{Z}_{\geq 0}$ with the same eigenfunctions: $P^2 e^{inx} = n^2 e^{inx}$.

Example 8.9. The next example is more interesting, and prototypical for the theory of self-adjoint extensions. Namely consider the same momentum operator $P := -i\frac{d}{dx}$, but now acting on the dense subspace $V \subset L^2[0, 2\pi]$ of smooth functions with vanishing derivatives of all orders on both ends of the interval. In this case, P is **not** essentially self-adjoint: the space $\overline{V}$ is the space of functions $f \in H^1[0, 2\pi]$ with $f(0) = f(2\pi) = 0$, while $V^\vee = H^1[0, 2\pi]$ with

$$B(f, g) = i(\overline{f(2\pi)}g(2\pi) - \overline{f(0)}g(0)).$$

So on the quotient $V^\vee/\overline{V} = \mathbb{C}^2$ we have

$$B((a, b), (a, b)) = i(|b|^2 - |a|^2),$$

where $a = f(0)$, $b = f(2\pi)$. Thus a Lagrangian subspace of $V^\vee$ is given by points $b \in \mathbb{C}$ with $|b| = 1$; namely, it is the space L_b of functions $f \in H^1[0, 2\pi]$ with $f(2\pi) = bf(0)$. The spectrum of the corresponding self-adjoint operator is again purely point, so we should look for eigenfunctions in the space L_b. Thus we get eigenfunctions $e^{i(n+s)x}$ where $b = e^{2\pi i s}$. So the set of eigenvalues is $\mathbb{Z} + s$, and we see that the spectrum depends on the choice of the self-adjoint extension.

Observe also that any complex number λ is the eigenvalue of the symmetric (non-self-adjoint!) operator $P^\dagger$ on $V^\vee$, with eigenvector $e^{i\lambda x}$.

Example 8.10. Now consider the same momentum operator $P := -i\frac{d}{dx}$ but acting on the space $V = C_0^\infty(\mathbb{R})$ of compactly supported smooth functions, a subspace of $\mathcal{H} = L^2(\mathbb{R})$. In this case P is essentially self-adjoint, with $\overline{V} = V^\vee$ being the subspace of $H^1(\mathbb{R})$ of $f \in L^2(\mathbb{R})$ such that $f' \in L^2(\mathbb{R})$. The spectral theorem realization of P is on $L^2(\mathbb{R})$ as the operator of multiplication by x, which is given by the Fourier transform. Thus the spectrum of this operator is purely continuous and constitutes the whole real line $\mathbb{R}$. Similarly, the operator $P^2 = -\frac{d^2}{dx^2}$ is also essentially self-adjoint on V, and its self-adjoint extension has purely continuous spectrum $\mathbb{R}_{\geq 0}$.

Example 8.11. And yet again, take $P := -i\frac{d}{dx}$, but now on the subspace V of $\mathcal{H} = L^2(\mathbb{R}_{\geq 0})$ of compactly supported smooth functions with vanishing derivatives at 0. This operator is not essentially self-adjoint: the space $\overline{V}$ is the space of $f \in H^1(\mathbb{R}_{\geq 0})$ with $f(0) = 0$, while

$V^\vee$ is the whole $H^1(\mathbb{R}_{\geq 0})$. Thus the space $V^\vee/\overline{V}$ is 1-dimensional with form B given by $B(f,g) = -i\overline{f(0)}g(0)$, and so there are no self-adjoint extensions (the deficiency indices are not equal: $n_+ = 1, n_- = 0$).

Let us find the eigenvalues of P on V. The eigenvector with eigenvalue λ is $e^{i\lambda x}$, and it belongs to V iff $\lambda \in \mathbb{C}_+$. Thus the set of eigenvalues of P is $\mathbb{C}_+$.

Example 8.12. Let $A = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$ with $V = C_0^\infty(\mathbb{R})$ (quantum harmonic oscillator). Then A is essentially self-adjoint and $\overline{A}$ has pure point spectrum $n + \frac{1}{2}$, $n \in \mathbb{N}$, with eigenvectors $H_n(x)e^{\frac{x^2}{2}}$, where H_n are Hermite polynomials (Theorem 4.13).

Remark 8.13. More generally, it is known that if $U(x)$ is a piecewise continuous potential on $\mathbb{R}$ which tends to $+\infty$ at $\pm\infty$ then the operator $A := -\frac{1}{2}\frac{d^2}{dx^2} + U(x)$ is essentially self-adjoint on $V = C_0^\infty(\mathbb{R})$ and has pure point spectrum, with eigenvalues $E_0 < E_1 \leq E_2 \leq ...$ (it is shown in Lemma 8.19 below that E_0 is always a simple eigenvalue and the corresponding eigenvector is a positive function).

Example 8.14. Let M be a compact Riemannian manifold with boundary ∂M, $\mathcal{H} = L^2(M)$, and $A = \Delta$ be the Laplace operator on M acting on the space V of smooth functions on M vanishing with all derivatives on the boundary. In this case $\overline{V}$ is the space of functions in the Sobolev space $H^2(M)$ (functions $f \in L^2(M)$ such that $\Delta f \in L^2(M)$) which vanish with first normal derivative on ∂M, and $V^\vee = H^2(M)$. By Stokes' formula

$$\int_M (u\Delta v - v\Delta u)dx = \int_{\partial M} (u\partial_{\mathbf{n}} v - v\partial_{\mathbf{n}} u)d\sigma,$$

where $\mathbf{n}$ denotes the normal derivative to ∂M, so we have

$$B(f,g) = i\int_{\partial M} (\overline{f}\partial_{\mathbf{n}} g - g\partial_{\mathbf{n}}\overline{f})d\sigma.$$

So if $\partial M = 0$, the operator A is essentially self-adjoint and has a unique self-adjoint extension, while if $\partial M \neq 0$, it is not and there are many self-adjoint extensions corresponding to various boundary conditions on ∂M. The most common ones are the Dirichlet boundary condition $f = 0$ and Neumann boundary condition $\partial_{\mathbf{n}} f = 0$. Of course, the spectra associated to these conditions (which are always purely point) are completely different.

The simplest example with non-trivial boundary is $M = [0, \pi]$, in which case we have $\dim V^\vee/\overline{V} = 4$ and

$$B(f,g) = i(\overline{f}g' - \overline{f'}g)|_0^\pi.$$

For the Dirichlet boundary conditions $f(0) = f(\pi) = 0$ we get eigenbasis $\sin nx$ with eigenvalues $-n^2$, $n \in \mathbb{Z}_{\geq 1}$, while for the Neumann boundary conditions $f'(0) = f'(\pi) = 0$ we get eigenbasis $\cos nx$ also with eigenvalues $-n^2$ but now for $n \in \mathbb{Z}_{\geq 0}$.

Let us consider the mixed boundary condition:

$$f(0) = 0,\ f'(\pi) - af(\pi) = 0$$

for some real number a. Then the eigenfunctions are $\sin \lambda x$ where

$$\lambda \cos \pi\lambda = a \sin \pi\lambda.$$

Thus the eigenvalues are $-\lambda^2$ where λ runs over solutions of the equation

$$\lambda \operatorname{cotan} \pi\lambda = a.$$

For example, in the limit $a \to \infty$ we recover the answer for the Dirichlet boundary condition.

Exercise 8.15. *Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + a\chi_{[-1,1]}(x)$ where χ is the indicator function and $a \in \mathbb{R}$, and let it be defined on $V = C_0^\infty(\mathbb{R}) \subset \mathcal{H} = L^2(\mathbb{R})$. Show that H is essentially self-adjoint and find the spectrum and eigenvalues of its self-adjoint extension (consider separately the cases $a \geq 0$ and $a < 0$).*

Hint. *As explained above, the spectrum consists of $E \in \mathbb{R}$ for which $H - E : \overline{V} \to \mathcal{H}$ is not surjective. So try to solve the equation*

$$(H - E)u = f$$

for $f \in \mathcal{H}$ as

$$f(x) = \int_{\mathbb{R}} G(x, y)dy,$$

where $G(x, y)$ is the fundamental solution of the equation

$$(H - E)f = \delta(x - y).$$

You should get that there are no eigenfunctions for $a \geq 0$ (purely continuous spectrum), while for $a < 0$ the spectrum is mixed: there is continuous spectrum and also some eigenfunctions with negative eigenvalues; they are called **bound states**.

8.3. **Hamiltonians in quantum mechanics.** The yoga of quantization says that to quantize classical mechanics on a manifold X, we need to replace the classical space of states T^*X by the quantum space of states – the Hilbert space $\mathcal{H} = L^2(X)$ on square integrable complex half-densities on X (or, more precisely, the corresponding projective space). Further, we need to replace classical observables, i.e. (sufficiently nice) real functions $f \in C^\infty(T^*X)$, by quantum observables

$\widehat{f}$, which are (unbounded, densely defined) operators on $\mathcal{H}$, not commuting with each other in general. Then the (expected) value of an observable A in a state $\psi \in \mathcal{H}$ of unit norm is, by definition, $\langle \psi, A\psi \rangle$ (provided that it is well defined).

The operators $\widehat{f}$ should linearly depend on f. Moreover, they should depend on a positive real parameter $\hbar$ called the *Planck constant*, and satisfy the following relation:

$$[\widehat{f}, \widehat{g}] = i\hbar\widehat{\{f, g\}} + O(\hbar^2),\ \hbar \to 0.$$

Since the role of Poisson brackets of functions is played in quantum mechanics by commutators of operators, this relation expresses the condition that classical mechanics should be the limit of quantum mechanics as $\hbar \to 0$.[13]

We must immediately disappoint the reader by confessing that there is no canonical choice of the quantization map $f \mapsto \widehat{f}$. Nevertheless, there are some standard choices of $\widehat{f}$ for particular f, which we will now discuss.

Let us restrict ourselves to the situation $X = \mathbb{R}$, so on the phase space we have coordinates q (position) and p (momentum). In this case we can naturally think of half-densities as functions and there are the following standard conventions.

1. $\widehat{f} = f(q)$ (multiplication operator by $f(q)$) when f is independent of p.

2. $\widehat{p^m} \to (-i\hbar\frac{d}{dq})^m$.

(Note that these conventions satisfy our condition, since $[\widehat{q}, \widehat{p}] = i\hbar$, while $\{q, p\} = 1$.)

Example 8.16. For the classical Hamiltonian $H = \frac{p^2}{2} + U(q)$ considered above, the quantization will be the *Schrödinger operator*

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q).$$

Remark 8.17. The extension of these conventions to other functions is not unique. However, such an extension will not be used, so we will not specify it.

Now let us see what the quantum analog of Hamilton's equations should be. In accordance with the outlined quantization yoga, Poisson brackets should be replaced in quantum theory by commutators

[13]Note that the assignment $f \mapsto \widehat{f}$ cannot possibly satisfy the identity $\widehat{fg} = \widehat{f}\widehat{g}$ since the product of functions is commutative but the product of operators is not.

(with coefficient $(i\hbar)^{-1} = -i/\hbar$). Thus, Hamilton's equations should be replaced by the equation

$$\frac{d}{dt}\langle\psi(t), A\psi(t)\rangle = \langle\psi(t), \tfrac{[A,\widehat{H}]}{i\hbar}\psi(t)\rangle = -\tfrac{i}{\hbar}\langle\psi(t), [A,\widehat{H}]\psi(t)\rangle,$$

where $\langle,\rangle$ is the Hermitian form on $\mathcal{H}$ and $\widehat{H}$ is some quantization of the classical Hamiltonian H. Since this equation must hold for any A, it is equivalent to the *Schrödinger equation*

$$\dot{\psi} = -\frac{i}{\hbar}\widehat{H}\psi$$

up to changing ψ by a time-dependent phase factor (check it!). Thus, the quantum analog of the Hamilton equations is the Schrödinger equation.

Remark 8.18. This "derivation" of the Schrödinger equation is definitely not a mathematical argument. It is merely a reasoning aimed to motivate a definition.

To solve the initial value problem for the Schrödinger equation, we need to make sense of the Hamiltonian $\widehat{H}$ as an unbounded self-adjoint operator on $\mathcal{H}$ in the sense of von Neumann, which in practice boils down to giving spatial boundary conditions for ψ, in addition to the initial value. The general solution of the Schrödinger equation then has the form

$$\psi(t) = e^{-\frac{it\widehat{H}}{\hbar}}\psi(0),$$

where $e^{-\frac{it\widehat{H}}{\hbar}}$ is the 1-parameter group of unitary operators attached to the self-adjoint operator $\widehat{H}$, which exists thanks to von Neumann's spectral theorem. Therefore, for any quantum observable A it is reasonable to define a new observable

$$A(t) := e^{\frac{it\widehat{H}}{\hbar}}A(0)e^{-\frac{it\widehat{H}}{\hbar}}$$

(such that to observe $A(t)$ is the same as to evolve for time t and then observe $A = A(0)$). The observable $A(t)$ satisfies the equation

$$A'(t) = -\frac{i}{\hbar}[A(t), \widehat{H}]$$

called the *operator Schrödinger equation*, and we have

$$\langle\psi(t), A\psi(t)\rangle = \langle\psi(0), A(t)\psi(0)\rangle.$$

The two sides of this equation represent two pictures of quantum mechanics: Schrödinger's (states change in time, observables don't) and Heisenberg's (observables change in time, states don't). The equation expresses the equivalence of the two pictures.

8.4. **Feynman-Kac formula.** Let us consider a 1-dimensional particle with potential $U(q)$. Let us assume that $U \geq 0$ and $U(q) \to \infty$ as $|q| \to \infty$. In this case, the operator $\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q)$ is essentially self-adjoint on Schwartz functions, positive definite, and its spectrum is purely point.

Lemma 8.19. *There is a unique eigenvector Ω of $\widehat{H}$ with smallest eigenvalue given by a positive function with norm 1.*

Proof. An eigenvector Ω of $\widehat{H}$ with smallest eigenvalue λ minimizes the "energy" functional

$$E(\phi) := \langle \phi, \widehat{H}\phi \rangle = \int_{\mathbb{R}} (\tfrac{\hbar^2}{2}\phi'(q)^2 + U(q)\phi(q)^2)dq$$

on the space of real C^1-functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} \phi(t)^2 dt = 1$. Suppose that $\Omega(a) = 0$, then the equation $\widehat{H}\Omega = \lambda\Omega$ implies $\Omega'(a) \neq 0$. But $E(\Omega) = E(|\Omega|)$, so, since $\Omega'(a) \neq 0$, this value can be reduced by smoothing out Ω in a small neighborhood of a and then normalizing it to have unit norm, a contradiction. This also implies that λ is a simple eigenvalue, hence Ω is unique. □

Remark 8.20. The vector Ω is called the *ground state*, or *vacuum state*, since it has lowest energy, and physicists often shift the Hamiltonian by a constant so that the energy of this state is zero (i.e. "there is no matter").

The correlation functions in the Hamiltonian setting are defined by the formula

$$\mathcal{G}_n^{\mathrm{Ham}}(t_1, ..., t_n) := \langle \Omega, q(t_1)...q(t_n)\Omega \rangle$$

where $q(t)$ is the operator quantizing the observable "coordinate of the particle at the time t".

Remark 8.21. Physicists usually write the inner product $\langle v, Aw \rangle$ as $\langle v|A|w\rangle$. In particular, Ω is written as $\langle 0|$ or $|0\rangle$ (the so-called Dirac *bra-ket notation*).

Theorem 8.22. *(Feynman-Kac formula) If $t_1 \geq ... \geq t_n$ then the function $\mathcal{G}_n^{\mathrm{Ham}}$ admits an asymptotic expansion in $\hbar$ (near $\hbar = 0$), which coincides with the path integral correlation function $\mathcal{G}_n^M$ constructed above. Equivalently, the Wick rotated function $\mathcal{G}_n^{\mathrm{Ham}}(-it_1, ..., -it_n)$ equals $\mathcal{G}_n^E(t_1, ..., t_n)$.*

This theorem plays a central role in quantum mechanics, and we will prove it below. Before we do so, let us formulate an analog of this theorem for "quantum mechanics on the circle".

Let $\mathcal{G}_{n,L}(t_1,...,t_n)$ denote the correlation function on the circle of length L (for $0 \leq t_n \leq ... \leq t_1 \leq L$), and let Z_L be the partition function on the circle of length L, defined from (Euclidean) path integrals. Also, let

$$Z_L^{\mathrm{Ham}} = \mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}}),$$

and

$$\mathcal{G}_{n,L}^{\mathrm{Ham}}(-it_1,...,-it_n) = \frac{\mathrm{Tr}(q(-it_n)...q(-it_1)e^{-\frac{L\widehat{H}}{\hbar}})}{\mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}})}.$$

Theorem 8.23. *(Feynman-Kac formula on the circle) The functions Z_L^{Ham}, $\mathcal{G}_{n,L}^{\mathrm{Ham}}$ admit asymptotic expansions in $\hbar$, which coincide with the functions Z_L and $\mathcal{G}_{n,L}$ computed from path integrals.*

Note that Theorem 8.22 is obtained from Theorem 8.23 by sending L to infinity. Thus, it is sufficient to prove Theorem 8.23.

Remark 8.24. As we mentioned before, the function $\mathcal{G}_n^E$ can be defined by means of the Wiener integral, and the equality

$$\mathcal{G}_n^{\mathrm{Ham}}(-it_1,...,-it_n) = \mathcal{G}_n^E(t_1,...,t_n)$$

actually holds for numerical values of $\hbar$, and not just in the sense of power series expansions. The same applies to the equalities $Z_L^{\mathrm{Ham}} = Z_L$, $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$. However, these results are technically more complicated (as they require non-trivial analytic input) and thus are beyond the scope of these notes.

Example 8.25. Consider the case of the quadratic potential. By renormalizing variables, we can assume that $\hbar = m = 1$, so $U = \frac{q^2}{2}$. In this case we know that $Z_L = \frac{1}{2\sinh(\frac{L}{2})}$. On the other hand, $\widehat{H}$ is the Hamiltonian of the quantum harmonic oscillator:

$$\widehat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}.$$

The eigenvectors of this operator are $H_n(x)e^{-\frac{x^2}{2}}$, where H_n are the Hermite polynomials ($k \geq 0$), and the eigenvalues are $n + \frac{1}{2}$ (see Theorem 4.13). Hence,

$$Z_L^{\mathrm{Ham}} = e^{-\frac{L}{2}} + e^{-\frac{3L}{2}} + ... = \frac{1}{e^{\frac{L}{2}} - e^{-\frac{L}{2}}} = Z_L,$$

as expected from the Feynman-Kac formula. (This shows the benefit of the choice $C = \frac{1}{2}$ in the normalization of Z_L).

8.5. **Proof of the Feynman-Kac formula in the free case (harmonic oscillator).** Consider again the quadratic Hamiltonian $\widehat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}$ of the quantum Harmonic oscillator. Note that it can be written in the form

$$\widehat{H} = a^\dagger a + \tfrac{1}{2},$$

where $a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q)$, $a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q)$. The operators $a, a^\dagger$ define a representation of the Heisenberg Lie algebra on (a dense subspace of) the Hilbert space $\mathcal{H}$:

$$[a, a^\dagger] = 1.$$

Thus the eigenvectors of $\widehat{H}$ are $(a^\dagger)^n\Omega$ where $\Omega = e^{-\frac{q^2}{2}}$ is the lowest eigenvector and the corresponding eigenvalues are $n + \frac{1}{2}$, $n \in \mathbb{Z}_{\geq 0}$ (as we already saw before in Theorem 4.13).

Remark 8.26. The operators a and $a^\dagger$ are called the *annihilation and creation operators*, since $a\Omega = 0$, while all eigenvectors of $\widehat{H}$ can be "created" from Ω by action of powers of $a^\dagger$.

Now, we have

$$q(0) = q = \frac{1}{\sqrt{2}}(a + a^\dagger).$$

Since $[a^\dagger a, a] = -a$, $[a^\dagger a, a^\dagger] = a^\dagger$, we have

$$q(t) = \frac{1}{\sqrt{2}}e^{ita^\dagger a}(a + a^\dagger)e^{-ita^\dagger a} = \frac{1}{\sqrt{2}}(e^{-it}a + e^{it}a^\dagger)$$

This shows that

$$\mathcal{G}^{\mathrm{Ham}}_{n,L}(-it_1, ..., -it_n) = 2^{-\frac{n}{2}}\frac{\mathrm{Tr}(\prod_{j=1}^n (e^{t_j}a^\dagger + e^{-t_j}a)e^{-L(a^\dagger a + \frac{1}{2})})}{\mathrm{Tr}(e^{-L(a^\dagger a+\frac{1}{2})})}.$$

Now we can easily prove Theorem 8.23. Indeed, let us move the terms $e^{t_1}a^\dagger$ and $e^{-t_1}a$ around the trace (using the cyclic property of the trace). This will yield, after a short calculation, using (7.3) :

$$\mathcal{G}^{\mathrm{Ham}}_{n,L}(-it_1, ..., -it_n) =$$

$$\sum_{j=2}^n \tfrac{1}{2}\mathcal{G}^{\mathrm{Ham}}_{n-2,L}(-it_2, ..., -it_{j-1}, -it_{j+1}, ..., -it_n)\left(\frac{e^{t_1-t_j}}{e^L - 1} - \frac{e^{t_j-t_1}}{e^{-L}-1}\right) =$$

$$\sum_{j=2}^n \mathcal{G}^{\mathrm{Ham}}_{n-2,L}(-it_2, ..., -it_{j-1}, -it_{j+1}, ..., -it_n)G_L(t_1 - t_j).$$

This implies the theorem by induction in n.

Remark 8.27. 1. In the quadratic case there is no formal expansions and the Feynman-Kac formula holds as an equality between usual functions.

2. Note that the equality $\frac{e^{t-s}}{e^L-1} - \frac{e^{s-t}}{e^{-L}-1} = G_L(t-s)$ used above holds only if $t \geq s$. In fact, the matrix coefficient $\langle \Omega, q(t_1)...q(t_n)\Omega \rangle$ is not symmetric in t_j, as the operators $q(t_j)$ do not commute. Thus the Feynman-Kac formula only holds if $t_1 \geq ... \geq t_n$. For this reason the correlation function $\mathcal{G}_n^M$ is called time-ordered - it corresponds to the matrix coefficient where the operators $q(t_j)$ are ordered chronologically.

8.6. **Proof of the Feynman-Kac formula (general case).** Now we consider an arbitrary potential $U(q) := \frac{m^2q^2}{2} - V(q)$, where

$$V(q) = \sum_{k \geq 3} \frac{g_k q^k}{k!}.$$

For simplicity we will assume that the coefficients g_j are formal parameters and $\hbar = 1$ (the latter condition does not cause a loss of generality, as this situation can be achieved by rescaling). Let us first consider the case of partition function. We have

$$Z_L^{\mathrm{Ham}} = \mathrm{Tr}(e^{-L\widehat{H}}) = \mathrm{Tr}(e^{-L(\widehat{H}_0 - V)}),$$

where $\widehat{H}_0 = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}m^2q^2$ is the free (=quadratic) part of the Hamiltonian. Since g_j are formal parameters, we have a series expansion

(8.2)
$$e^{-L(\widehat{H}_0 - V)} = e^{-L\widehat{H}_0} + \sum_{N \geq 1} \int_{L \geq s_1 \geq ... \geq s_N \geq 0} e^{-(L-s_1)\widehat{H}_0} V e^{-(s_1-s_2)\widehat{H}_0} V ... e^{-(s_{n-1}-s_n)\widehat{H}_0} V e^{-s_n \widehat{H}_0} d\mathbf{s}$$

This follows from the general fact that in the (completed) free algebra with generators A, B, one has

(8.3)
$$e^{A+B} = e^A + \sum_{N \geq 1} \int_{1 \geq s_1 \geq ... \geq s_N \geq 0} e^{(1-s_1)A} B e^{(s_1-s_2)A} B ... e^{(s_{N-1}-s_N)A} B e^{s_N A} d\mathbf{s}$$

(check this identity!).

Equation (8.2) shows that

$$Z_L^{\mathrm{Ham}} =$$

$$\sum_{N \geq 0} \sum_{j_1,...,j_N=3}^{\infty} \frac{g_{j_1}...g_{j_N}}{j_1!...j_N!} \int_{1 \geq s_1 \geq ... \geq s_N \geq 0} \mathrm{Tr}(q_0(-is_1)^{j_1}...q_0(-is_N)^{j_N} e^{-L\widehat{H}_0}) d\mathbf{s},$$

where $q_0(t)$ is the operator $q(t)$ in the free theory associated to the potential $\frac{m^2q^2}{2}$.

Since the Feynman-Kac formula for the free theory has already been proved, we know that the trace on the right hand side can be evaluated as a sum over matchings. To see what exactly is obtained, let us collect the terms corresponding to all permutations of $j_1, ..., j_N$ together. This means that the summation variables will be the numbers $i_3, i_4, ..$ of occurences of $3, 4, ..$ among $j_1, ..., j_N$. Further, to every factor $q_0(-is)^j$ will be assigned a j-valent vertex, with a variable s attached to it, and it is easy to see that Z_L^{Ham} equals the sum over all ways of connecting the vertices (i.e. Feynman diagrams Γ) of integrals

$$\int_{0\leq s_1,...,s_N\leq L} \prod_{v-w} G_L(s_v - s_w) d\mathbf{s},$$

multiplied by the coefficients $\frac{\prod_k g_k^{i_k}}{|\mathrm{Aut}\Gamma|}$. Thus, $Z_L^{\mathrm{Ham}} = Z_L$, as desired.

Now let us consider correlation functions. Thus we have to compute

$$\mathrm{Tr}(e^{-(L-t_1)\widehat{H}} q e^{-(t_1-t_2)\widehat{H}} q ... q e^{-t_n \widehat{H}}).$$

Explanding each exponential inside the trace as above, we will clearly get the same Feynman diagram sum, except that the Feynman diagrams will contain n external vertices marked by variables $t_1, ..., t_n$. This implies that $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$, and we are done.

8.7. **The massless case.** Consider now the massless case, $m = 0$, in the Hamiltonian setting. For maps $q : \mathbb{R} \to \mathbb{R}$, we have $\mathcal{H} = L^2(\mathbb{R})$, and $\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2}$. This operator has continuous spectrum, and there is no lowest eigenvector Ω (more precisely, there is a lowest eigenvector $\Omega = 1$, but it is not in L^2), which means that we cannot define the correlation functions in the usual way, i.e. as $\langle \Omega, q(t_1)...q(t_n)\Omega \rangle$. (This is the reflection, in the Hamiltonian setting, of the difficulties related to the growth of the Green's function at infinity, i.e., infrared divergences, which we encountered in the Lagrangian setting).

Consider now the case $q : \mathbb{R} \to S^1 = \mathbb{R}/2\pi r\mathbb{Z}$. In this case, we have the same Hamiltonian but acting in the space $\mathcal{H} := L^2(S^1)$. The eigenvectors of this operator are $e^{\frac{iNq}{r}}$, with eigenvalues $\hbar^2 \frac{N^2}{2r^2}$. In particular, the lowest eigenvector is $\Omega = 1$. Thus the Hamiltonian correlation functions (in the Euclidean setting, for $t_1 \geq ... \geq t_n$) are

$$\langle\, \Omega, e^{\frac{t_1\widehat{H}}{\hbar}} e^{\frac{ip_1 q}{r}} e^{\frac{(t_2-t_1)\widehat{H}}{\hbar}} ... e^{\frac{ip_n q}{r}} e^{-\frac{t_n\widehat{H}}{\hbar}} \Omega \,\rangle =$$

$$e^{\frac{\hbar}{2r^2}\sum_j (t_j - t_{j+1})(p_1+...+p_j)^2},$$

which is equal to the correlation function in the Lagrangian setting. Thus the Feynman-Kac formula holds.

Now we pass to the case of circle-valued quantum mechanics on the circle. In this case, we have

$$\mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}}) = \sum_{N\in\mathbb{Z}} e^{-\frac{N^2 L\hbar}{2r^2}}$$

and

$$\mathrm{Tr}(e^{\frac{t_1\widehat{H}}{\hbar}} e^{\frac{ip_1 q}{r}} e^{\frac{(t_2-t_1)\widehat{H}}{\hbar}} ... e^{\frac{ip_n q}{r}} e^{\frac{(L-t_n)\widehat{H}}{\hbar}}) = \sum_{N\in\mathbb{Z}} e^{\frac{\hbar}{2r^2}\sum_{j=0}^{n}(t_j-t_{j+1})(N-p_1-...-p_j)^2},$$

where $t_{n+1} := L$, $t_0 := 0$. Simplifying this expression, we obtain

$$e^{\frac{\hbar}{2r^2}\sum_j (t_j-t_{j+1})(p_1+...+p_j)^2} \sum_{N\in\mathbb{Z}} e^{-\frac{\hbar}{2r^2}(LN^2+2N\sum_j p_j t_j)} =$$

$$e^{\frac{\hbar}{2r^2}\sum_j (t_j-t_{j+1})(p_1+...+p_j)^2} \theta(\frac{\hbar}{2\pi i r^2}\sum_j p_j t_j, \frac{L\hbar}{2\pi r^2}).$$

Comparing with (7.7), we see that the Feynman-Kac formula reduces to the modular invariance of the theta-function:

$$\theta(\frac{u}{iT}, \frac{1}{T}) = \sqrt{T} e^{\frac{\pi u^2}{T}} \theta(u, T)$$

with $T = \frac{2\pi r^2}{\hbar L}$ (which follows from the Poisson summation formula applied to the Gaussian).

Note that the Feynman-Kac formula in this example would have been false if in the Lagrangian setting we had ignored the topologically nontrivial maps. Thus we may say that the Feynman-Kac formula "sees topology". This ability of the Feynman-Kac formula to "see topology" (in much more complex situations) lies at the foundation of many interrelations between geometry and quantum field theory.

Remark 8.28. It should be noted that the contributions of topologically nontrivial maps from the source circle to the target circle are, strictly speaking, beyond our usual setting of perturbation theory, since they are exponentially small in $\hbar$. To be specific, the contribution from maps of degree N mostly comes from those maps which are close to the minimal action map $q_N(t) = \frac{2\pi t N r}{L}$, so it is of the order $e^{-\frac{2\pi^2 N^2 r^2}{L\hbar}}$. The maps $q_N(t)$ are the simplest examples of "instantons" – nonconstant solutions of the classical equations of motion, which have finite action (and are nontrivial in the topological sense). Exponentially small contributions to the path integral coming from integration over neighborhoods of instantons are called "instanton corrections to the perturbation series".

Remark 8.29. This calculation allows us to give sense to the partition function $Z(L)$ of the line-valued massless quantum mechanics on the circle. To this end, we just need to look at the asymptotics $r \to \infty$ of the partition function

$$Z(r, L) = \theta(0, \tfrac{\hbar L}{2\pi r^2}) = r\sqrt{\tfrac{2\pi}{\hbar L}}\theta(0, \tfrac{2\pi r^2}{\hbar L}).$$

Since $\theta(0, T) \to 1$ as $T \to \infty$, for the leading coefficient of the asymptotics we have (up to numerical scaling, which we are free to choose):

$$Z(L) \sim \frac{1}{\sqrt{\hbar L}}.$$

Note however that in this case we cannot write $Z(L) = \mathrm{Tr}(e^{-\frac{L\widehat{H}}{\hbar}})$ since this operator is not trace class. Also the vector $\Omega = 1$ is not normalizable. Thus this theory is somewhat ill-defined, as already mentioned above.

8.8. **Spectrum of the Schrödinger operator for a piecewise constant periodic potential.** In this subsection we demonstrate the behavior of the spectrum of a 1-dimensional Schrödinger operator on the example of a piecewise constant periodic potential, when the eigenvalues and eigenfunctions can be computed fairly explicitly.

We consider the Schrödinger operator on the circle $\mathbb{R}/2\pi\mathbb{Z}$ given by $H := -\frac{\hbar^2}{2}\partial^2 + U(x)$, where U is a piecewise continuous 2π-periodic potential. Clearly, without loss of generality we may assume that $\int_0^{2\pi} U(x)dx = 0$, otherwise we can shift $U(x)$ by a constant. By a standard result in analysis (the theory of Sturm-Liouville operators), the operator H has discrete spectrum, i.e., eigenvalues $E_0 < E_1 \leq E_2 \leq ...$ going to $+\infty$ with the corresponding eigenfunctions $\Psi_0, \Psi_1, \Psi_2, ...$ For example, if $U = 0$ then $E_0 = 0$ and $E_{2m-1} = E_{2m} = \frac{\hbar^2 m^2}{2}$ for $m > 0$, with eigenfunctions $\Psi_0 = 1, \Psi_{2m-1} = \sin mx, \Psi_{2m} = \cos mx$.

Consider now the simplest non-trivial example – the piecewise constant potential

$$U(x) = \begin{cases} Mb, \ 0 \leq x < a \\ -Ma, a \leq x < 2\pi \end{cases} \tag{8.4}$$

where $a, b, M > 0$, $a + b = 2\pi$.

For every $p \in \mathbb{R}$, we have a basis f_p, g_p of solutions of the equation $H\Psi = E\Psi$ on $[p, \infty]$ such that $f_p(p) = g_p'(p) = 1, g_p(p) = f_p'(p) = 0$.

For example,

$$f_0(x) = \cos\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}x,\ g_0(x) = \frac{\sin\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}x}{\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}}$$

for $0 \le x < a$ and

$$f_a(x) = \cos\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}(x-a),\ g_a(x) = \frac{\sin\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}(x-a)}{\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}(x-a)}$$

for $a \le x < 2\pi$. Thus the monodromy matrices along the intervals $[0,a]$, $[a,2\pi]$ in these bases are

$$A := \begin{pmatrix} \cos\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a & \frac{\sin\sqrt{\frac{2}{\hbar^2}(E-Mb)}a}{\sqrt{\frac{2}{\hbar^2}(E-Mb)}} \\ -\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}\sin\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a & \cos\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a \end{pmatrix},$$

$$B := \begin{pmatrix} \cos\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b & \frac{\sin\sqrt{\frac{2}{\hbar^2}(E+Ma)}b}{\sqrt{\frac{2}{\hbar^2}(E+Ma)}} \\ -\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}\sin\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b & \cos\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b \end{pmatrix}.$$

The condition for a periodic solution is that the matrix AB (monodromy around the circle) has an eigenvalue 1. Since $\det A = \det B = 1$, in this case the second eigenvalue of AB is also 1 (generically this matrix is a unipotent Jordan block), so the condition is $\mathrm{Tr}(AB) = 2$, which gives

(8.5)
$$\cos\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a\cos\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b - \tfrac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}}\sin\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a\sin\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b = 1.$$

Thus the eigenvalues of H are the solutions E of (8.5).

If $a < E < b$ then $\sqrt{\frac{2}{\hbar^2}(E-Mb)}$ is imaginary, so (8.5) can be written in terms of real parameters as

(8.6)
$$\cosh\sqrt{\tfrac{2}{\hbar^2}(Mb-E)}a\cos\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b - \tfrac{E+M\frac{a-b}{2}}{\sqrt{(Mb-E)(E+Ma)}}\sinh\sqrt{\tfrac{2}{\hbar^2}(Mb-E)}a\sin\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b = 1.$$

As mentioned above, if $M = 0$, then for each $n \ge 1$ the operator H double eigenvalue $\frac{1}{2}\hbar^2 n^2$. We would like to see what happens to this eigenvalue for large n as we turn on M and keep the product $\hbar n$ in a bounded interval $[C^{-1}, C]$ (so $\hbar \to 0$).

Let us rewrite (8.5) in the form

$$1 - \cos\left(\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a + \sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b\right) =$$

$$\left(1-\frac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}}\right)\sin\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a\sin\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b$$

and look for solutions

$$E=\frac{1}{2}(\hbar^2n^2+\varepsilon),$$

where $|\varepsilon|\ll\frac{1}{n}$. We have

$$\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}=\sqrt{n^2+\tfrac{\varepsilon-2Mb}{\hbar^2}}=n\left(1+\tfrac{\varepsilon-2Mb}{2\hbar^2n^2}-\tfrac{(\varepsilon-2Mb)^2}{8\hbar^4n^4}...\right),$$

$$\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}=\sqrt{n^2+\tfrac{\varepsilon+2Ma}{\hbar^2}}=n\left(1+\tfrac{\varepsilon+2Ma}{2\hbar^2n^2}-\tfrac{(\varepsilon+2Ma)^2}{8\hbar^4n^4}...\right),$$

so

$$\sqrt{\tfrac{2}{\hbar^2}(E-Mb)}a+\sqrt{\tfrac{2}{\hbar^2}(E+Ma)}b=2\pi n\left(1+\tfrac{\varepsilon}{2\hbar^2n^2}-\tfrac{M^2ab}{2\hbar^4n^4}+...\right).$$

Thus the left hand side of the above equation has the form

$$LHS=\frac{\pi^2}{2}\left(\frac{\varepsilon}{\hbar^2n}-\frac{M^2ab}{2\hbar^4n^3}\right)^2+...$$

We also have

$$1-\frac{E+M\frac{a-b}{2}}{\sqrt{(E-Mb)(E+Ma)}}=-\frac{\pi^2M^2}{2\hbar^4n^4}+...$$

So we get

$$RHS=\frac{\pi^2M^2}{2\hbar^4n^4}\sin^2na+...$$

Thus we obtain

$$\frac{\varepsilon}{\hbar^2n}-\frac{M^2ab}{2\hbar^4n^3}=\pm\frac{M|\sin na|}{\hbar^2n^2},$$

which yields

$$\varepsilon=\frac{M^2ab}{2\hbar^2n^2}\pm\frac{M|\sin na|}{n},$$

We see that the double eigenvalue $\Lambda_n=\frac{\hbar^2n^2}{2}$, $n>0$ for $M=0$ bifurcates into two eigenvalues

(8.7)

$$\Lambda_n^{\pm}(M)=\Lambda_n+\frac{M^2a(2\pi-a)}{8\Lambda_n}\pm\frac{M|\sin na|}{2n}+o((M+\tfrac{1}{n})^2),\ M\to0,n\to\infty.$$

8.9. **WKB approximation and the Weyl law.** The goal of this subsection is to explain how to compute semiclassical asymptotics of eigenvalues and eigenfunctions of quantum hamiltonians. This method is called the **Wentzel-Kramers-Brillouin (WKB) approximation**, named after the authors of three separate papers which introduced it independently in 1926.

We start with a general discussion of WKB approximation for linear ODE. Suppose we have an equation

$$\hbar\frac{dF}{dx} = AF \tag{8.8}$$

for a vector-function of one variable $F(x) \in \mathbb{C}^n$, where $A(x) \in \mathrm{Mat}_n(\mathbb{C})$ is a matrix-valued function (smooth on a certain interval $I \subset \mathbb{R}$). We would like to understand the asymptotic behavior of solutions of this equation as $\hbar \to 0$. To this end, assume for simplicity that $A(x)$ has simple spectrum for generic x, and let $v_1(x), ..., v_n(x)$ be its column eigenvectors with eigenvalues $\lambda_1(x), ..., \lambda_n(x)$, and $v_1^*(x), ..., v_n^*(x)$ the dual basis of row eigenvectors. Let us now look for solutions of (8.8) in the form

$$F(x) = e^{\frac{\phi(x)}{\hbar}}(\psi_0(x) + \hbar\psi_1(x) + \hbar^2\psi_2(x)...),$$

where $\psi_0(x) \neq 0$ and the series in parentheses is formal. Substituting, we get

$$(\hbar\partial_x + \phi' - A)(\psi_0 + \hbar\psi_1 + \hbar^2\psi_2 + ...) = 0,$$

which in degree 0 with respect to $\hbar$ yields the equation

$$A\psi_0 = \phi'\psi_0.$$

Thus $\phi' = \lambda_j$ is an eigenvalue of A, so

$$\phi(x) = \int \lambda_j(x)dx,\ \psi_0(x) = f(x)v_j(x),$$

where f is a scalar function.

Further, in degree 1 in $\hbar$ we obtain the equation

$$\psi_0' = (A - \lambda)\psi_1,$$

i.e.,

$$f'v_j + fv_j' = (A - \lambda)\psi_1.$$

For this to have a solution ψ_1, we need $(v_j^*, f'v_j + fv_j') = 0$, i.e.,

$$f' = -(v_j^*, v_j')f.$$

Thus

$$f(x) = \exp\left(-\int (v_j^*, v_j')dx\right).$$

Now we can recursively solve for $\psi_1, \psi_2, \ldots$. This leads to the following result.

Theorem 8.30. *There is a unique, up to scaling, basis of formal solutions of equation* (8.8) *of the form*

$$F_j(x) = \exp\left(\frac{\int \lambda_j(x)dx}{\hbar}\right)\left(\exp\left(-\int (v_j^*(x), v_j'(x))dx\right) v_j(x) + O(\hbar)\right).$$

Let us now apply this theorem to the stationary Schrödinger equation

$$(-\tfrac{\hbar^2}{2}\partial_x^2 + U(x))\Psi = E\Psi. \tag{8.9}$$

Set $p(x) := \sqrt{2(E - U(x))}$, then (8.9) takes the form

$$\hbar^2\partial_x^2\Psi = -p^2\Psi.$$

This can be written as the system of equations

$$\hbar\partial_x\begin{pmatrix}\Psi\\ \hbar\Psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ -p^2 & 0\end{pmatrix}\begin{pmatrix}\Psi\\ \hbar\Psi'\end{pmatrix}.$$

Thus we have equation (8.8) with $A = \begin{pmatrix}0 & 1\\ -p^2 & 0\end{pmatrix}$. So we have

$$\lambda_1 = ip,\ \lambda_2 = -ip$$

and we may take

$$v_1 = \begin{pmatrix}1\\ ip\end{pmatrix},\ v_2 = \begin{pmatrix}1\\ -ip\end{pmatrix},$$

so that

$$v_1^* = \frac{1}{2}(1, -ip^{-1}),\ v_2^* = \frac{1}{2}(1, ip^{-1}).$$

Thus we obtain the following formal solutions of (8.9):

$$\Psi_\pm = \exp\left(\pm\frac{i\int pdx}{\hbar}\right)\left(\exp\left(-\frac{1}{2}\int p^{-1}p'dx\right) + O(\hbar)\right) =$$

$$p^{-\frac{1}{2}}\exp\left(\pm\frac{i\int pdx}{\hbar}\right)(1 + O(\hbar)).$$

We get

Theorem 8.31. *(local WKB approximation) Equation* (8.9) *has a basis of formal solutions*

$$\Psi_\pm(x) = (2(E - U(x))^{-\frac{1}{4}}\exp\left(\pm\frac{i\int\sqrt{2(E - U(x))}dx}{\hbar}\right)(1 + O(\hbar)).$$

The WKB approximation can also be used to find asymptotic distribution of eigenvalues of a Schrödinger operator when it has discrete spectrum. Let us explain, somewhat informally, how this works.

As an example, consider the stationary Schrödinger equation (8.9) on the circle $\mathbb{R}/2\pi\mathbb{Z}$ with piecewise continuous 2π-periodic potential $U(x)$. We would like to write an asymptotic formula for the n-th eigenvalue $E_n(\hbar)$ of the operator $H = -\frac{1}{2}\hbar^2\partial^2 + U(x)$ when $n \sim \frac{A}{\hbar}$ for a given constant A. This is equivalent to determining the number $\nu(E)$ of eigenvalues of H satisfying the inequality $\Lambda \leq E$ for a given constant E.

To this end, we will use Theorem 8.31. Assume first that

$$E > \sup U(x).$$

The periodicity condition for the solutions $\Psi_\pm$ in Theorem 8.31 (called the *quantization condition* in quantum mechanics) in the zeroth approximation is that

$$\int_0^{2\pi} \sqrt{2(E - U(x))}dx = 2\pi n\hbar,\ n \in \mathbb{Z}_{\geq 0}. \tag{8.10}$$

It follows that if 8.10 holds then the number of eigenvalues of H which are $\leq E$ is about $2n$. So we get

Proposition 8.32.

$$\nu(E) \sim \frac{A(E)}{\hbar},\ \hbar \to 0,\ \textit{where } A(E) := \frac{1}{\pi}\int_0^{2\pi} \sqrt{2(E - U(x))}dx.$$

Thus for sufficiently large A, we have

$$E_{[\frac{A}{\hbar}]}(\hbar) \sim E(A),$$

where $E(A)$ is the solution of the equation

$$A = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2(E - U(x))}dx.$$

Note that $A(E)$ is the area of the region in the classical phase space $T^*S^1 = S^1 \times \mathbb{R}$ defined by the inequality

$$H_{\text{cl}} \leq E,$$

where $H_{\text{cl}} := \frac{1}{2}p^2 + U(x)$ is the corresponding classical hamiltonian. Moreover, one can show that with this definition of $A(E)$, the formula

$$\nu(E) \sim \frac{A(E)}{\hbar}$$

in fact holds in a much larger generality, whenever H has discrete spectrum (namely, for the operator $-\frac{1}{2}\hbar^2\Delta + U(x)$ on any compact

Riemannian manifold, or even on a non-compact one when one has $U(x) \to +\infty$ as $x \to \infty$). This formula is known as the *Weyl law*.

Exercise 8.33. *Prove the Weyl law on the circle for* $E \leq \sup U(x)$.

Finally, let $U(x) := MU_0(x)$ where U_0 is a fixed potential and consider the asymptotics of eigenvalues for small M, assuming that $\hbar \ll M$ (i.e., $\frac{1}{n} \ll M$). In this case we can write equation (8.10) as

$$\sqrt{2E} \int_0^{2\pi} \left(1 - \frac{MU_0(x)}{2E} - \frac{M^2 U_0(x)^2}{8E^2} + o(M^2)\right) dx = 2\pi n\hbar. \tag{8.11}$$

As before, we assume without loss of generality that $\int_0^{2\pi} U_0(x)dx = 0$. Let $I := \frac{1}{2\pi}\int_0^{2\pi} U_0(x)^2 dx$. Then we obtain

$$\sqrt{2E} = n\hbar + \tfrac{M^2 I}{2(2E)^{\frac{3}{2}}} + o(M^2) = n\hbar(1 + \tfrac{M^2 I}{2n^4\hbar^4} + ...) \tag{8.12}$$

It follows that

$$E = \tfrac{1}{2}n^2\hbar^2(1 + \tfrac{M^2 I}{n^4\hbar^4} + ...) = \Lambda_n + \frac{M^2 I}{8\Lambda_n} + ...$$

This gives the first correction of the eigenvalue $\Lambda_n := \frac{1}{2}n^2\hbar^2$ as we turn on M.

For example, if $U(x)$ is given by (8.4) then $I = a(2\pi - a)$ and we recover the asymptotics (8.7) without the last (bifurcation) term (which is negligible compared to $\frac{M^2 a(2\pi - a)}{8\Lambda_n}$ in the range $\frac{1}{n} \ll M$).

9. Fermionic integrals

9.1. **Bosons and fermions.** In physics there exist two kinds of particles – bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions: e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is $\mathcal{H}$ then the space of states of the system of k such particles is $S^k\mathcal{H}$ for bosons and $\Lambda^k\mathcal{H}$ for fermions. In particular, in the fermionic case, if $\dim \mathcal{H} = n$ then the space of states of $\geq n+1$ identical particles is zero, which is the *Pauli exclusion principle* (leading, for instance, to the fact that the number of electrons in an atom at the m-th energy level is bounded by $2m^2$). In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or, more generally, a manifold), while for a fermionic particle it is an *odd vector space*. Mathematically "odd" means that the algebra of smooth functions on this space (i.e. the algebra of classical observables) is an *exterior* algebra (unlike the case of a usual, *even* space, for which the algebra of polynomial functions is a *symmetric* algebra).

More generally, one may consider systems of classical particles or fields some of which are bosonic and some fermionic. In this case, the space of states will be a supervector space, i.e. the direct sum of an even and an odd space (or, more generally, a supermanifold – a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this section.

9.2. **Supervector spaces.** Let k be a field of characteristic zero. A *supervector space* (or shortly, superspace) over k is just a $\mathbb{Z}/2$-graded vector space: $V = V_0 \oplus V_1$. If $V_0 = k^n$ and $V_1 = k^m$ then V is denoted by $k^{n|m}$. The notions of a linear operator, direct sum, tensor product, dual space for supervector spaces are defined in the same way as for $\mathbb{Z}/2$-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of $\mathbb{Z}/2$-graded vector spaces.

However, the notions of a supervector space and a $\mathbb{Z}/2$-graded vector space are *not* the same. The difference is as follows. The category of vector (and hence $\mathbb{Z}/2$-graded vector) spaces has a *symmetric structure*, which is the standard isomorphism $V \otimes W \to W \otimes V$ (given by $v \otimes w \to w \otimes v$). This isomorphism allows one to define symmetric powers S^iV, exterior powers Λ^iV, etc. For supervector spaces, there is also

a symmetry $V \otimes W \to W \otimes V$, but it is defined differently. Namely, $v \otimes w$ goes to $(-1)^{ij} w \otimes v$, $v \in V_i, w \in V_j$ $(i, j \in \{0, 1\})$. In other words, it is the same as usual except that if v, w are both odd then $v \otimes w \mapsto -w \otimes v$. As a result, we can define the superspaces $S^i V$ and $\Lambda^i V$ for a superspace V, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if V is purely odd ($V = V_1$), then $S^i V$ is the i-th exterior power of V, and $\Lambda^i V$ is the i-th symmetric power of V (purely even for even i and purely odd for odd i). Thus in general for $V = V_0 \oplus V_1$, we have the following expressions for the symmetric algebra $SV := \oplus_{i \geq 0} S^i V$ and exterior algebra $\Lambda V := \oplus_{i \geq 0} \Lambda^i V$:

$$SV = SV_0 \otimes \Lambda V_1, \ \Lambda V = \Lambda V_0 \otimes SV_1.$$

For a superspace V, let ΠV be the same space with opposite parity, i.e. $(\Pi V)_j = V_{1-j}$, $j = 0, 1$. Then we have

$$S^i V = \Pi^i(\Lambda^i \Pi V), \ \Lambda^i V = \Pi^i(S^i \Pi V).$$

Let $V = V_0 \oplus V_1$ be a finite dimensional superspace. Define the algebra of polynomial functions on V, $\mathcal{O}(V)$, to be the algebra SV^* (where symmetric powers are taken in the supersense). Thus, $\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$, where V_0 and V_1 are regarded as usual spaces. More explicitly, if $x_1, ..., x_n$ are linear coordinates on V_0, and $\xi_1, ..., \xi_m$ are linear coordinates on V_1, then $\mathcal{O}(V) = k[x_1, ..., x_n, \xi_1, ..., \xi_m]$, with defining relations

$$x_i x_j = x_j x_i, \ x_i \xi_r = \xi_r x_i, \ \xi_r \xi_s = -\xi_s \xi_r$$

(in particular, $\xi_r^2 = 0$). Note that this algebra is itself a (generally, infinite dimensional) supervector space, and is commutative in the supersense. Also, if V, W are two superspaces, then $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$, where the tensor product of algebras is understood in the supersense, i.e.

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd),$$

where $p(x)$ is the parity of x.

9.3. **Supermanifolds.** Now assume that $k = \mathbb{R}$. Then by analogy with the above for any supervector space V we can define the algebra of smooth functions, $C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$. In fact, this is a special case of the following more general setting.

Definition 9.1. A *supermanifold* M is a usual manifold M_0 with a sheaf C_M^∞ of $\mathbb{Z}/2\mathbb{Z}$ graded algebras (called the *structure sheaf*), which is locally isomorphic to $C_{M_0}^\infty \otimes \Lambda(\xi_1, ..., \xi_m)$.

The manifold M_0 is called the *reduced manifold* of M. The dimension of M is the pair of integers $\dim M_0|m$.

For example, a supervector space V is a supermanifold of dimension $\dim V_0|\dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e. a domain $U_0 \subset V_0$ together with the sheaf $C^\infty_{U_0} \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is "locally isomorphic" to a superdomain.

Let M be a supermanifold. An *open set* U in M is the supermanifold $(U_0, C^\infty_M|_{U_0})$, where U_0 is an open subset in M_0.

By the definition, supermanifolds form a category $\mathcal{S}$. Let us describe explicitly morphisms in this category, i.e. maps $F : M \to N$ between supermanifolds M and N. By the definition, it suffices to assume that M, N are superdomains, with global coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$, and $y_1, ..., y_p, \eta_1, ..., \eta_q$, respectively (here x_i, y_i are even variables, and ξ_i, η_i are odd variables). Then the map F is defined by the formulas:

$$y_i = f_{0,i}(x_1, ..., x_n) + f^{j_1 j_2}_{2,i}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2} + ...,$$

$$\eta_i = a^j_{1,i}(x_1, ..., x_n)\xi_j + a^{j_1 j_2 j_3}_{3,i}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + ...,$$

where $f_{0,i}, f^{j_1 j_2}_{2,i}, ..., a^j_{1,i}, a^{j_1 j_2 j_3}_{3,i}, ...$ are usual smooth functions, and we assume summation over repeated indices. These formulas, determine F completely, since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor's formula. For example, if $M = N = \mathbb{R}^{1|2}$, $F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)$, and $g = g(x)$, then

$$g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2.$$

9.4. **Supermanifolds and vector bundles.** Let M_0 be a manifold, and E be a real vector bundle on M_0. Then we can define the supermanifold $M := \mathrm{Tot}(\Pi E)$, the total space of E with changed parity. Namely, the reduced manifold of M is M_0, and the structure sheaf C^∞_M is the sheaf of sections of ΛE^*. This defines a functor $S : \mathcal{B} \to \mathcal{S}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor S_* in the opposite direction: namely, $S_*(M)$ is the manifold M_0 with the vector bundle $(R/R^2)^*$, where R is the nilpotent radical of C^∞_M.

The following proposition (whose proof we leave as an exercise) gives a classification of supermanifolds.

Proposition 9.2. *(i)* $S_* \circ S = \mathrm{Id}$*;*

(ii) $S \circ S_* = \mathrm{Id}$ *on isomorphism classes of objects.*

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds,

$S \circ S_*$ is *not* the identity on morphisms (e.g. it maps the automorphism $x \to x + \xi_1\xi_2$ of $\mathbb{R}^{1|2}$ to Id), and hence, S is not an equivalence of categories. In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds "has more morphisms").

Remark 9.3. 1. The relationship between these two categories is quite similar to the relationship between the categories of (finite dimensional) filtered and graded vector spaces, respectively (namely, for them we also have functors S, S_* with the same properties – check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a splitting of a filtered space.

2. In the definition of a supermanifold one can replace the real exterior algebra $\Lambda(\xi_1, ..., \xi_m)$ with the complexified exterior algebra $\Lambda_{\mathbb{C}}(\xi_1, ..., \xi_m)$. This gives a notion of a $\mathbb{C}$-supermanifold, which generalizes the notion of an ordinary smooth manifold with the sheaf of complex-valued (as opposed to real-valued) smooth functions. Similarly to Proposition 9.2, isomorphism classes of $\mathbb{C}$-supermanifolds with reduced submanifolds M_0 are in bijection with isomorphism classes of *complex* vector bundles on M_0, so they are more general (as not every complex vector bundle is the complexification of a real one). Otherwise, the theory of $\mathbb{C}$-supermanifolds (which does actually arise in quantum field theory, see Remark 11.3 below) is completely parallel to the theory of usual supermanifolds.

One may also similarly define complex analytic and algebraic supermanifolds, but this is a different story which we will not discuss here.

9.5. **Supertrace and superdeterminant (Berezinian).** Before proceeding further, we need to generalize to the supercase the basic notions of linear algebra, such as trace and determinant of a matrix.

Let $R := R_0 \oplus R_1$ be a supercommutative $\mathbb{C}$-algebra. Fix two nonnegative integers m, n. Let $\mathrm{Mat}_{n|m}(R)$ be the algebra of $n+m$ by $n+m$ matrices over R which have the block decomposition

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

so that A_{00} is n by n, A_{11} is m by m, and A_{00}, A_{11} have even entries (i.e., in R_0), while A_{01}, A_{10} have odd entries (i.e., in R_1). We would like to define the *supertrace* of A as a linear function

$$\mathrm{sTr}(A) = \sum_{i,j=1}^{n+m} \lambda_{ij} a_{ij}, \ \lambda_{ij} \in \mathbb{Z},$$

so that $\mathrm{sTr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = n$ and $\mathrm{sTr}(AB) = \mathrm{sTr}(BA)$ for any R and $A, B \in \mathrm{Mat}_{n|m}(R)$. Thus we must have $\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) + \varepsilon\mathrm{Tr}(A_{11})$ for some $\varepsilon \in \mathbb{Z}$, and taking all blocks of A, B except A_{01}, B_{10} to be zero, we get $\varepsilon = -1$. So the supertrace of A has to be defined by the formula

$$\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) - \mathrm{Tr}(A_{11}).$$

Now let us generalize to the supercase the definition of determinant. For a finite dimensional algebra R and $C \in \mathrm{Mat}_{n|m}(\mathbb{R})$ we would like to have

$$\mathrm{sdet}(e^C) = e^{\mathrm{sTr}C} = e^{\mathrm{Tr}(C_{00})-\mathrm{Tr}(C_{11})}, \tag{9.1}$$

which generalizes the usual property of trace and determinant. So in the case of a block-diagonal matrix $C = C_{00} \oplus C_{11}$ we get

$$\mathrm{sdet}(e^C) = \frac{\det(e^{C_{00}})}{\det(e^{C_{11}})}.$$

Thus if $A = A_{00} \oplus A_{11}$ is block-diagonal, we must have

$$\mathrm{sdet}A = \frac{\det A_{00}}{\det A_{11}}.$$

This shows that we cannot hope that the superdeterminant will be a polynomial in the entries of A – it has to be a rational function defined only on some open subset. In fact, if we want to have the usual property $\mathrm{sdet}(AB) = \mathrm{sdet}(A)\mathrm{sdet}(B)$ then there is just one possibility. Indeed, suppose that

$$A = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_+ & 0\\ 0 & a_-\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = \begin{pmatrix}a_+ + ba_-c & ba_-\\ a_-c & a_-\end{pmatrix}.$$

By (9.1), we must have

$$\mathrm{sdet}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \mathrm{sdet}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = 1,$$

hence

$$\mathrm{sdet}(A) = \frac{\det a_+}{\det a_-}.$$

In other words, the superdeterminant has to be defined by the formula

$$\mathrm{sdet}(A) = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det(A_{11})}$$

provided that A_{11} is invertible; otherwise the superdeterminant is not defined.

This function is also called the *Berezinian* of A and denoted $\mathrm{Ber}(A)$. So for $m = 0$ one has $\mathrm{Ber}(A) = \det(A)$, and for $n = 0$ one has $\mathrm{Ber}(A) = (\det A)^{-1}$.

Remark 9.4. Recall for comparison that if A is a purely even block matrix then

$$\det(A) = \det(A_{00} - A_{01}A_{11}^{-1}A_{10})\det(A_{11}).$$

Proposition 9.5. *(i) For any $A, B \in \mathrm{Mat}_{n|m}(R)$ with A_{11}, B_{11} invertible, we have*

$$\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B).$$

(ii) If R is finite dimensional and $A(t) \in \mathrm{Mat}_{n|m}(R)$ is a C^1-function near 0 with $A(0)$ invertible then

$$\tfrac{d}{dt}|_{t=0}\mathrm{Ber}(A(t)) = \mathrm{sTr}(A'(0)A(0)^{-1})\mathrm{Ber}(A(0)).$$

(iii) If R is finite dimensional then for any $C \in \mathrm{Mat}_{n|m}(R)$ we have

$$\mathrm{Ber}(e^C) = e^{\mathrm{sTr}C}.$$

Proof. (i) From the triangular factorization, it is clear that it suffices to consider the case

$$A = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix},$$

where X, Y are matrices with odd elements, so that

$$AB = \begin{pmatrix} 1 & Y \\ X & 1 + XY \end{pmatrix}.$$

Then the required identity is

$$\det(1 - Y(1 + XY)^{-1}X) = \det(1 + XY).$$

To prove this identity, recall that $X : V_0 \to V_1 \otimes R$ and $Y : V_1 \to V_0 \otimes R$. We have

$$\det(1 - Y(1 + XY)^{-1}X) = \sum_{k \geq 0}(-1)^k \mathrm{Tr}(Y(1 + XY)^{-1}X|_{\Lambda^k V_0}) =$$

$$= \sum_{k \geq 0}(-1)^k \mathrm{sTr}(Y(1 + XY)^{-1}|_{\Lambda^k V_1} \circ X|_{\Lambda^k V_0}) = \sum_{k \geq 0}(-1)^k \mathrm{sTr}(XY(1 + XY)^{-1}|_{\Lambda^k V_1})$$

$$\sum_{k \geq 0} \mathrm{Tr}(XY(1 + XY)^{-1}|_{S^k \Pi V_1}) = \det(1 - XY(1 + XY)^{-1})^{-1} = \det(1 + XY).$$

(ii) By (i) we may replace $A(t)$ by $A(t)A(0)^{-1}$, so it suffices to consider the case $A(0) = 1$, where the statement easily follows from the definition.

(iii) Consider the function $f(t) := \mathrm{Ber}(e^{Ct})$. By (ii) it satisfies the differential equation $f'(t) = \mathrm{sTr}(C)f(t)$ with $f(0) = 1$. Thus $f(t) = e^{\mathrm{sTr}(C)t}$, and the statement follows by setting $t = 1$. □

9.6. **Integration on superdomains.** We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.

2. Find the transformation formula for the integral under change of coordinates (i.e. discover the factor $|J|$, where J is the Jacobian).

3. Define a *density* on a manifold to be a quantity which is locally the same as a function, but multiplies by $|J|$ under coordinate change (unlike true functions, which don't multiply by anything). Then define integral of compactly supported densities on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The *Berezinian* of V is the line $\Lambda^{\mathrm{top}} V_0^* \otimes \Lambda^{\mathrm{top}} V_1$ (where V_0, V_1 are treated as usual spaces). Suppose that V is equipped with a nonzero element dv of its Berezinian (called a *supervolume element*).

Let U_0 be an open set in V_0, and $f \in C^\infty(U_0) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$ (i.e. $f = \sum f_i \otimes \omega_i$, $f_i \in C^\infty(U_0)$, $\omega_i \in \Lambda V_1^*$, and f_i are compactly supported). Let dv_0, dv_1 be volume forms on V_0, V_1 such that $dv = dv_0/dv_1$.

Definition 9.6. The integral $\int_U f(v)dv$ is $\int_{U_0} (f(v), (dv_1)^{-1})dv_0$.

It is clear that this quantity depends only on dv and not on dv_0 and dv_1 separately.

Thus, $\int_U f(v)dv$ is defined as the integral of the suitably normalized top coefficient of f (expanded with respect to some homogeneous basis of ΛV_1^*). To write it in coordinates, let $x_1, ..., x_n, \xi_1, ..., \xi_m$ be a linear system of coordinates on V such that $dv = \frac{dx_1 ... dx_n}{d\xi_1 ... d\xi_m}$ (such coordinate systems will be called unimodular with respect to dv). Then $\int_U f(v)dv$ equals $\int_{U_0} f_{\mathrm{top}}(x_1, ..., x_n)dx_1...dx_n$, where f_{top} is the coefficient of $\xi_1...\xi_m$ in the expansion of f.

9.7. **Berezin's change of variable formula.** Let V be a vector space, $f \in \Lambda V^*$, $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of f with v.

Let U, U' be superdomains, and $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$ on U and $y_1, ..., y_p, \eta_1, ..., \eta_q$ on U', we can describe F by expressing y_i and η_j as

functions of x_i and ξ_j. Define the *Berezin matrix* of F, $A := DF(x,\xi)$ by the formulas:

$$A_{00} = (\frac{\partial y_i}{\partial x_k}),\ A_{01} = (\frac{\partial y_i}{\partial \xi_\ell}),\ A_{10} = (\frac{\partial \eta_j}{\partial x_k}),\ A_{11} = (\frac{\partial \eta_j}{\partial \xi_\ell}).$$

Clearly, this is a superanalog of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

Theorem 9.7. *(Berezin) Let g be a smooth function with compact support on U', and $F : U \to U'$ be an isomorphism. Let dv, dv' be supervolume elements on U, U'. Then*

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\mathrm{Ber}(DF(v))|dv,$$

where the Berezinian is computed with respect to unimodular coordinate systems.

Here if $f(\xi) = a$+terms containing ξ_j, $a \in \mathbb{R}$, $a \neq 0$ then by definition $|f(\xi)| := f(\xi)$ is $a > 0$ and $|f(\xi)| := -f(\xi)$ if $a < 0$.

Proof. The chain rule of the usual calculus extends verbatim to supercalculus. Thus, since $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B)$, if we know the statement for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, ..., x_n, \xi_1, ..., \xi_m) = (x'_1, ..., x'_n, \xi'_1, ..., \xi'_m)$. We see that it suffices to consider the following cases.

1. x'_i depend only on x_k, $k = 1, ..., n$, and $\xi'_j = \xi_j$.
2. $x'_i = x_i + z_i$, where z_i lie in the ideal generated by ξ_j, and $\xi'_j = \xi_j$.
3. $x'_i = x_i$.

Indeed, it is clear that any isomorphism F is a composition of isomorphisms of types $1, 2, 3$.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by F, i.e. $x'_1 = x_1 + z$, and $x'_i = x_i$ for $i \geq 2$. In this case we have to show that the integral of

$$g(x_1 + z, x_2, ..., x_n, \xi)(1 + \tfrac{\partial z}{\partial x_1}) - g(x_1, x_2, ..., x_n, \xi)$$

is zero. But this follows easily upon expansion in powers of z, since all the terms are manifestly total derivatives with respect to x_1.

In case 3, we can also assume $\xi'_j = \xi_j$, $j \geq 2$, and a similar (actually, even simpler) argument proves the result. □

9.8. **Integration on supermanifolds.** Now we will define densities on supermanifold, and $\{U_\alpha\}$ be an open cover of M together with isomorphisms $f_\alpha : U_\alpha \to U'_\alpha$, where U'_α is a superdomain in $\mathbb{R}^{n|m}$. Let $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density s on M is a choice of an element $s_\alpha \in C^\infty_M(U_\alpha)$ for each α, such that on $U_\alpha \cap U_\beta$ one has $s_\beta(z) = s_\alpha(z)|\mathrm{Ber}(g_{\alpha\beta})(f_\beta(z))|$.

Remark 9.8. It is clear that a density on M is a global section of a certain sheaf on M, called the sheaf of densities.

Now, for any (compactly supported) density ω on M, the integral $\int_M \omega$ is well defined. Namely, it is defined as in usual calculus: one uses a partition of unity ϕ_α such that $\mathrm{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$ (where the summands can be defined using f_α). Berezin's theorem guarantees then that the final answer will be independent on the choices made.

9.9. **Gaussian integrals in an odd space.** Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of *Pfaffian*. Let A be a skew-symmetric matrix of even size. Then the determinant of A is the square of a polynomial in the entries of A. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of A and denoted $\mathrm{Pf}A$. The Pfaffian obviously has the property $\mathrm{Pf}(X^T AX) = \mathrm{Pf}(A)\det(X)$ for any matrix X.

Let now V be a $2m$-dimensional vector space with a volume element dv, and B a skew-symmetric bilinear form on V. We define the Pfaffian $\mathrm{Pf}B$ of B to be the Pfaffian of the matrix of B in any unimodular basis (by the above transformation formula, it does not depend on the choice of the basis). It is easy to see (by reducing B to the canonical form) that

$$\frac{\Lambda^m B}{m!} = \mathrm{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well known) formula for the Pfaffian of a skew symmetric matrix of size $2m$:

$$\mathrm{Pf}(A) = \sum_{\sigma\in\Pi_m} \varepsilon_\sigma \prod_{i\in\{1,\dots,2m\}, i<\sigma(i)} a_{i\sigma(i)},$$

where Π_m is the set of matchings of $\{1, ..., 2m\}$, and ε_σ is the sign of the permutation sending $1, ..., 2m$ to $i_1, \sigma(i_1), ..., i_m, \sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all r). For example, for $m = 2$ (i.e. a 4 by 4 matrix),

$$\mathrm{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space V of dimension $2m$ with a volume element $d\xi$. Let B be a symmetric bilinear form on V (i.e. a skewsymmetric form on ΠV). Let $\xi_1, ..., \xi_{2m}$ be unimodular linear coordinates on V (i.e. $d\xi = d\xi_1 \wedge ... \wedge d\xi_{2m}$). So if $\xi = (\xi_1, ..., \xi_{2m})$ then $B(\xi, \xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where b_{ij} is a skewsymmetric matrix.

Proposition 9.9.

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(B).$$

Proof. The integral equals $\frac{1}{m!}\frac{\wedge^m B}{d\xi}$, which is precisely $\mathrm{Pf}(B)$. □

This formula has the following important special case. Let Y be a finite dimensional odd vector space, and $V = Y \oplus Y^*$. The space Y has a canonical volume element $dv = dydy^*$, defined as follows: if $e_1, ..., e_m$ is a basis of Y and $e_1^*, ..., e_m^*$ is the dual basis of Y^* then $dydy^* = e_1 \wedge e_1^* \wedge ... \wedge e_n \wedge e_n^*$.

Let $A : Y \to Y$ be a linear operator. Then we can define an even smooth function S on the odd space Y as follows: $S(y, y^*) = (Ay, y^*)$. More explicitly, if ξ_i are coordinates on Y corresponding to the basis e_i, and η_i the dual system of coordinates on Y^*, then

$$S(\xi_1, ..., \xi_m, \eta_1, ..., \eta_m) = \sum_{i,j} a_{ij}\xi_j\eta_i,$$

where (a_{ij}) is the matrix of A in the basis e_i.

Proposition 9.10.

$$\int_V e^S (dv)^{-1} = (-1)^{\frac{n(n-1)}{2}} \det A.$$

Proof. We have $S(y, y_*) = \frac{1}{2}B((y, y_*), (y, y_*))$, where B is the skewsymmetric form on ΠV given by the formula

$$B((y, y^*), (w, w^*)) = (Ay, w^*) - (Aw, y^*).$$

It is easy to see that $\mathrm{Pf}(B) = (-1)^{\frac{n(n-1)}{2}} \det(A)$, so Proposition 9.10 follows from Proposition 9.9.

Another proof can be obtained by direct evaluation of the top coefficient. □

9.10. **The Wick formula in the odd case.** Let V be a $2m$-dimensional odd space with a volume form $d\xi$, and $B \in S^2V^*$ a non-degenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let $\lambda_1, ..., \lambda_n$ be linear functions on V. Then $\lambda_1, ..., \lambda_n$ can be regarded as odd smooth functions on the superspace V.

Theorem 9.11.

$$\int_V \lambda_1(\xi)...\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(-B)\mathrm{Pf}(B^{-1}(\lambda_i,\lambda_j)).$$

(By definition, this is zero if n is odd). In other words, we have:

$$\int_V \lambda_1(\xi)...\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} =$$

$$\mathrm{Pf}(-B)\sum_{\sigma\in\Pi_m}\varepsilon_\sigma \prod_{i\in\{1,...,2m\}, i<\sigma(i)} B^{-1}(\lambda_i,\lambda_{\sigma(i)}).$$

Proof. We prove the second formula. Choose a basis e_i of V with respect to which the form B is standard: $B(e_j, e_l) = 1$ if $j = 2i-1, l = 2i$, and $B(e_j, e_l) = 0$ for other pairs $j < l$. Since both sides of the formula are polylinear with respect to $\lambda_1, ..., \lambda_n$, it suffices to check it if $\lambda_1 = e^*_{i_1}$,..., $\lambda_n = e^*_{i_n}$. This is easily done by direct computation (in the sum on the right hand side, only one term may be nonzero). □

Exercise 9.12. *Let $Y = \mathbb{R}^{\frac{n(n+1)}{2}+\frac{m(m-1)}{2}|mn}$ be the real superspace of matrices*

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

(where A_{00} is n by n and A_{11} is m by m) which are symmetric in the supersense, i.e., A_{00} is symmetric, A_{11} is skew-symmetric, and $A^T_{01} = A_{10}$. Let $Y_+ \subset Y$ be the superdomain of those matrices for which $A_{00} > 0$. Let dA be a supervolume element on Y. Let f be a compactly supported smooth function on Y_+. Show that

$$\int_{Y_+\times\mathbb{R}^{n|m}} f(A)e^{-x^TA_{00}x-2x^TA_{01}\xi-\xi^TA_{11}\xi}dAdx(d\xi)^{-1} =$$

$$= C\int_{Y_+} f(A)\mathrm{Ber}(A)^{-1/2}dA.$$

(C is a constant). What is C?

Exercise 9.13. *Prove the Amitsur-Levitzki identity: if $X_1, ..., X_{2n}$ are n by n matrices over a commutative ring, then*

$$\sum_{\sigma\in S_{2n}}(-1)^\sigma X_{\sigma(1)}...X_{\sigma(2n)} = 0.$$

Hint. *(a) Show that for any n by n matrix X with anticommuting entries, $X^{2n} = 0$ (namely, show that traces of X^{2k} vanish for all positive k, then use the Cayley-Hamilton theorem for X^2).*

(b) Apply this to $X = \sum_{i=1}^{2n} X_i \xi_i$, where ξ_i are anticommuting variables.

10. Quantum mechanics for fermions

10.1. **Feynman calculus in the supercase.** Wick's theorem allows us to extend Feynman calculus to the supercase. Namely, let

$$V = V_0 \oplus V_1$$

be a finite dimensional real superspace with a supervolume element $dv = dv_0(dv_1)^{-1}$, equipped with a symmetric non-degenerate form $B = B_0 \oplus B_1$ ($B_0 > 0$). Let

$$S(v) = \frac{1}{2}B(v,v) - \sum_{r\geq 3} \frac{B_r(v,v,...,v)}{r!}$$

be an even function on V (the action). Note that B_r, $r \geq 3$ can contain mixed terms involving both odd and even variables, e.g. $x\xi_1\xi_2$ (the so called "Yukawa term"). We will consider the integral

$$I(\hbar) = \int_V \ell_1(v_0)...\ell_n(v_0)\lambda_1(v_1)...\lambda_p(v_1)e^{-\frac{S(v)}{\hbar}}dv,$$

where v_0, v_1 are the even and odd components of v. Then this integral has an expansion in $\hbar$ written in terms of Feynman diagrams. Since v has both odd and even part, these diagrams will contain "odd" and "even" edges (which are usually depicted by straight and wiggly lines, respectively). More precisely, let us write

$$B_r(v,v,...,v) = \sum_{s=0}^{r} \binom{r}{s} B_{s,r-s}(v_1,...,v_1,v_0,...,v_0),$$

where $B_{s,r-s}$ has homogeneity degree s with respect to v_1 and $r-s$ with respect to v_0 (i.e. it will be nonzero only for even s). Then to each term $B_{s,r-s}$ we assign an $(s,r-s)$-valent flower, i.e. a flower with s odd and $r-s$ even outgoing edges, and for the set of odd outgoing edges, specify which orderings are even. Then, given an arrangement of flowers, for every matching σ of outgoing edges, we can define an amplitude $\mathbb{F}(\sigma)$ by contracting the tensors $B_{s,r-s}$ (and being careful with the signs). It is easy to check that all matchings giving the same graph will contribute to $I(\hbar)$ with the same sign, and thus we have almost the same formula as in the bosonic case:

$$I(\hbar) = (2\pi)^{\frac{\dim V_0}{2}}\hbar^{\frac{\dim V_0 - \dim V_1}{2}}\frac{\mathrm{Pf}(-B_1)}{\sqrt{\det B_0}}\sum_{\Gamma}\frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\mathbb{F}_\Gamma(\ell_1,...,\ell_n,\lambda_1,...,\lambda_p),$$

where the summation is taken over graphs with n even and p odd outgoing edges.

Remark 10.1. More precisely, we can define the sign ε_σ of a matching σ as follows: label outgoing edges by $1, 2, ...$, starting from the fisrt flower, then second, etc., so that the labeling is even on each flower. Then write the labels in a sequence, enumerating (in any order) the pairs defined by σ (the element with the smaller of the two labels goes first). The sign ε_σ is by definition the sign of this ordering (as a permutation of $1, 2, ...$). Then $\mathbb{F}_\Gamma$ is $\mathbb{F}(\sigma)$ for any matching σ yielding Γ which is *positive*, i.e. such that $\varepsilon_\sigma = 1$. For a negative matching, $\mathbb{F}_\Gamma = -\mathbb{F}(\sigma)$.

In most (but not all) situations considered in physics, the action is quadratic in the fermionic variables, i.e.

$$S(v) = S_b(v_0) - \tfrac{1}{2} S_f(v_0)(v_1, v_1),$$

where $S_f(v_0)$ is a skew-symmetric bilinear form on ΠV_1. In this case, using fermionic Wick's theorem, we can perform exact integration with respect to v_1, and reduce $I(\hbar)$ to a purely bosonic integral. For example, if we have only ℓ_i and no λ_i, then

$$I(\hbar) = \hbar^{-\frac{\dim V_1}{2}} \int_{V_0} \ell_1(v_0)...\ell_n(v_0) e^{-\frac{S_b(v_0)}{\hbar}} \mathrm{Pf}(S_f(v_0)) dv_0.$$

In this situation, all vertices which have odd outgoing edges, will have only two of them, and therefore in any Feynman diagram with even outgoing edges, odd lines form nonintersecting simple curves, called *fermionic loops* (in fact, the last formula is nothing but the result of regarding these loops as a new kind of vertices – convince yourself of this!). In this case, there is the following simple way of assigning signs to Feynman diagrams. For each vertex with two odd outgoing edges, we orient the first edge inward and the second one outward. We allow only connections (matchings) that preserve orientations (so the fermionic loops become oriented). Then the sign is $(-1)^r$, where r is the number of fermionic loops (i.e. each fermionic loop contributes a minus sign). This follows from the fact that an even cycle is an odd permutation.

10.2. **Fermionic quantum mechanics.** Let us now pass from finite dimensional fermionic integrals to quantum mechanics, i.e. integrals over fermionic functions of one (even) real variable t.

Let us first discuss fermionic classical mechanics, in the Lagrangian setting. Its difference with the bosonic case is that the "trajectory" of the particle is described by an *odd-valued* function of one variable, i.e. $\psi : \mathbb{R} \to \Pi V$, where V is a vector space. Mathematically this means that the space of fields (=trajectories) is an odd vector space

$\Pi C^\infty(\mathbb{R}, V)$. A Lagrangian $\mathcal{L}(\psi)$ is a local expression in such a field (i.e. a polynomial in $\psi, \dot{\psi}, ...$), and an action is the integral $S = \int_{\mathbb{R}} \mathcal{L}dt$. This means that the action is an element of the space $\Lambda(C_0^\infty(\mathbb{R}, V)^*)$.

Consider for example the theory of a single scalar-valued free fermion $\psi(t)$. By definition, the Lagrangian for such a theory is

$$\mathcal{L} = \frac{1}{2}\psi\dot{\psi},$$

i.e. the action is

$$S = \frac{1}{2}\int \psi\dot{\psi}dt.$$

This Lagrangian is the odd analog of the Lagrangian of a free particle, $\frac{\dot{q}^2}{2}$.

Remark 10.2. Note that $\psi\dot{\psi} \neq \frac{d}{dt}(\frac{\psi^2}{2}) = 0$, since $\psi\dot{\psi} = -\dot{\psi}\psi$, so this Lagrangian is "reasonable". On the other hand, the same Lagrangian would be unreasonable in the bosonic case, as it would be a total derivative, and hence the action would be zero. Finally, note that it would be equally unreasonable to use in the fermionic case the usual bosonic Lagrangian $\frac{1}{2}(\dot{q}^2 - m^2q^2)$; it would identically vanish if q were odd-valued.

The Lagrangian $\mathcal{L}$ is invariant under the group of reparametrizations $\mathrm{Diff}_+(\mathbb{R})$, and the Euler-Lagrange equation for this Lagrangian is

$$\dot{\psi} = 0$$

(i.e. no dynamics). Theories with such properties are called *topological quantum field theories.*

Let us now turn to quantum theory in the Lagrangian setting, i.e. the theory given by the Feynman integral $\int \psi(t_1)...\psi(t_n)e^{\frac{iS(\psi)}{\hbar}}D\psi$. In the bosonic case, we "integrated" such expressions over the space $C_0^\infty(\mathbb{R})$. This integration did not make immediate sense because of difficulties with measure theory in infinite dimensions. So we had to make sense of this integration in terms of $\hbar$-expansion, using Wick's formula and Feynman diagrams. In the fermionic case, the situation is analogous. Namely, now we must integrate functions over $\Pi C_0^\infty(\mathbb{R})$, which are elements of $\Lambda\mathcal{D}(\mathbb{R})$, where $\mathcal{D}(\mathbb{R})$ is the space of distributions on $\mathbb{R}$. Although in the fermionic case we don't need measure theory (as integration is completely algebraic), we still have trouble defining the integral: recall that by definition the integral should be the top coefficient of the integrand as the element of $\Lambda\mathcal{D}(\mathbb{R})$, which makes no sense since in the exterior algebra of an infinite dimensional space there is

no top component. Thus we have to use the same strategy as in the bosonic case, i.e. Feynman diagrams.

Let us, for instance, define the quantum theory for a free scalar valued fermion, i.e one described by the Lagrangian $\mathcal{L} = \frac{1}{2}\psi\dot{\psi}$. According to the yoga we used in the bosonic case, the two-point function of this theory $\langle\psi(t_1)\psi(t_2)\rangle$ should be the function $G(t_1 - t_2)$, where G is the solution of the differential equation

$$\frac{dG}{dt} = i\delta(t).$$

(the factor i comes from the exponent in the Feynman integral; note that in the fermionic case it does *not* go away under Wick rotation).

The general solution of this equation has the form

$$G(t) = \frac{i}{2}\text{sign}(t) + C.$$

Because of the fermionic nature of the field $\psi(t)$, it is natural to impose the requirement that $G(-t) = -G(t)$, i.e. that the correlation functions are antisymmetric; this singles out the solution $G(t) = \frac{i}{2}\text{sign}(t)$ (we also see from this condition that we should set $G(0) = 0$). As usual, the $2n$-point correlation functions are defined by the Wick formula. That is, for distinct t_j,

$$\langle\psi(t_1)...\psi(t_{2n})\rangle = (-1)^\sigma(2n-1)!!(\tfrac{i}{2})^n,$$

where σ is the permutation that orders t_j in the decreasing order. If at least two points coincide, the correlation function is zero.

Thus we see that the correlation functions are invariant under $\text{Diff}_+(\mathbb{R})$. In other words, using physical terminology, we have a *topological quantum field theory*.

Note that the correlation functions in the Euclidian setting for this model are the same as in the Minkowski setting, since they are (piecewise) constant in t_j. In particular, they don't decay at infinity, and hence our theory does not have the clustering property.

We have considered the theory of a massless fermionic field. Consider now the massive case. This means, we want to add to the Lagrangian a quadratic term in ψ which does not contain derivatives. If we have only one field ψ, the only choice for such term is ψ^2, which is zero. So in the massive case we must have at least two fields. Let us therefore consider the theory of two fermionic fields ψ_1, ψ_2 with (Euclidean) Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 - m\psi_1\psi_2),$$

where $m > 0$ is a mass parameter. The Green's function for this model satisfies the differential equation

$$\frac{dG}{dt} - MG = i\delta(t),$$

where $M = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$ and G is a 2 by 2 matrix-valued function. The general solution of this equation is

$$G(t) = \begin{cases} e^{Mt}Q_-, \ t < 0 \\ e^{Mt}Q_+, \ t > 0 \end{cases}$$

where $Q_+ - Q_- = i$. Now, we want the Wick rotated Green's function $G(-it)$ to have the clustering property. Thus we want

$$\lim_{t\to+\infty} e^{-iMt}Q_+ = 0, \ \lim_{t\to-\infty} e^{-iMt}Q_- = 0.$$

This implies that $Q_+ = iP_+$, $Q_- = -iP_-$, where $P_\pm$ are the orthogonal projectors to the eigenspaces of iM with eigenvalues $\pm m$ (and $G(0) = 0$).

Remark 10.3. It is easy to generalize this analysis to the situation when ψ takes values in a positive definite inner product space V, and $M : V \to V$ is a skewsymmetric operator, since such a situation is a direct sum of the situations considered above.

In the case when M is non-degenerate, one can define the corresponding theory with interactions, i.e. with higher than quadratic terms in ψ. Namely, one defines the correlators as sums of amplitudes of appropriate Feynman diagrams. We leave it to the reader to work out this definition, by analogy with the finite dimensional case which we have discussed above.

10.3. **Super Hilbert spaces.** The space of states of a quantum system is a Hilbert space. As we plan to do Hamiltonian quantum mechanics for fermions, we must define a superanalog of this notion.

Suppose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a $\mathbb{Z}/2$-graded complex vector space.

Definition 10.4. (i) A Hermitian form on $\mathcal{H}$ is an even sesquilinear form $\langle \, , \, \rangle$, such that $\langle \, x, y \, \rangle = \overline{\langle \, y, x \, \rangle}$ for even x, y, and $\langle \, x, y \, \rangle = -\overline{\langle \, y, x \, \rangle}$ for odd x, y.

(ii) A Hermitian form is positive definite if $\langle \, x, x \, \rangle > 0$ for even $x \neq 0$, and $-i\langle \, x, x \, \rangle > 0$ for odd $x \neq 0$. A super Hilbert space is a superspace with a positive definite Hermitian form $\langle \, , \, \rangle$, which is complete in the corresponding norm.

(iii) Let $\mathcal{H}$ be a super Hilbert space, and $T : \mathcal{H}_0 \oplus \Pi\mathcal{H}_1 \to \mathcal{H}_0 \oplus \Pi\mathcal{H}_1$ be a homogeneous linear operator between the underlying purely even spaces. The Hermitian adjoint operator $T^\dagger$ is defined by the equation $\langle x, T^\dagger y \rangle = (-1)^{p(x)p(T)} \langle Tx, y \rangle$, where p denotes the parity.

10.4. **The Hamiltonian setting for fermionic quantum mechanics.** Let us now discuss what should be the Hamiltonian picture for the theory of a free fermion. More precisely, let V be a positive definite finite dimensional real inner product space, and consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}((\psi, \dot{\psi}) - (\psi, M\psi)),$$

where $\psi : \mathbb{R} \to \Pi V$, and $M : V \to V$ is a skew-symmetric operator.

To understand what the Hamiltonian picture should be, let us compare with the bosonic case. Namely, consider the Lagrangian

$$\mathcal{L}_b = \frac{1}{2}(\dot{q}^2 - m^2 q^2),$$

where $q : \mathbb{R} \to V$. In this case, the classical space of states is

$$Y := T^*V = V \oplus V^*.$$

The equations of motion are Newton's equations

$$\ddot{q} = -m^2 q,$$

which can be reduced to Hamilton's equations

$$\dot{q} = p,\ \dot{p} = -m^2 q.$$

The algebra of classical observables is $C^\infty(Y)$, with Poisson bracket defined by $\{a, b\} = (a, b)$, $a, b \in Y^*$, where $(,)$ is the form on Y^* inverse to the natural symplectic form on Y. The hamiltonian H is determined (up to adding a constant) by the condition that the equations of motion are $\dot{f} = \{f, H\}$; in this case it is $H = \frac{1}{2}(p^2 + m^2 q^2)$.

The situation in the fermionic case is analogous, with some important differences which we will explain below. Namely, it is easy to compute that the equation of motion (i.e. the Euler-Lagrange equation) is

$$\dot{\psi} = M\psi.$$

The main difference with the bosonic case is that this equation is of first and not of second order, so the space of classical states is just ΠV (no momentum or velocity variables are introduced). Hence the algebra of classical observables is $C^\infty(\Pi V) = \Lambda V^*$. To define a Poisson bracket on this algebra, recall that ΠV has a natural "symplectic structure", defined by the *symmetric* form $(,)$ on V. Thus we can define a Poisson bracket on ΛV^* by the same formula as above: $\{a, b\} = (a, b)$ when

$a, b \in V^*$. More precisely, $\{,\}$ is a unique skew symmetric (in the supersense) bilinear operation on ΛV^* which restricts to (a, b) for $a, b \in V^*$, and is a derivation with respect to each variable:

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)}b\{a, c\},$$

where $p(a)$ denotes the parity of a.

Now it is easy to see what should play the role of the Hamiltonian. More precisely, the definition with Legendre transform is not valid in our situation, since the Legendre transform was done with respect to the velocity variables, which we don't have in the fermionic case. On the other hand, as we discussed in Section 8, in the bosonic case the equation of motion

$$\dot{f} = \{f, H\}$$

determines H uniquely, up to a constant. The situation is the same in the fermionic case. Namely, by looking at the equation of motion $\dot{\psi} = M\psi$, it is easy to see that the Hamiltonian equals

$$H = \frac{1}{2}(\psi, M\psi).$$

In particular, if $M = 0$ (massless case), the Hamiltonian is zero (a characteristic property of topological field theories).

Now let us turn to quantum theory. In the bosonic case the algebra of quantum observables is a noncommutative deformation of the algebra $C^\infty(Y)$ in which the relation $\{a, b\} = (a, b)$ is replaced with its quantum analog

$$ab - ba = i(a, b)$$

(up to the Planck constant factor which here we will set to 1). In particular, the subalgebra of polynomial observables is the Weyl algebra $W(Y)$, generated by Y^* with this defining relation. By analogy with this, we should define the algebra of quantum observables in the fermionic case to be generated by V^* with the relation

$$ab + ba = i(a, b)$$

(it deforms the relation $ab + ba = 0$ which defines ΛV^*). So we recall the following definition.

Definition 10.5. Let V be a vector space over a field k with a symmetric bilinear form Q. The *Clifford algebra* $\mathrm{Cl}(V, Q)$ is generated by V with defining relations $ab + ba = Q(a, b)$, $a, b \in V$.

We see that the algebra of quantum observables should be $\mathrm{Cl}(V^*_{\mathbb{C}}, i(,))$. Note that like in the classical case, this algebra is naturally $\mathbb{Z}/2$ graded, so that we have even and odd quantum observables.

Now let us see what should be the Hilbert space of quantum states. In the bosonic case it was $L^2(V)$, which is, by the well known Stone-von Neumann theorem, the unique irreducible unitary representation of $W(Y)$. By analogy with this, in the fermionic case the Hilbert space of states should be an irreducible unitary representation of $\mathrm{Cl}(V_{\mathbb{C}}^*)$ on a supervector space $\mathcal{H}$.

The structure of the Clifford algebra $\mathrm{Cl}(V_{\mathbb{C}}^*)$ is well known. Namely, consider separately the cases when $\dim V$ is odd and even.

In the even case, $\dim V = 2d$, $\mathrm{Cl}(V_{\mathbb{C}}^*)$ is simple (i.e., isomorphic to a matrix algebra), and has a unique irreducible representation $\mathcal{H}$, of dimension 2^d. This representation is constructed as follows: choose a decomposition $V_{\mathbb{C}} = L \oplus L^*$, where L, L^* are Lagrangian subspaces; then $\mathcal{H} = \Lambda L$, where $L \subset V_{\mathbb{C}}^*$ acts by multiplication and L^* by differentiation (multiplied by $-i$). The structure of the superspace on $\mathcal{H}$ is the standard one on the exterior algebra.

In the odd case, $\dim V = 2d + 1$, choose a decomposition

$$V_{\mathbb{C}} = L \oplus L^* \oplus K,$$

where L, L^* are maximal isotropic, and K is a non-degenerate 1-dimensional subspace orthogonal to L and L^*. Let $\mathcal{H} = \Lambda(L \oplus K)$, where L, K act by multiplication and L^* by ($-i$ times) differentiation. This is a representation of $\mathrm{Cl}(V_{\mathbb{C}}^*)$ with a $\mathbb{Z}/2$ grading. This representation is not irreducible, and decomposes in a direct sum of two non-isomorphic irreducible representations $\mathcal{H}_+ \oplus \mathcal{H}_-$ (this is related to the fact that the Clifford algebra for odd $\dim V$ is not simple but is a direct product of two simple, i.e. matrix, algebras). However, this decomposition is not consistent with the $\mathbb{Z}/2$-grading, and therefore as superrepresentation, $\mathcal{H}$ is irreducible.

Now, it is easy to show that both in the odd and in the even case the space $\mathcal{H}$ carries a unique up to scaling Hermitian form, such that $V^* \subset V_{\mathbb{C}}^*$ acts by self-adjoint operators. This form is positive definite. So the situation is similar to the bosonic case for any $\dim V$.

Let us now see which operator on $\mathcal{H}$ should play the role of the Hamiltonian of the system. The most natural choice is to define the quantum Hamiltonian to be the obvious quantization of the classical Hamiltonian $H = \frac{1}{2}(\psi, M\psi)$. Namely, if ε_i is an orthonormal basis of V^* and a_{ij} is the matrix of M in this basis, then one sets

$$\widehat{H} = \frac{1}{2} \sum_{i,j} a_{ij} \varepsilon_i \varepsilon_j.$$

To compute this operator more explicitly, we will assume (without loss of generality) that the decomposition of $V_{\mathbb{C}}$ that we chose is stable

under M. Let ξ_j be an eigenbasis of M in L with eigenvalues im_j where $m_j \geq 0$, and ∂_j be differentiations along the vectors of this basis. Then

$$\widehat{H} = \sum_j m_j(\xi_j \partial_j - \partial_j \xi_j) = \sum_j m_j(2\xi_j \partial_j - 1).$$

This shows that if $\dim V$ is even then the partition function on the circle of length L for our theory is

$$Z = \mathrm{sTr}(e^{-L\widehat{H}}) = \prod_j (e^{m_j L} - e^{-m_j L}).$$

If the dimension of V is odd then the partition function is zero.

Now we would like to consider the fermionic analog of the Feynman-Kac formula. For simplicity consider the fully massive case, when $\dim V$ is even and $m_j \neq 0$ (i.e. M is non-degenerate). In this case, we have a unique up to scaling lowest eigenvector of $\widehat{H}$, namely $\Omega = 1$.

Let $\psi(0) \in V \otimes \mathrm{End}(\mathcal{H})$ be the element corresponding to the action map $V^* \to \mathrm{End}(\mathcal{H})$ (the *Clifford multiplication*), and $\psi(t) = e^{it\widehat{H}}\psi(0)e^{-it\widehat{H}}$. Also, denote by $\langle \psi(t_1)...\psi(t_n)\rangle$, $t_1 \geq ... \geq t_n$, the correlation function for the free theory in the Lagrangian setting, taking values in $V^{\otimes n}$ (so in this expression $\psi(t_j)$ is a formal symbol and not an operator).

Theorem 10.6. *(Feynman-Kac formula) (i) For the free theory on the line we have*

$$\langle \psi(t_1)...\psi(t_n)\rangle = \langle\, \Omega, \psi(t_1)...\psi(t_n)\Omega \,\rangle.$$

(ii) For the free theory on the circle of length L we have

$$\langle \psi(t_1)...\psi(t_n)\rangle = \frac{\mathrm{sTr}(\psi(t_1)...\psi(t_n)e^{-L\widehat{H}})}{\mathrm{sTr}(e^{-L\widehat{H}})}.$$

Exercise 10.7. *Prove this theorem. (The proof is analogous to Theorem 8.22 in the free case).*

It should now be straightforward for the reader to formulate and prove the Feynman-Kac formula for an interacting (i.e., not necessarily free) quantum-mechanical model which includes both bosonic and fermionic massive fields. We leave this as an instructive exercise.

Exercise 10.8. *(i) Consider quantum mechanics with Yukawa coupling. That is, we have a scalar boson $\phi(t)$ and two fermions $\psi_1(t), \psi_2(t)$, and the Euclidean Lagragian is*

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2 + \psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 - \mu\psi_1\psi_2) + g\phi\psi_1\psi_2.$$

Compute the 2-point function $\langle\phi(t)\phi(0)\rangle$ modulo g^3 (in the Euclidean setting).

Hint. *The correction to the free theory answer is given by one Feynman diagram. Remember about automorphism groups and the minus sign corresponding to fermionic loops.*

(ii) In the same theory, compute the two-point function $\langle\psi_1(t)\psi_1(0)\rangle$ modulo g^3 (in the Euclidean setting). Does the corresponding diagram have non-trivial automorphisms?

11. Free field theories in higher dimensions

11.1. **Minkowski and Euclidean space.** Now we pass from quantum mechanics to quantum field theory in dimensions $d \geq 1$. As we explained above, we have two main settings.

1. Minkowski space. Fields are functions on a spacetime $V = V_M$, which is a real inner product space of signature $(1, d-1)$. This is where physical processes actually "take place". The symmetry group of V, $G = SO(1, d-1)$, is called the *Lorentz group*; it is the group of transformations of spacetime in special relativity. Therefore, field theories in Minkowski space which are in an appropriate sense "compatible" with the action of G are called *relativistic*.

Recall some standard facts and definitions. The *light cone* in V is the cone described by the equation $|\mathbf{v}|^2 = 0$, where $|\mathbf{v}|^2 := (\mathbf{v}, \mathbf{v})$. Vectors belonging to the light cone are called *lightlike*. The light cone divides the space V into *spacelike vectors* $|\mathbf{v}|^2 < 0$ (outside the cone), and *timelike vectors* $|\mathbf{v}|^2 > 0$ (inside the cone). We will choose one of the two components of the interior of the cone and call it positive; it will be denoted by V_+. The opposite (negative) component is denoted by V_-. The group of $g \in SO(V) = SO(1, d-1)$ which preserve V_+ is denoted by $SO_+(1, d-1)$; it is the connected component of the identity of the group $SO(1, d-1)$ (which has two connected components).

Often (e.g. when doing Hamiltonian field theory) it is necessary to split V in an orthogonal direct sum $V = V_s \oplus \mathbb{R}$ of space and time. In this decomposition, the space V_s is required to be spacelike (i.e. negative definite), which implies that the time axis $\mathbb{R}$ has to be timelike (positive definite). Note that such a splitting is not unique, and that fixing it breaks the Lorentz symmetry $SO_+(1, d-1)$ down to the usual rotation group $SO(d-1)$.

To do explicit calculations, one further chooses Cartesian coordinates $x_1, ..., x_{d-1}$ on V_s and t on the time axis $\mathbb{R}$, so that $\mathbf{v} = (t, x_1, ..., x_{d-1})$. In these coordinates the inner product takes the form

$$|\mathbf{v}|^2 = c^2 t^2 - \sum_{j=1}^{d-1} x_j^2$$

where c is the speed of light. This explains the origin of the term "light cone" – it consists of worldlines of free photons (particles of light) traveling in space in some direction at speed c. To simplify notation, we will chose units of measurement so that $c = 1$.

2. Euclidean space. Fields are functions on a spacetime V_E, which is a positive definite inner product space. It plays an auxiliary role and

has no direct physical meaning, although path integrals computed in this space are similar to expectation values in statistical mechanics.

The two settings are related by the "Wick rotation". Namely the Euclidean space V_E corresponding to the Minkowski space V_M is the real subspace in $(V_M)_{\mathbb{C}}$ consisting of vectors $(it, x_1, ..., x_{d-1})$, where t and x_j are real. In other words, to pass to the Euclidean space, one needs to make a change of variable $t \mapsto it$. Note that under this change, the standard metric on the Minkowski space, $dt^2 - \sum_j dx_j^2$ goes into a negative definite metric $-dt^2 - \sum_j dx_j^2$. However, the minus sign is traditionally dropped and one considers instead the positive metric $dt^2 + \sum_j dx_j^2$ on V_E.

11.2. **Free scalar boson.** Consider the theory of a free scalar bosonic field ϕ of mass m. The procedure of quantization of this theory in the Lagrangian setting is a straightforward generalization from the case of quantum mechanics. Namely, the Lagrangian for this theory in Minkowski space is

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2),$$

and the Euler-Lagrange equation is the *Klein-Gordon equation*

$$(\Box + m^2)\phi = 0,$$

where $\Box$ is the D'Alembertian (wave operator),

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial x_j^2}.$$

Thus to define the corresponding quantum theory, we should invert the operator $\Box + m^2$. This operator is essentially self-adjoint on compactly supported smooth functions and thus defines a self-adjoint operator, but as in the quantum mechanics case, it is not invertible – its spectrum is the whole $\mathbb{R}$, as can be easily seen by taking the Fourier transform. So as before, it is best to proceed using the Wick rotation.

After the Wick rotation (i.e. the transformation $t \mapsto it$), we arrive at the Euclidean Lagrangian

$$\mathcal{L}_E = \frac{1}{2}((d\phi)^2 + m^2\phi^2),$$

and the Euler-Lagrange equation is the Euclidean Klein-Gordon equation

$$(-\Delta + m^2)\phi = 0.$$

So to define the quantum theory, i.e. the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

where $S = \int \mathcal{L}$, we now need to invert the self-adjoint operator $A = -\Delta + m^2$ (initially defined as an essentially self-adjoint operator on smooth compactly supported functions), whose spectrum is $[m^2, \infty)$, so it is invertible when $m > 0$. The operator A^{-1} is an integral operator whose Schwartz kernel is $G(x-y)$, where $G(x)$ is the Green's function, i.e. the fundamental solution of the Klein-Gordon equation:

$$-\Delta G + m^2 G = \delta.$$

To solve this equation, note that the solution is rotationally invariant. Therefore, outside of the origin, $G(x) = g(|x|)$, where g is a function on $(0, \infty)$ such that

$$-g'' - \frac{d-1}{r} g' + m^2 g = 0$$

(where the left hand side is the radial part of the operator A). This is a version of the Bessel equation. If $m > 0$, the two basic solutions are $r^{\frac{2-d}{2}} J_{\pm\frac{2-d}{2}}(imr)$, where J is the Bessel function. (Actually, these functions are elementary for odd d). Since we want G to decay at infinity (clustering property), we should pick the unique up to scaling linear combination which decays at infinity, namely,

$$g = C r^{\frac{2-d}{2}} (J_{\frac{2-d}{2}}(imr) + i^d J_{-\frac{2-d}{2}}(imr)), \ d \neq 2. \tag{11.1}$$

For $d = 2$, this expression is zero, and one should instead take the limit of the right hand side divided by $d - 2$ as $d \to 2$. The normalizing constant can be found from the condition that $AG = \delta$.

Remark 11.1. It is easy to check that for $d = 1$ this function equals the familiar Green's function for quantum mechanics, $\frac{e^{-mr}}{2m}$.

If $m = 0$ (massless case), the basis of solutions is: $1, r$ for $d = 1$, $1, \log r$ for $d = 2$, and $1, r^{2-d}$ for $d > 2$. Thus, if $d \leq 2$, we don't have a decaying solution and thus the corresponding quantum theory will be deficient: it will not satisfy the clustering property. On the other hand, for $d > 2$ we have a unique up to scaling decaying solution $g = C r^{1-d}$. The normalizing constant is found as in the massive case.

The higher correlation functions are found from the 2-point function via the Wick formula, as usual.

We should now note a fundamental difference between quantum mechanics and quantum field theory in $d > 1$ dimensions. This difference comes from the fact that while for $d = 1$, the Green's function $G(x)$ is continuous at $x = 0$, for $d > 1$ it is singular at $x = 0$. Namely, $G(x)$ behaves like $C|x|^{2-d}$ as $x \to 0$ for $d > 2$, and as $C \log |x|$ as $d = 2$.

Thus for $d > 1$, unlike the case $d = 1$, the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

(as defined above) makes sense only if $x_i \neq x_j$. In other words, this path integral should be regarded not as a function but rather as a distribution. Luckily, there is a canonical way to do it, since the Green's function $G(x)$ is locally L^1.

Now we can Wick rotate this theory back into the Minkowski space. It is clear that the Green's function will then turn into

$$G_M(x) = g(\sqrt{-|x|^2 - i\varepsilon}),$$

which involves Bessel functions of both real and imaginary argument (depending on whether x is timelike or spacelike) and has a singularity on the light cone $|x|^2 = 0$. In particular, it is easy to check that $G_M(x)$ is real-valued for spacelike x, while for timelike x it is not. The function $G_M(x)$ satisfies the equation

$$(\Box + m^2)G_M = i\delta.$$

The higher correlation functions, as before, are determined from this by the Wick formula.

Actually, it is more convenient to describe this theory "in momentum space", where the Green's function can be written more explicitly. Namely, the Fourier transform $\widehat{G}(p)$ of the distribution $G(x)$ is a solution of the equation

$$p^2\widehat{G} + m^2\widehat{G} = 1,$$

obtained by Fourier transforming the differential equation for G. Thus,

$$\widehat{G}(p) = \frac{1}{p^2 + m^2},$$

as in the quantum mechanics case. Therefore, like in quantum mechanics, the Wick rotation produces the distribution

$$\widehat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon},$$

which is the Fourier transform of $G_M(x)$.

11.3. **Spinors.** To consider field theory for fermions, we must generalize to the case of $d > 1$ the basic fermionic Lagrangian $\frac{1}{2}\psi\frac{d\psi}{dt}$. To do this, we must replace $\frac{d}{dt}$ by some differential operator on V. This operator should be of first order, since in fermionic quantum mechanics it was important that the equations of motion are first order equations. Clearly, it is impossible to define such an operator so that the Lagrangian is $SO_+(V)$-invariant, if ψ is a scalar-valued (odd) function

on V. Thus, a fermionic field in field theory of dimension $d > 1$ cannot be scalar-valued, but rather must take values in a real representation S of $SO_+(V)$, such that there exists a nonzero intertwining operator $V \to \mathrm{Sym}^2 S^*$. This property is satisfied by *spinor representations*. They are indeed basic in fermionic field theory, and we will now briefly discuss them (for more detail see "Spinors" by P.Deligne, in "QFT and string theory: a course for mathematicians").

First consider the complex case. Let V be a complex inner product space of dimension $d > 1$. Let $\mathrm{Cl}(V)$ be the Clifford algebra of V, defined by the relation $\xi\eta + \eta\xi = 2(\xi, \eta)$, $\xi, \eta \in V$. As we discussed, for even d it is simple and has a unique irreducible representation S of dimension $2^{\frac{d}{2}}$, while for odd d it has two such representations S', S'' of dimension $2^{\frac{d-1}{2}}$. It is easy to show that the space $\mathrm{Cl}_2(V)$ of quadratic elements of $\mathrm{Cl}(V)$ (i.e. the subspace spanned elements of the form $\xi\eta - \eta\xi,\ \xi, \eta \in V$) is closed under bracket, and constitutes the Lie algebra $\mathfrak{o}(V)$. Thus $\mathfrak{o}(V)$ acts on S (respectively, S', S''). This action does not integrate to an action of $SO(V)$, but integrates to an action of its double cover $\mathrm{Spin}(V)$.

If d is even, the representation S of $\mathrm{Spin}(V)$ is not irreducible. Namely, recall that S is the exterior algebra of a Lagrangian subspace of V. Thus it splits in a direct sum $S = S_+ \oplus S_-$ (odd and even elements). The subspaces S_+, S_- are subrepresentations of S, which are irreducible. They are called the *half-spinor representations*. The half-spinor representations are interchanged by the adjoint action of $O(V)$ on $\mathrm{Spin}(V)$ ($SO(V)$ clearly acts trivially, so this is, in fact, and action of $O(V)/SO(V) = \mathbb{Z}/2$ on the set of irreducible representations of $SO(V)$). Note that in contrast, for odd d we have $O(V) = SO(V) \times \mathbb{Z}/2$, so the $\mathbb{Z}/2$ acts on representations of $\mathrm{Spin}(V)$ trivially.

If d is odd, the representations S' and S'' of $\mathrm{Spin}(V)$ are irreducible and isomorphic. Any of them will be denoted by S and called the *spinor representation*. Thus, we have the spinor representation S for both odd and even d, but for even d it is reducible.

An important structure attached to the spinor representation S is the intertwining operator $\Gamma : V \to \mathrm{End}S$ called *Clifford multiplication*, given by the action of $V \subset \mathrm{Cl}(V)$ in S, which we already encountered above. This intertwiner allows us to define the *Dirac operator*

$$\mathbf{D} = \sum_i \Gamma_i \frac{\partial}{\partial x_i} \tag{11.2}$$

where x_i are coordinates on V associated to an orthornormal basis e_i, and $\Gamma_i = \Gamma(e_i)$. This operator acts on functions from V to S, and $\mathbf{D}^2 = \Delta$, so $\mathbf{D}$ is a square root of the Laplacian. The matrices Γ_i are called Γ-*matrices*.

Note that for even d, one has $\Gamma(v) : S_\pm \to S_\mp$, so $\mathbf{D}$ acts from functions with values in $S_\pm$ to functions with values in $S_\mp$.

By a *polyspinor representation* of $\mathrm{Spin}(V)$ we will mean any linear combination of S_+, S_- for even d, and any multiple of S for odd d. For even d and a polyspinor representation $Y = Y_+ \otimes S_+ \oplus Y_- \otimes S_-$ (i.e., $Y_\pm = \mathrm{Hom}(S_\pm, Y)$) where Y_+, Y_- are vector spaces, set $Y' := Y_+ \otimes S_- \oplus Y_- \otimes S_+$, while for odd d and $Y = Y_0 \otimes S$ we set $Y' := Y$; thus $Y \mapsto Y'$ is an endofunctor on the category of polyspinor representations. Then for every polyspinor representation Y and $v \in V$ we have the Clifford multiplication operator $\Gamma(v) : Y \to Y'$.

Now assume that V is a real inner product space with Minkowski metric. In this case we can define the group $\mathrm{Spin}_+(V)$ to be the preimage of $SO_+(V)$ under the map $\mathrm{Spin}(V_\mathbb{C}) \to SO(V_\mathbb{C})$. It is a double cover of $SO_+(V)$ (if $d = 2$, this double cover is disconnenced and actually a direct product by $\mathbb{Z}/2$).

By a *real polyspinor representation* of $\mathrm{Spin}_+(V)$ we will mean a real representation Y of this group such that $Y_\mathbb{C}$ is a polyspinor representation of $\mathrm{Spin}(V_\mathbb{C})$.

Remark 11.2. Note that in all dimensions except $d = 2$, the group $\mathrm{Spin}(d)$ is the universal cover of $SO(d)$, which means that spins of all particles are either integers or half-integers. On the other hand, the universal cover of $SO(2)$ is not $\mathrm{Spin}(2)$, but rather $\mathbb{R}$. This creates in two dimensions a possibility of particles whose spin is any positive real number. Such particles are called *anyons* (particles of any spin), and we will see how they appear in 2-dimensional conformal field theory.

11.4. **Fermionic Lagrangians.** Now let us consider Lagrangians for a spinor field ψ with values in a polyspinor representation Y. Note that in even dimensions such fields are split into fields valued in S_+ and S_-, respectively. Such spinors are called *chiral*.

As the Lagrangian is supposed to be real in the Minkowski setting, we will require in that case that Y be real. First of all, let us see what we need in order to write the "kinetic term" $(\psi, \mathbf{D}\psi)$. Clearly, to define such a term (so that the corresponding term in the action does not reduce to zero via integration by parts), we need an invariant non-degenerate pairing $(,)$ between Y and Y' (i.e., an isomorphism of representations $Y' \cong Y^*$) such that for any $v \in V$, the bilinear form $(x, \Gamma(v)y)$ on Y is symmetric.

Let us find for which Y this is possible (for complex V). The behavior of Spin groups depends on d modulo 8 (*real Bott periodicity*). Thus we will list the answers labeling them by d mod 8 (they are easily extracted from the tables given in Deligne's text). First we summarize properties of spin representations.

0. $S_\pm$ orthogonal.
1. S orthogonal, $S \otimes S \to V$ symmetric.
2. $S_+^* = S_-$, $S_\pm \otimes S_\pm \to V$ symmetric.
3. S symplectic, $S \otimes S \to V$ symmetric.
4. $S_\pm$ symplectic.
5. S symplectic, $S \otimes S \to V$ antisymmetric.
6. $S_+ = S_-^*$, $S_\pm \otimes S_\pm \to V$ antisymmetric.
7. S orthogonal, $S \otimes S \to V$ antisymmetric.

Thus the possibilities for the kinetic term are:

0. $n(S_+ \oplus S_-)$; $(,)$ gives a perfect pairing between Y_+ and Y_-.
1. nS; $(,)$ gives a symmetric inner product on Y_0.
2. $nS_+ \oplus kS_-$; $(,)$ gives symmetric inner products on $Y_\pm$.
3. nS; $(,)$ gives a symmetric inner product on Y_0.
4. $n(S_+ \oplus S_-)$; $(,)$ gives a perfect pairing between Y_+ and Y_-.
5. $2nS$; $(,)$ gives a skew-symmetric inner product on Y_0.
6. $2nS_+ \oplus 2kS_-$; $(,)$ gives skew-symmetric inner products on $Y_\pm$.
7. $2nS$; $(,)$ gives a skew-symmetric inner product on Y_0.

Let us now find when we can also add a mass term. Recall that the mass term has the form $(\psi, M\psi)$, so it corresponds to an invariant skew-symmetric operator $M : Y \to Y^* \cong Y'$ (note that by definition, Γ_i commute with M). Let us list those Y from the above list for which such a non-degenerate operator exists.

0. $2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are skew-symmetric under $(,)$.
1. $2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under $(,)$.
2. $n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under $(,)$.
3. nS; $M : Y_0 \to Y_0$ is symmetric under $(,)$.
4. $n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are symmetric under $(,)$.
5. $2nS$; $M : Y_0 \to Y_0$ is symmetric under $(,)$.
6. $2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under $(,)$.
7. $2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under $(,)$.

To pass to the real Minkowski space (in both massless and massive case), one should put the additional requirement that Y should be a real representation.

We note that upon Wick rotation to Minkowski space, it may turn out that a real spinor representation Y will turn into a complex representation which has no real structure. Namely, this happens for massless spinors that take values in $S_\pm$ if $d = 2 \bmod 8$. These representations have a real structure for Minkowskian V (i.e. for $\mathrm{Spin}_+(1, d-1)$), but no real structure for Euclidean V (i.e. for $\mathrm{Spin}(d)$). This is quite obvious, for example, when $d = 2$ (check!).

Remark 11.3. One may think that this causes a problem in quantum field theory, where we would be puzzled what to integrate over – real or complex space. However, the problem in fact does not arise, since we have to integrate over fermions, and integration over fermions (say, in the finite dimensional case) is purely algebraic and does not make a distinction between real and complex.

11.5. **Free fermions.** Let us now consider a free theory for a spinor field $\psi : V \to \Pi Y$, where Y is a polyspinor representation, defined by a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D} - M)\psi),$$

where M is allowed to be degenerate (we assume that Y is such that this expression makes sense). The equation of motion in Minkowski space is

$$\mathbf{D}\psi = M\psi.$$

Thus, to define the corresponding quantum theory, we need to invert the operator $\mathbf{D} - M$. As usual, this cannot be done because of a singularity, and it is best to use the Wick rotation.

The Wick rotation produces the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D}_E + M)\psi)$$

(note that the i in the kinetic term is hidden in the definition of the Euclidean Dirac operator). We invert $\mathbf{D}_E + M$ to obtain the Euclidean Green's function. To do this, it is convenient to go to momentum space, i.e. perform a Fourier transform. Namely, after Fourier transform $\mathbf{D}_E$ turns into the operator $i\mathbf{p}$, where $\mathbf{p} = \sum_j p_j \Gamma_j$, and p_j are the operators of multiplication by the momentum coordinates p_j. Thus, the Green's function (i.e. the 2-point function) $G(x) \in \mathrm{Hom}(Y^*, Y)$ is the Fourier transform of the matrix-valued function $\frac{1}{i\mathbf{p}+M}$.

In the Euclidean case the group $\mathrm{Spin}(V)$ is compact and the spinor representations carry natural positive invariant Hermitian forms. So in this case without loss of generality we may consider polyspinor representations equipped with such positive forms, and on every polyspinor

representation such a form is unique up to isomorphism. Let

$$M^\dagger : Y^* \to Y$$

be the Hermitian adjoint operator to M. Then the reality condition is that M is Hermitian: $M^\dagger = M$. Thus

$$(-i\mathbf{p} + M)(i\mathbf{p} + M) = p^2 + M^2$$

so that

$$\widehat{G}(p) = (p^2 + M^2)^{-1}(-i\mathbf{p} + M).$$

This shows that $G(x)$ is expressed through the Green's function in the bosonic case by differentiations (how?). After Wick rotation back to the Minkowski space, we get

$$\widehat{G}_M(p) = (p^2 - M^2 + i\varepsilon)^{-1}(\mathbf{p} + iM).$$

Finally, the higher correlation functions, as usual, are found from the Wick formula.

11.6. **Hamiltonian formalism of classical field theory.** Let us now develop the hamiltonian approach to QFT, extending the hamiltonian formalism of quantum mechanics. We start with classical field theory, extending the hamiltonian formalism of classical mechanics. As in the Lagrangian setting, this can be done by formalizing the idea that field theory is mechanics of a continuum of particles occupying each point of the space $\mathbb{R}^{d-1}$.

Namely, consider a free scalar bosonic field $\phi(x)$ on a Minkowski space $\mathbb{R}^d$. As we have discussed, its Lagrangian is $\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$ and the equation of motion is the Klein-Gordon equation

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0,$$

where Δ_s is the spacial Laplacian. This is a second order equation with respect to t, so the initial value problem for this equation has the form

$$\phi(0, x) = q(x),\ \phi_t(0, x) = p(x)$$

(there is a standard explicit formula for solution of this problem, expressing it via the fundamental solution of the Klein-Gordon equation). Thus it is natural to introduce the phase space

$$Y := T^*C_0^\infty(\mathbb{R}^{d-1}) := C_0^\infty(\mathbb{R}^{d-1}) \oplus C_0^\infty(\mathbb{R}^{d-1})$$

of pairs (q, p) of smooth functions with compact support, on which the dynamics of the Klein-Gordon equation takes place (note that the space $C_0^\infty(\mathbb{R}^{d-1})$ is invariant under this dynamics since the speed of wave

propagation is finite, namely equals 1). Note that the phase space is an infinite dimensional symplectic space with constant symplectic form

$$\omega((q_1,p_1),(q_2,p_2)) = \int_{\mathbb{R}^{d-1}} (p_1(x)q_2(x) - p_2(x)q_1(x))dx.$$

Also for any point $x \in \mathbb{R}^{d-1}$ we have the local linear functionals

$$(q,p) \mapsto q(x),\ (q,p) \mapsto p(x)$$

which we will denote by $\phi(x)$ and $\phi_t(x)$, respectively. From these functionals we can make other linear functionals: for example, given $\rho \in C_0^\infty(\mathbb{R}^{d-1})$, we can define the functionals

$$\phi(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} q(x)\rho(x)dx,\ \phi_t(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} p(x)\rho(x)dx.$$

The Poisson bracket between such functionals can be computed by the formulas

$$\{\phi(\rho_1),\phi(\rho_2)\} = 0,\ \{\phi_t(\rho_1),\phi_t(\rho_2)\} = 0,$$

$$\{\phi(\rho_1),\phi_t(\rho_2)\} = \int_{\mathbb{R}^{d-1}} \rho_1(x)\rho_2(x)dx.$$

This can be written as a *field-theoretic Poisson bracket*:

$$\{\phi(x),\phi(y)\} = 0,\ \{\phi_t(x),\phi_t(y)\} = 0,\ \{\phi(x),\phi_t(y)\} = \delta(x-y);$$

then the previous formulas can be recovered by integrating both sides against $\rho_1(x)\rho_2(y)$. In other words, the linear local functionals $\phi(x)$ and $\phi_t(x)$ should be thought of not as smooth functions on Y depending on a point $x \in \mathbb{R}^{d-1}$ but rather as distributions on $\mathbb{R}^{d-1}$ with values in smooth functions on Y.

Similarly, one may consider non-linear polynomial local functionals, given by differential polynomials $P(\phi,\phi_t)$ evaluated at a point x, such as $\phi^n, \phi_t^2, (d_s\phi)^2, \phi^2(d_s\phi)^2$ (where d_s is the spatial differential), etc., and even non-polynomial ones depending on finitely many derivatives of ϕ, such as $e^\phi(d_s\phi)^2, \cos\phi$, and so on. They are called local because they depend only on the derivatives of ϕ at a single point x. Each of them is a distribution on $\mathbb{R}^{d-1}$ with values in smooth functions on Y, and can be applied to any density $\rho(x)$ to produce a smooth function on Y. Poisson brackets of such functionals are computed using the chain rule, the Leibniz rule, and the fact that taking Poisson brackets commutes with differentiation by x. So given two local functionals P and Q, we obtain

$$\{P(\phi)(x),Q(\phi)(y)\} = \sum_\alpha \{P,Q\}_\alpha(\phi)(x)\partial_x^\alpha\delta(x-y)$$

for some local functionals $\{P, Q\}_\alpha$, where ∂^α are monomials in the derivatives. For example, for $d = 2$

$$\{\phi_{tx}(u)\phi_t(u), \tfrac{1}{3}\phi^3(v)\} =$$

$$-(\phi_{tx}(u)\phi^2(u) + 2\phi_{tx}(u)\phi(u)\phi_x(u))\delta(u-v) - \phi_t(u)\phi^2(u)\delta'(u-v).$$

This Poisson bracket can of course be extended to products of local functionals at different points using the Leibniz rule.

The Hamiltonian of the theory is then given by integrating a local functional against the constant density:

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} (\phi_t^2 + (d_s\phi)^2 + m^2\phi^2)dx.$$

Namely, it is determined (up to a constant) by the condition that the Hamilton equation

$$F_t = \{F, H\}$$

for local functionals ϕ, ϕ_t is equivalent to the Klein-Gordon equation.

The Hamiltonian dynamics allows us to define the local functionals not just at a point $x \in \mathbb{R}^{d-1}$ but actually at any point $(t, x) \in \mathbb{R}^d$. When we do, by definition we get $\phi_t(t, x) = \frac{d}{dt}\phi(t, x)$ and the local functional $\phi(t, x)$ becomes a solution of the Klein-Gordon equation:

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0.$$

This can be used to compute the Poisson brackets: for example, we see that

$$\{\phi(t_1, x_1), \phi(t_2, x_2)\} = \mathbf{G}(t_2 - t_1, x_2 - x_1)$$

where $\mathbf{G}(t, x)$ solves the Klein-Gordon equation with initial conditions

$$\mathbf{G}(0, x) = 0,\ \mathbf{G}_t(0, x) = \delta(x).$$

To find it, take the Fourier transform. Then we get a distribution $\widehat{\mathbf{G}}$ supported on the two-sheeted hyperboloid X_m given by the equation $E^2 = p^2 + m^2$, of the form

$$\widehat{\mathbf{G}}(E, p) = f_+(p)\delta_{X_m^+} + f_-(p)\delta_{X_m^-},$$

where $X_m^\pm$ are the sheets of X_m. Moreover, the initial conditions give (up to appropriate normalization)

$$\int_{\mathbb{R}} \widehat{\mathbf{G}}(E, p)dE = 0,\ \int_{\mathbb{R}} \widehat{\mathbf{G}}(E, p)EdE = 1,$$

which yields

$$f_+(p) + f_-(p) = 0,\ \sqrt{p^2 + m^2}(f_+(p) - f_-(p)) = 1.$$

Thus $f_+ = -f_- = \frac{1}{2\sqrt{p^2+m^2}}$ and we have

$$\widehat{\mathbf{G}}(E,p) = \frac{1}{2\sqrt{p^2+m^2}}(\delta_{X_m^+} - \delta_{X_m^-}).$$

Now $\mathbf{G}$ can be found by taking the inverse Fourier transform (it expresses via the Bessel functions).

Note that since the speed of wave propagation is 1, this distribution $\mathbf{G}$ is supported on the solid light cone, so $\{\phi(t_1,x_1),\phi(t_2,x_2)\} = 0$ if the points (t_1,x_1) and (t_2,x_2) are *spacelike separated*, meaning that the vector (t_1-t_2, x_1-x_2) is spacelike. This property is called *space locality*, a mathematical expression of *causality* in special relativity.

Remark 11.4. A part of this analysis extends straightforwardly to the case of non-free theories, for example the ϕ^4-*theory*, having the Lagrangian

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2) - \frac{g}{4}\phi^4.$$

In this case the Klein-Gordon equation is replaced by its non-linear deformation

$$\phi_{tt} - \Delta_s\phi + m^2\phi + g\phi^3 = 0,$$

so there is a nontrivial issue of existence of solutions of the initial value problem for this non-linear PDE. However, this issue is irrelevant if we just want to consider Poisson brackets of local functionals on $\mathbb{R}^{d-1}$ or its formal neighborhood, since then the computations are purely formal (algebraic).

An important fact is that this structure is invariant under the *Poincaré group* $\mathbf{P} := SO_+(V) \ltimes V$ generated by Minkowski rotations and translations, where $V = \mathbb{R}^d$ is the spacetime (the semidirect product of the *Lorentz group* $SO_+(V)$ and the group of translations V). This follows from the fact that the Lagrangian of the theory is relativistically invariant. Namely, for $g \in \mathbf{P}$ given by

$$g(t,x) = (at + bx + c, \alpha t + \beta x + \gamma)$$

we have

$$(\phi g)(x)(q,p) = \phi(bx + c, \beta x + \gamma)$$

and

$$(\phi_t g)(x)(q,p) = (\partial_\alpha \phi)(bx + c, \beta x + \gamma) + a\phi_t(bx + c, \beta x + \gamma).$$

where $\phi(t,x)$ is the solution of the Klein-Gordon equation with initial conditions $(q(x), p(x))$.

In particular, note that the *Galileo subgroup* $SO(\mathbb{R}^{d-1}) \ltimes \mathbb{R}^{d-1}$ acts by manifest geometric symmetries, while time translations act by the Hamiltonian flow.

Finally, note that this discussion extends in a straighforward way to theories including fermions. In this case, as in fermionic classical mechanics, we get a field theoretic *super*-Poisson bracket on classical fields, which is symmetric rather than skew-symmetric if both fields are odd. Also, since odd fields take values in polyspinor representations, the Poincaré group should be replaced by its double cover $\widetilde{\mathbf{P}} := \mathrm{Spin}_+(V) \ltimes V$. We leave the details to the reader.

11.7. **Hamiltonian formalism of QFT: the Wightman axioms.** To quantize this picture, we need to define a Hilbert space $\mathcal{H}$ and lift classical observables (local functionals and their integrals) to (densely defined) operators on $\mathcal{H}$, notably lift the classical hamiltonian H to a quantum hamiltonian $\widehat{H}$ depending on the Planck constant $\hbar$ which should be a self-adjoint (in general, unbounded) operator on $\mathcal{H}$. Moreover, this should be done in such a way that commutators vanish at $\hbar = 0$ and in first order in $\hbar$ recover the Poisson bracket. We should also have a unitary representation of the double cover $\widetilde{\mathbf{P}}$ of the Poincaré group on the space $\mathcal{H}$ such that the 1-parameter subgroup of time translations acts by the quantum dynamics 1-parameter group $e^{-it\widehat{H}}$. This generalization of Hamiltonian quantum mechanics can be accomplished by means of so called *Wightman axioms*, which we now describe.

First of all, for the quantum theory to have good properties, we want the energy to be bounded below. Thus we introduce the following definition. Let us fix an orthogonal decomposition $V = \mathbb{R} \oplus V_s$ into space and time and consider the self-adjoint operator

$$\widehat{H}_\pi := i\tfrac{d}{dt}|_{t=0}\pi(t,0).$$

Definition 11.5. A unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\mathcal{H}$ is said to be *positive energy* if the spectrum of $\widehat{H}_\pi$ is bounded below.

Note that every unitary representation π of V has a spectrum $\sigma(\pi)$, which is a closed subset of $V^* \cong V$; namely, $\sigma(\pi)$ is the set of characters of V that occur (discretely or continuously) in π (i.e., the smallest set containing the support of the Fourier transform of the distribution $\langle\, w_1, \pi(v)w_2 \,\rangle$, $v \in V$, for any $w_1, w_2 \in \mathcal{H}$).

Lemma 11.6. *Suppose* $\dim V \geq 2$. *Then* π *is positive energy if and only if* $\sigma(\pi)$ *is contained in the positive part of the solid light cone,* $\overline{V}_+$.

Proof. By definition, π is of positive energy iff the orthogonal projection of $\sigma(\pi)$ onto the dual of the time axis is bounded below. Since $\sigma(\pi)$ is

invariant under $SO_+(V)$, this implies the statement (an $SO_+(V)$-orbit on V has bounded below projection iff it is contained in $\overline{V}_+$). $\square$

Note that this is false for $d = 1$ (quantum mechanics), where the hamiltonian can be shifted by a constant without any effect on the theory. But the latter is longer so in quantum field theory on a Minkowski space of dimension > 1.

We are now ready to give Wightman's definition of a QFT. Let $\mathcal{S} = \mathcal{S}(V)$ be the Schwartz space of V.

Definition 11.7. A *Wightman QFT* on a Minkowski space V entails the following data:

1. A finite dimensional real super-representation $R = R_0 \oplus R_1$ of $\mathrm{Spin}_+(V)$ (the *field space*).

2. A super Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ carrying a positive energy unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\mathcal{H}$ of the double cover of the Poincaré group, $\widetilde{\mathbf{P}} = \mathrm{Spin}_+(V) \ltimes V$.

3. A dense $\widetilde{\mathbf{P}}$-stable subspace $\mathcal{D} \subset \mathcal{H}$.

4. A $\widetilde{\mathbf{P}}$-invariant unit vector $\Omega \in \mathcal{D}$ called the *vacuum vector*.

5. A $\widetilde{\mathbf{P}}$-invariant even linear map: $\mathcal{S} \otimes R^* \to \mathrm{End}\mathcal{D}$ called the *field map*.

This data is subject to the following axioms.

A1. If f is real then $\phi(f)$ is Hermitian symmetric (in the supersense).

A2. ϕ is weakly continuous, i.e. for every $w_1, w_2 \in \mathcal{D}$, the functional $\mathcal{S} \otimes R^* \to \mathbb{C}$ defined by $f \mapsto \langle\, w_1, \phi(f) w_2 \,\rangle$ is continuous.

A3. $\mathcal{D}$ is spanned (algebraically) by vectors $\phi(f_1)...\phi(f_n)\Omega$.

A4. *Space locality*: If f_1, f_2 have spacelike separated supports, i.e., for any $v_1 \in \mathrm{supp} f_1$, $v_2 \in \mathrm{supp} f_2$ we have $|v_1 - v_2|^2 < 0$, then

$$[\phi(f_1), \phi(f_2)] = 0$$

(with commutator understood in the supersense).

In addition, if $\mathcal{H}^{\widetilde{\mathbf{P}}} = \mathbb{C}\Omega$, one says that we have a Wightman QFT with a *unique vacuum*.

We will also always assume that our QFT is *nondegenerate*, i.e., for every irreducible subrepresentation $E \subset R_j^*$, $j = 0, 1$, one has $\phi|_{\mathcal{S}\otimes E} \neq 0$; otherwise we can simply remove this subrepresentation without any effect on the theory.

A fundamental fact about Wightman QFT is the following theorem, which we will not prove here. Let ζ be the generator of the kernel of the map $\mathrm{Spin}_+(V) \to \mathrm{SO}_+(V)$, so $\zeta^2 = 1$.

Theorem 11.8. *(The spin-statistics theorem) If $E \subset R_j^*$ is a subrepresentation then $\zeta|_E = (-1)^j$.*

In other words, there is a relationship between the *spin* (mod integers) of a quantum field (essentially, the eigenvalue of ζ) and its *statistics*, i.e., whether it is bosonic (even) or fermionic (odd). Namely, the theorem says that all bosonic fields must have $\zeta = 1$ (integer spin) and all fermionic fields must have $\zeta = -1$ (half-integer spin).

Remark 11.9. We will see that the theory of free bosons and fermions can be naturally formulated as a Wightman QFT. Moreover, this is also the case for a number of non-free theories, which is the subject of a difficult area of mathematical physics called *constructive field theory*. Still, most theories that physicists really care about are either not known to be Wightman QFT, or simply fail to be ones for various reasons (perturbative theories, low energy effective theories, non-unitary theories, Euclidean theories, theories living on compact manifolds, etc.) Thus we will view Wightman axioms just as one (somewhat limited) rigorous model for our mathematical understanding of QFT.

11.8. **Wightman functions.**

Proposition 11.10. *In a Wightman QFT on a Minkowski space V, for every $n \geq 1$ there exists a unique tempered distribution W_n on V^n valued in $R^{*\otimes n}$ such that*

$$W_n(f_1 \boxtimes ... \boxtimes f_n) = \langle\, \Omega, \phi(f_1)...\phi(f_n)\Omega \,\rangle.$$

We leave the proof of this proposition as an exercise.

We will therefore think of W_n as a (generalized) function on $V^{\otimes n}$ valued in $R^{*\otimes n}$, denoted $W_n(x_1, ..., x_n)$, so that

$$W_n(f_1 \boxtimes ... \boxtimes f_n) = \int_{V^n} W_n(x_1, ..., x_n) f_1(x_1)...f_n(x_n) dx_1...dx_n$$

where the product on the right hand side involves contraction of corresponding copies of R and R^*. Thus, given $u_1, ..., u_n \in R$, we have the scalar-valued distribution

$$W_n^{u_1,...,u_n}(x_1, ..., x_n) := (W_n(x_1, ..., x_n), u_1 \otimes ... \otimes u_n).$$

In other words, we may define an operator-valued distribution $\phi(x)$ such that

$$\phi(f) = \int_V \phi(x) f(x) dx;$$

then

$$W_n(x_1, ..., x_n) = \langle\, \Omega, \phi(x_1)...\phi(x_n)\Omega \,\rangle.$$

Definition 11.11. The generalized functions $W_n(x_1, ..., x_n)$ are called the *Wightman (correlation) functions* of the Wightman QFT.

Note that Wightman functions completely determine the Wightman QFT as follows. Let $\widetilde{\mathcal{D}} := T(\mathcal{S} \otimes R)$ (the tensor algebra), so it is spanned by elements $f_1 \otimes f_2 \otimes ... \otimes f_n$, $f_i \in \mathcal{S} \otimes R$. Define the inner product on $\widetilde{\mathcal{D}}$ by

$$\langle f_1 \otimes ... \otimes f_n, g_1 \otimes ... \otimes g_m \rangle := (-1)^{\sum_{i<j} p(f_i)p(f_j)} W_{n+m}(\overline{f}_n \boxtimes ... \boxtimes \overline{f}_1 \boxtimes g_1 \boxtimes ... \boxtimes g_m).$$

It is easy to see that this inner product is well defined, and

$$\langle f_1 \otimes ... \otimes f_n, g_1 \otimes ... \otimes g_m \rangle = \langle \phi(f_1)...\phi(f_n)\Omega, \phi(g_1)...\phi(g_m)\Omega \rangle$$

(where f_i are purely odd or purely even). Thus the inner product $\langle , \rangle$ on $\widetilde{\mathcal{D}}$ is nonnegative definite, the Hilbert space $\mathcal{H}$ can be recovered as the completion of $\widetilde{\mathcal{D}}$ with respect to $\langle , \rangle$, and $\mathcal{D}$ is the image of $\widetilde{\mathcal{D}}$ in $\mathcal{H}$ (note that the map $\widetilde{\mathcal{D}} \to \mathcal{H}$ need not be injective). Moreover, the vector Ω is the image of $1 \in \widetilde{\mathcal{D}}$ in $\mathcal{D}$, and the representation π is obtained by extending the action of $\widetilde{\mathbf{P}}$ on $\mathcal{D}$ (which descends from $\widetilde{\mathcal{D}}$) by continuity.

So we can ask: what conditions should Wightman functions satisfy to define a Wightman QFT? Let us list some necessary conditions, which follow from the above discussion. To this end, denote by

$$W : T(\mathcal{S} \otimes R) \to \mathbb{C}$$

the natural liner map and by $* : T(\mathcal{S} \otimes R) \to T(\mathcal{S} \otimes R)$ the antilinear map given by $(f_1 \otimes ... \otimes f_n)^* = (-1)^{\sum_{i<j} p(f_i)p(f_j)} \overline{f}_n \otimes ... \otimes \overline{f}_1$.

Proposition 11.12. *The Wightman functions W_n of a Wightman QFT satisfy the following properties.*

1. W_n are $\widetilde{\mathbf{P}}$-invariant.

2. Positive energy: the Fourier transform of W_n is supported on the set of $(p_1, ..., p_n) \in V^n$ such that $\sum_i p_i = 0$ and $p_{i+1} - p_i \in \overline{V}_+$.

3. $W_n(f^*) = \overline{W_n(f)}$.

4. Space locality:

$$W_n^{u_1,...,u_n}(x_1, ..., x_i, x_{i+1}, ..., x_n) = (-1)^{p(u_i)p(u_{i+1})} W_n(x_1, ..., x_{i+1}, x_i, ..., x_n)$$

if $|x_i - x_{i+1}|^2 < 0$.

5. Positivity: $W(f^* \otimes f) \geq 0$ for any $f \in T(\mathcal{S} \otimes R)$.

Proof. (1) follows from the invariance of the vacuum vector and the field map. (3) follows from the fact that for real f, $\phi(f)$ is hermitian symmetric. (4) follows from the space locality axiom. (5) follows from positivity of the inner product on $\mathcal{H}$. So it remains to prove (2). Let us do so for $n = 2$, the general proof is similar.

By translation invariance we have

$$W_2(v_1, v_2) = \mathbb{W}(v)$$

where $v = v_2 - v_1$. Thus our job is to show that the Fourier transform of $\mathbb{W}$ is supported on $\overline{V}_+$. We have

$$\mathbb{W}(v) = \langle\, \Omega, \phi(0)\phi(v)\Omega \,\rangle =$$
$$= \langle\, \Omega, \phi(0)\pi(v)\phi(0)\pi(-v)\Omega \,\rangle = \langle\, \phi(0)\Omega, \pi(v)\phi(0)\Omega \,\rangle.$$

So the statement follows from the fact that every character of V which occurs in $\mathcal{H}$ belongs to $\overline{V}_+$. □

In fact, it turns out that these necessary conditions are also sufficient, and we have the following theorem, which can be proved by following the above reconstruction procedure (but we will not give a proof):

Theorem 11.13. *If a collection of distributions W_n satisfies conditions (1)-(5) of Proposition 11.12 then they define a Wightman QFT.*

Remark 11.14. The 1-point Wightman function $W_1(x) = \langle\, \Omega, \phi(x)\Omega \,\rangle$ is a constant c by translation invariance, i.e. it is an element of R^*, and by invariance under rotations it is in $(R^*)^{\mathrm{Spin}_+(V)}$. Thus we may (and will) assume without loss of generality that $c = 0$ (otherwise we can replace $\phi(x)$ by $\phi(x) - c$). So we may assume without loss of generality that $W_1 = 0$.

Remark 11.15. The positivity property for the 2-point function can be written as

$$\int_{V^2} \mathbb{W}(x_2 - x_1)\overline{f(x_1)}f(x_2)dx_1dx_2 \geq 0,$$

where $\mathbb{W}(x) = W_2(0, x)$. Thus, taking Fourier transforms, we have

$$\int_V \widehat{\mathbb{W}}(p)\overline{\widehat{f}(p)}\widehat{f}(p)dp \geq 0.$$

This shows that $\widehat{\mathbb{W}}(p)dp$ is a measure concentrated on $\overline{V}_+$ and valued in nonnegative hermitian forms on $R_{\mathbb{C}}$.

11.9. **The mass spectrum of a Wightman QFT.** Let $\mathcal{H}^{(1)} \subset \mathcal{H}$ be the closure of the span of vectors $\phi(x)\Omega$, $x \in V$. It is called the space of 1-*particle states*, and it is clearly a $\widetilde{\mathbf{P}}$-subrepresentation of $\mathcal{H}$. The *mass spectrum* of the theory is determined by the structure of this representation. So we need to discuss the representation theory of $\widetilde{\mathbf{P}}$.

Since $\widetilde{\mathbf{P}}$ is a semidirect product, its irreducible unitary representations are unitarily induced. Namely, let $\mathcal{O}$ be an orbit of $\mathrm{Spin}_+(V)$ on V and ρ be an irreducible unitary representation of the stabilizer $\widetilde{\mathbf{P}}_0$ of a point $v_0 \in \mathcal{O}$. Then ρ defines an equivariant Hilbert bundle on $\mathcal{O}$ with total space $(\widetilde{\mathbf{P}} \times \rho)/\widetilde{\mathbf{P}}_0$ where $\widetilde{\mathbf{P}}_0$ acts diagonally. Thus we can consider the space $\mathcal{H}_{\mathcal{O},\rho}$ of square integrable half-densities on $\mathcal{O}$ with

values in this bundle. This space carries a unitary representation of $\widetilde{\mathbf{P}}$. A theorem of Mackey then says that this unitary representation is irreducible, and all irreducible unitary representations of $\widetilde{\mathbf{P}}$ are obtained uniquely in this way. For example, if $\mathcal{O} = \{0\}$, then $\mathcal{H}_{0,\rho}$ is just a unitary irreducible representation of $\mathrm{Spin}_+(V)$.

Now we are ready to discuss the structure of the representation $\mathcal{H}^{(1)}$. By taking Fourier transforms (see Remark 11.15), we see that if $\mathcal{H}_{\mathcal{O},\rho}$ occurs in $\mathcal{H}^{(1)}$ then ρ needs to be finite dimensional. For example, for $d \geq 3$ and $\mathcal{O} = \langle 0 \rangle$ the only choice is the trivial representation, as the group $\mathrm{Spin}_+(V)$ is a connected semisimple non-compact Lie group. Moreover, if the theory has a unique vacuum then the trivial representation occurs in $\mathcal{H}$ discretely with multiplicity 1, as the span of the vacuum vector Ω. As $\mathcal{H}^{(1)}$ is orthogonal to Ω (since $W_1 = 0$), we see that the trivial representation does not occur in $\mathcal{H}^{(1)}$.

Let us now consider what happens with other orbits. By the positive energy condition, the only orbits that can occur are X_m^+ defined by $E = \sqrt{p^2 + m^2}$, $E > 0$ (where for $d = 2$ the set X_0^+ falls into two orbits X_0^{++} and X_0^{+-} defined by $p = \pm E > 0$). For $m > 0$ this is the upper sheet of a 2-sheeted hyperboloid and for $m = 0$ it is the upper part of the light cone (which is a union of two orbits for $d = 2$).

In the case $m > 0$, we may take $v_0 = (m, 0)$, then $\widetilde{\mathbf{P}}_0 = \mathrm{Spin}(d-1)$, so ρ is a (necessarily finite dimensional) unitary representation of this compact Lie group. Physicists say that this representation corresponds to a *massive particle of mass m and type ρ*. Particles arising in physically relevant quantum field theories are usually *scalars* ($\rho = \mathbb{C}$), *spinors* (ρ is a spinor representation of $\mathrm{Spin}(d-1)$) and *vectors* ($\rho = \mathbb{C}^{d-1}$ is the vector representation $\mathrm{Spin}(d-1)$). Note that by the spin-statistics theorem, scalars and vectors are bosons and spinors are fermions.

If $m = 0$, $d \geq 3$, then we can take $v_0 = (1, 1, 0, ..., 0)$, and the stabilizer is the non-reductive Lie group $\mathrm{Spin}(d-2) \ltimes \mathbb{R}^{d-2}$. Since ρ is finite dimensional, $\mathbb{R}^{d-2}$ has to act trivially, so ρ is an irreducible representation of the compact Lie group $\mathrm{Spin}(d-2)$. Physicists say that this representation corresponds to a *massless particle of type ρ*. The classification of massless particles is the same as for massive ones; however, note that since for massless particles ρ is a representation of $\mathrm{Spin}(d-2)$ rather than $\mathrm{Spin}(d-1)$, they in general have fewer components than massive ones; for example, a massless vector has one fewer component than a massive one.

If $m = 0, d = 2$ then there are two choices for v_0: $(1, 1)$ and $(1, -1)$. They have trivial stabilizer, so $\rho = \mathbb{C}$. Thus we have two types of

massless particles: *right-moving* and *left-moving*, corresponding to the two choices of v_0. These particles are called this way since the corresponding operators $\phi(x)$ satisfy the conditions $\phi(t,x) = \phi(0,x-t)$, $\phi(t,x) = \phi(0,x+t)$, respectively, which classically would be right-moving and left-moving waves.

The set M of numbers m corresponding to representations $\mathcal{H}_{X_m^+,\rho}$ (or $\mathcal{H}_{X_0^{+\pm},\rho}$ for $d=2$) occurring in $\mathcal{H}^{(1)}$ is called the *mass spectrum* of the theory. One says that the theory has a *mass gap* when $\inf M = m > 0$. In this case the spectrum of $\widehat{H}$ is $\{0\} \cup [m, +\infty]$, so there is a gap between 0 and m. To find the mass spectrum, it suffices to look at the function $\widehat{\mathbb{W}}$: the mass spectrum is just the intersection of its support with the time axis (this follows from Remark 11.15).

11.10. **Free theory of a scalar boson.** Let us now construct a Wightman QFT corresponding to a scalar boson of mass $m > 0$. Recall that in the Lagrangian setting we had a 2-point function $G_M(x_2 - x_1)$, where $G_M(x)$ is a distribution satisfying the Klein-Gordon equation

$$(\Box + m^2)G_M = i\delta.$$

So at first sight for the corresponding Wightman QFT we want to have $\mathbb{W}(x) = G_M(x)$, so that the Lagrangian and Hamiltonian approach agree. However, the function $G_M(x)$ is even, while for $\mathbb{W}(x)$ we are supposed to have $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$, so our equality needs to be relaxed. In fact, the correct condition is that the identity $\mathbb{W}(x) = G_M(x)$ only needs to hold when x is spacelike or when $x \in \overline{V}_+$. When $x \in \overline{V}_-$, we should rather have $\mathbb{W}(x) = \overline{G_M(x)}$. In other words,

$$G_M(x_2 - x_1) = W_2^T(x_1, x_2)$$

is the so-called *time ordered* 2-point function, i.e. one obtained from $W_2(x_1, x_2)$ when x_1, x_2 are put in the chronological order (where in the spacelike separated case the order does not matter due to space locality).

We claim that with this definition the function $\mathbb{W}(x)$ satisfies the Klein-Gordon equation

$$(\Box + m^2)\mathbb{W} = 0$$

on the nose (without the delta-function on the right hand side). Indeed, we have $\mathrm{Re}\mathbb{W}(x) = \mathrm{Re}G(x)$, which satisfies the Klein-Gordon equation, so it remains to show that $\mathrm{Im}\mathbb{W}(x)$ satisfies it as well. But it is easy to see that $(\Box + m^2)\mathrm{Im}\mathbb{W}(x)$ is a distribution supported at the origin of homogeneity degree $-d$, so it is a multiple of δ. Since $\mathrm{Im}\mathbb{W}(x)$ is an odd function, this distribution must be zero, as claimed.

Also, since $G_M(x)$ is real for spacelike x, we get $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$. Thus the Fourier transform $\widehat{\mathbb{W}}(p)$ is real valued, supported on the hyperboloid X_m and invariant under $SO_+(V)$. It follows that

$$\widehat{\mathbb{W}}(p) = c_+\delta_{X_m^+} + c_-\delta_{X_m^-}.$$

where $c_\pm \in \mathbb{R}$. but in fact it can be shown that only $\delta_{X_m^+}$ occurs (this follows from the exponential decay of the Euclidean 2-point correlation function at infinity). Thus

$$\widehat{\mathbb{W}}(p) = c\delta_{X_m^+}.$$

In fact, one can show that $c = 2\pi$.

Similarly, we define higher W_n for $n > 2$ by the Wick formula, and this analysis implies after some work that these functions define a Wightman QFT.

In this case, $\mathcal{H}^{(1)} = L^2(X_m^+)$, so we have a single particle of mass m.

The theory of a free massless scalar, as well as massive and massless spinor is defined similarly.

11.11. **Normal ordering, composite operators and operator product expansion in a free QFT.** In classical field theory, given a classical scalar field $\phi(x)$, we may consider arbitrary polynomials and even any smooth functions of ϕ. The same is true for quantum mechanics, where $\phi(t)$ is a self-adjoint (possibly unbounded) operator on the Hilbert space $\mathcal{H}$ of quantum states, so using its spectral decomposition, we may define functions of ϕ. However, in quantum field theory in $d+1$ dimensions with $d \geq 1$ the situation is more complicated. Indeed, in this case $\phi(x)$ is not a usual operator-valued function of x, but rather a generalized one – an operator-valued distribution, and we know that for singular distributions, such as $\delta(x)$, we cannot even define the square $\delta(x)^2$.

Indeed, let $\phi(x)$ be a quantum scalar boson. Then the 2-point correlation function

$$\langle\phi(x)\phi(y)\rangle = \langle\Omega, \phi(x)\phi(y)\Omega\rangle = G(x-y)$$

blows up when $|x-y|^2 = 0$ (so in Euclidean signature, when $x = y$), so the operator $\phi^2(x)$ cannot possibly be well defined.

Thus, if we want to quantize the classical field $\phi^2(x)$, we need to regularize the corresponding operator product. This can be done by a standard regularization procedure called the *normally ordered product*.

For example, in Euclidean signature, the operator product $\phi(x)\phi(y)$ is well defined when $x \neq y$: indeed, by Wick's formula

$$\langle\phi(x)\phi(y)\phi(z_1)...\phi(z_k)\rangle =$$

$$G(x-y)\langle\phi(z_1)...\phi(z_k)\rangle+\sum_{i\neq j}G(x-z_i)G(y-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle,$$

where the hat indicates omissions (here $x, y, z_1, ..., z_k$ are distinct). Now, when $x \to y$, the first summand in this formula blows up while the second one does not. So it is natural to define the *normally ordered product* $:\phi(x)\phi(y):$ just by throwing away the singular terms, i.e. by the condition that its correlation function with $\phi(z_1)...\phi(z_k)$ is

$$\langle :\phi(x)\phi(y): \phi(z_1)...\phi(z_k)\rangle = \sum_{i\neq j}G(x-z_i)G(y-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle.$$

This is equivalent to just saying that

$$:\phi(x)\phi(y): \ = \phi(x)\phi(y) - G(x-y).$$

Note that while $\phi(x)\phi(y)$ blows up when $x = y$, the normally ordered product $:\phi(x)\phi(y):$ does not:

$$\langle :\phi^2(x): \phi(z_1)...\phi(z_k)\rangle = \sum_{i\neq j}G(x-z_i)G(x-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle.$$

This defines a *composite operator* $:\phi^2(x):$, which is a well defined operator-valued distribution.

Similarly one may define the normally ordered product $:\phi(x_1)...\phi(x_m):$ of any number of factors, by removing all the singular terms from the correlators. For example,

$$:\phi(x)\phi(y)\phi(z): \ = \phi(x)\phi(y)\phi(z) - G(x-y)\phi(z) - G(y-z)\phi(x) - G(z-x)\phi(y).$$

Such a product is well defined for all values of $x_1, ..., x_k$ and is commutative (independent of ordering of factors) and associative. We can also differentiate by x_j any number of times, to define the normally ordered product of arbitrary derivatives of ϕ. Evaluating such products on the diagonal (when all points are the same), we obtain composite operators attached to any differential monomials (hence polynomials) with respect to ϕ, such as $:\phi^3(x):$, $:\phi_{x_i}\phi_{x_j}:$, etc.

Exercise 11.16. *Derive a formula for the correlation function of several composite operators (evaluated at different points) in the theory of the scalar boson.*

In particular, we can now consider the product of two composite operators, e.g. $:\phi^2(x):\phi(y)$. Of course, this has a singularity at $x = y$, and an important problem is to understand the nature of this singularity. This is achieved by the procedure called the *operator product expansion*, which replaces the non-existent multiplication of composite operators.

To explain this procedure, consider first the simplest example of operator product:

$$\phi(x)\phi(y) = G(x-y) + :\phi(x)\phi(y):\ .$$

Using Taylor's formula, this can be rewritten so that the right hand side only contains $\phi(y)$ and no $\phi(x)$:

$$\phi(x)\phi(y) = G(x-y) + \sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y):\ ,$$

where $\mathbf{n} := (n_1, ..., n_{d+1})$, $(x-y)^{\mathbf{n}} := \prod_i (x_i - y_i)^{n_i}$, $\partial^{\mathbf{n}} := \prod_i \partial_{x_i}^{n_i}$, and $\mathbf{n}! := \prod_i n_i!$. In this sum, all terms except the first one are regular (i.e., continuous) at $x = y$.

Let us now try to write down a similar expansion for a more complicated example of operator product, $:\phi^2(x):\phi(y)$. We have

$$\langle :\phi^2(x):\phi(y)\phi(z_1)...\phi(z_k)\rangle = 2G(x-y)\langle\phi(x)\phi(z_1)...\widehat{\phi(z_j)}...\phi(z_k)\rangle +$$

$$\sum_{j,m,n \text{ distinct}} G(x-z_j)G(x-z_m)G(y-z_n)\langle\phi(z_1)...\widehat{\phi(z_j)}...\widehat{\phi(z_m)}...\widehat{\phi(z_n)}...\phi(z_k)\rangle.$$

Thus we get

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + :\phi^2(x)\phi(y):\ .$$

As before, using Taylor's formula, this can be rewritten so that the right hand side only contains $\phi(y)$ and no $\phi(x)$:

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + \sum_{\mathbf{n},\mathbf{m}} \frac{(x-y)^{\mathbf{n}+\mathbf{m}}}{\mathbf{n}!\mathbf{m}!} :\partial^{\mathbf{n}}\phi(y)\cdot\partial^{\mathbf{m}}\phi(y)\cdot\phi(y):\ .$$

And again, all terms except the first one are regular at $x = y$.

As a final example, consider the product $:\phi^2(x):\cdot:\phi^2(y):$. A similar computation yields

$$:\phi^2(x):\cdot:\phi^2(y): \ = 2G^2(x-y) + 4G(x-y):\phi(x)\phi(y): + :\phi^2(x)\phi^2(y):\ ,$$

and as before we can expand this to remove $\phi(x)$ using Taylor's formula. Namely, expanding the second summand, we get

$$:\phi^2(x):\cdot:\phi^2(y): \ =$$

$$2G^2(x-y) + 4G(x-y)\sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y): + :\phi^2(x)\phi^2(y):\ ,$$

and the last summand can be expanded similarly. We now see that there are many singular terms: $G(x)$ behaves as $|x|^{1-d}$ for $d > 1$ and as $\log|x|$ for $d = 1$, so the singular terms are the ones with $|\mathbf{n}| \le d-1$,

where $|\mathbf{n}| := \sum_i n_i$. For example, for $d = 2$ for the massless boson we have

$$: \phi^2(x) : \cdot : \phi^2(y) : \ =$$

$$\frac{2}{|x-y|^2} + \frac{4}{|x-y|} : \phi^2(y) : + \sum_{j=1}^{3} \frac{4}{|x-y|}(x_j - y_j) : \partial_{x_i}\phi(y) \cdot \phi(y) : + \text{ regular.}$$

Yet we see that the number of singular terms is finite. In fact, it is not hard to prove the following proposition (see [QFS], vol 1, p.449).

Proposition 11.17. *Let A, B be two composite operators in the theory of scalar boson. Then there exist a unique collection of functions $F_j(y)$ and composite operators $C_j(y)$ such that we have an asymptotic expansion*

$$A(x)B(y) \sim \sum_j F_j(x-y)C_j(y), \ x \to y$$

such that for every N we have $|F_j(z)| = O(|z|^N)$, $z \to 0$, for all but finitely many j. In particular, there are finitely many singular terms (not continuous at $x = y$).

The expansion of Proposition 11.17 is called the *operator product expansion*. It is not hard to show that it exists in any free quantum field theory.

11.12. **Symmetries in quantum field theory.** In studying any physical system, it is crucial to find all its symmetries and use them to their full potential. For example, the equations of motion of a particle in a rotationally symmetric potential field can be fully solved by utilizing the rotational symmetry (see [A]).

The most fundamental fact about symmetries in classical or quantum mechanics is that for any 1-parameter group of symmetries of the system there is an (essentially unique) observable responsible for this symmetry, which is conserved in this system; i.e., every 1-parameter symmetry corresponds to a conservation law, and vice versa. This statement is called *Noether's theorem*.

Let us first explain the precise meaning of Noether's theorem in the setting of classical mechanics. Suppose we have a system with phase space a symplectic manifold (M, ω) (typically $M = T^*X$, where X is the configuration space, and $\omega = d\alpha$ is the differential of the Liouville form) and hamiltonian $H \in C^\infty(M)$. Let g^t be a 1-parameter group of symmetries of this system, i.e., of symplectic diffeomorphisms of M which preserve H. Let $v := \frac{d}{dt}|_{t=0} g^t$ be the vector field generating the flow g^t. Then we have $L_v\omega = 0$ (i.e., v is a symplectic vector field, so $\omega_v := \omega(v, ?)$ is a closed 1-form), and $L_v H = 0$. Let us assume

that M is simply connected (for example, we can restrict ourselves to a neighborhood of a point in M or X). In this case ω_v is exact, so there exists $Q \in C^\infty(M)$ (unique up to adding a constant) such that $\omega_v = dQ$. Then for any observable $F \in C^\infty(M)$ we have $L_v F = \{Q, F\}$. Moreover $\{Q, H\} = L_v H = 0$. The observable Q is thus conserved under the hamiltonian flow and is the conservation law corresponding to the 1-parameter group g^t. It is called (especially in the setting of field theory) the *Noether (conserved) charge* of the symmetry.

A trivial example of this is the hamiltonian flow h^t defined by the hamiltonian H itself, i.e., the time translation symmetry; in this case $Q = H$, so the corresponding conserved quantity is H (the energy). Other examples include the momenta $p_1, ..., p_n$ which corresponds to translation symmetry (for $X = \mathbb{R}^n$) and *angular momenta* $M_{kj} := x_k p_j - x_j p_k$ corresponding to rotational symmetries around the codimension 2 hyperplanes $x_k = x_j = 0$.

More generally, suppose G is a Lie group acting (on the right) by symmetries of the system. Let $\mathfrak{g} = \mathrm{Lie}G$ be the Lie algebra of G. Any element $y \in \mathfrak{g}$ gives rise to a 1-parameter subgroup $e^{ty} \in G$, so defines a conserved quantity Q_y such that

$$\{Q_y, F\} = y \cdot F := \tfrac{d}{dt}|_{t=0} e^{ty} \cdot F$$

for each $F \in C^\infty(M)$, where $(g \cdot F)(m) = F(mg)$, $m \in M$. More precisely, Q_y is defined only up to adding a constant, so let us fix some linear assignment $y \mapsto Q_y$.

Moreover, it is clear that for $y, z \in \mathfrak{g}$

$$\{Q_y, Q_z\} = Q_{[y,z]} + C(y, z),$$

where $C(y, z)$ is a skew-symmetric bilinear form on $\mathfrak{g}$ which arises because Q_y is uniquely determined by y only up to adding a constant. Furthermore, by the Jacobi identity, the form C is a 2-cocycle :

$$C([x, y], z) + C([y, z], x) + C([z, x], y) = 0.$$

It follows that the assignment $y \mapsto Q_y$ is almost a homomorphism $\mathfrak{g} \to C^\infty(M)$, but not quite: rather, it defines a homomorphism

$$\mu : \widehat{\mathfrak{g}} \to C^\infty(M),$$

where $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ is a 1-dimensional central extension of $\mathfrak{g}$ with commutator

$$[(y, a), (z, b)] = ([y, z], C(y, z)).$$

Namely, $\mu(y, a) = Q_y + a$.

The map μ may be viewed as an element of $C^\infty(M) \otimes \widehat{\mathfrak{g}}^*$, i.e., geometrically as a C^∞-map

$$\mu : M \to \widehat{\mathfrak{g}}^*.$$

This map is called the *moment map* and plays a fundamental role in symplectic geometry.

The following example shows that the cohomology class of C may be nonzero, which means that we may not be able to choose Q_y to make $C = 0$.

Example 11.18. The group $\mathbb{R}^{2n}$ acts on $M = T^*\mathbb{R}^n$ (with trivial hamiltonian $H = 0$) by translations. So we have $\mathfrak{g} = \mathbb{R}^{2n}$ and $C(y, z) = \omega(y, z)$. Thus $\widehat{\mathfrak{g}}$ is the *Heisenberg Lie algebra* $\mathbb{R}^{2n} \oplus \mathbb{R}$ with commutation relations

$$[(y, a), (z, b)] = ([y, z], C(y, z)),$$

which is a non-trivial central extension of $\mathfrak{g}$.

However, in many examples the cohomology class $[C] \in H^2(\mathfrak{g})$ is, in fact, zero, i.e., $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ as Lie algebras. For instance, this is automatically so if $H^2(\mathfrak{g}) = 0$ (e.g., if G is a compact Lie group). In this case, we may choose Q_y so that $C = 0$, and we have a moment map

$$\mu : M \to \mathfrak{g}^*.$$

For example, for translation symmetries of the free particle, μ is the momentum $\mathbf{p}$ of the particle, which explains the terminology "moment map".

A similar discussion applies to classical field theory, using the formalism of Subsection 11.6. Namely, in this case, the Noether charge is given by the integral over the space of a certain local field called *Noether current*.

For example, consider the free massive boson ϕ on the spacetime $\mathbb{R}^d \times \mathbb{R}$. The Hamiltonian is

$$H = \frac{1}{2} \int_X (\phi_t^2 + |d_x \phi|^2 + m^2 \phi^2) dx.$$

Thus $H = \int_{\mathbb{R}^d} J dx$ where

$$J = \frac{1}{2}(\phi_t^2 + |d_x \phi|^2 + m^2 \phi^2) = \frac{1}{2}(: \phi_t^2 : + \sum_{j=1}^{d} : \phi_{x_j}^2 : + m^2 : \phi^2 :) \tag{11.3}$$

is the Noether current associated to the time translation symmetry.

Similarly, the Noether current for the spacial translation in the i-th coordinate is

$$J_k = \phi_t \phi_{x_k}. \tag{11.4}$$

Indeed, using the formulas of Subsection 11.6, we have

$$\{J_k(x), \phi(y)\} = -\phi_{x_k}(x)\delta(x-y),\ \{J_k(x), \phi_t(y)\} = \phi_t(x)\delta_{x_k}(x-y).$$

Thus defining the charge

$$P_k = \int_{\mathbb{R}^d} J_k(x)dx,$$

using integration by parts, we get

$$\{P_k, \phi(y)\} = -\phi_{x_k}(y),\ \{P_k, \phi_t(y)\} = -\phi_{tx_k}(y),$$

as needed.

Furthermore, this discussion extends to quantum theory, with observables replaced by operators as usual. Namely, in this case, we have a unitary projective representation $\pi : G \to \mathrm{Aut}(\mathcal{H})$ of the Lie group G of symmetries on the Hilbert space $\mathcal{H}$ of quantum states of the system, so that $[\pi(g), \widehat{H}] = 0$, where $\widehat{H} : \mathcal{H} \to \mathcal{H}$ is the hamiltonian (an unbounded self-adjoint operator). The quantum Noether charges corresponding to these symmetries simply define the corresponding Lie algebra representation $\pi_* : \mathfrak{g} \to \mathrm{End}(\mathcal{S})$, where $\mathcal{S}$ is a certain dense subspace of $\mathcal{H}$ (of smooth vectors) on which all the operators $\pi_*(y)$ are defined. For instance, in quantum mechanics, like in classical one, the time translation corresponds to the Hamiltonian $\widehat{H}$, the spacial translations to the momentum operators $\widehat{p}_j := -i\hbar\partial_{x_j}$, and rotations around $x_k = x_j = 0$ to the angular momentum operators

$$\widehat{M}_{kj} := -i\hbar(x_k\partial_j - x_j\partial_k).$$

Finally, in quantum field theory, by analogy with classical one, a quantum Noether charge is an operator of the form

$$Q = \int_{\mathbb{R}^d} J(x)dx,$$

where $J(x)$ is a quantum local operator called the *quantum Noether current*. For example, in the case of a free massive boson, the currents $J(x)$ and $J_k(x)$ for time and space translations are given by the same formulas (11.3),(11.4), but now with $\phi(x,t)$ being the quantum field corresponding to the massive boson (say, in the setting of Wightman axioms) rather than the classical field, and with normal ordered product instead of the usual product:

$$J = \frac{1}{2}(:\phi_t^2: + \sum_{j=1}^{d} :\phi_{x_j}^2: + m^2 :\phi^2:),$$

$$J_k =: \phi_t\phi_{x_k} : ,$$

and the corresponding charges, as in the classical case, are given by integration of the current over the space. For example, for the free boson

$$\widehat{H} = \int_{\mathbb{R}^d} J(x)dx \tag{11.5}$$

is the quantum hamiltonian, and

$$\widehat{P}_k := \int_{\mathbb{R}^d} J_k(x)dx$$

are the quantum momentum operators.

11.13. **Field theories on manifolds.** As already mentioned above, an important feature of classical and quantum field theory is the possibility to consider them not just on a Euclidean or Minkowskian space, but more generally on Riemannian and Lorentzian manifolds. The main examples are theories on $X \times \mathbb{R}$, where X is a Riemannian d-dimensional space manifold with metric $g_{ij}dx^i dx^j$ (Einstein summation) and $\mathbb{R}$ is the time line, with Lorentzian metric

$$|dx|^2 := (dt)^2 - g_{ij}dx^i dx^j,$$

and Euclidean theories on a Riemannian $d+1$-dimensional spacetime manifold M.

Here we will consider only classical field theories on manifolds. These theories can then be quantized using either Lagrangian or Hamiltonian approach, but we will not discuss this, except in some examples. The story is parallel to the case of flat space considered above, but we should make sure that the kinetic term and other terms in the Lagrangian are defined canonically (i.e., do not depend on the choice of coordinates). For simplicity consider the Euclidean case (in the Lorentzian case the story is similar). We restrict ourselves to reviewing the most common types of classical fields in such theories, as well as the corresponding kinetic and other terms in their Lagrangians. A more complete discussion can be found in [QFS].

1. **Scalar (bosonic) fields.** In the simplest case a scalar field is just a real function on M (real scalar), but one can also consider scalars valued in a finite dimensional real vector space with a positive inner product (for example, $\mathbb{C}$, for complex scalars) or, more generally, valued in a real vector bundle on M. The kinetic term for a scalar $\phi : M \to E$ valued in a vector space $E \cong E^*$ with inner product is $|d\phi(x)|^2$, the squared norm of the vector $d\phi \in T_{\phi(x)}M \otimes E$ with respect to the inner products on $T_{\phi(x)}M$ and E. Thus if this vector has components $(d\phi)_{ij}$

in orthonormal bases then

$$|d\phi|^2 = \sum_{i,j} (d\phi)_{ij}^2.$$

More generally, if E is a vector bundle on M then we need to fix an inner product on E (i.e., E should be an orthogonal bundle) and also a connection A preserving this inner product, which gives rise to the covariant derivative operator ∇_A; if E is trivialized on a local chart $U \subset M$ then A becomes a 1-form on U with values in $\mathfrak{o}(E)$ and we have $\nabla_A = d + A$. In this case, an E-valued scalar field ϕ is a section of E over M, and the kinetic term is $|\nabla_A \phi|^2$, which in local trivialization has the form $|d\phi + A\phi|^2$.

Note that for a scalar field ϕ, we can always add to the kinetic term a mass term $m^2|\phi|^2$, where m^2 is a real number. More generally, we can add a mass term $(\phi, Q\phi)$, where Q is a self-adjoint endomorphism of E.

2. **Spinor (fermionic) fields.** Spinor fields can be defined on a spin manifold M, i.e., an oriented manifold equipped with a spin structure (a lift of the tangent bundle from $SO(n)$ to $\mathrm{Spin}(n)$). For such a manifold, we have the canonically defined *spin bundle* S_M, which is the associated bundle to the above $\mathrm{Spin}(n)$ bundle via the spin representation $\mathrm{Spin}(n) \to \mathrm{Aut}(S)$. This bundle carries a natural inner product and a connection induced by the Levi-Civita connection of M that preserves this inner product. Moreover, as explained in Subsection 11.3, in even dimensions we have $S = S_+ \oplus S_-$, where S_+, S_- are irreducible representations of $\mathrm{Spin}(n)$, so we have $S_M = S_{M+} \oplus S_{M-}$, an orthogonal decomposition of S_M into two subbundles.

Spinor fields, in the most basic case, are sections of the Spin bundle S_M. The sections of S_{M+} and S_{M-}, as noted in Subsection 11.4, are called *chiral spinors*.

The possible kinetic and mass terms for spinors on the flat space are described in Subsection 11.4, and the story on the curved manifold is similar. The only new feature is that we have to define the Dirac operator $\mathbf{D}$ for a spinor field on an arbitrary spin manifold. To this end, all we have to do is replace ordinary partial derivatives in formula (11.2) by the covariant ones with respect to the Levi-Civita connection:

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{LC}. \tag{11.6}$$

More generally, similar to the scalar field case, we may consider spinors valued in a vector bundle E with an inner product and an orthogonal connection A, i.e., sections of the bundle $S_M \otimes E$. This

bundle carries a tensor product connection $\nabla^{\text{total}} = \nabla^{LC} \otimes \nabla_A$, and the Dirac operator is defined by the formula

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{\text{total}}.$$

3. **Gauge fields.** Let G be a compact Lie group $\mathfrak{g} = \text{Lie}G$ equipped with a positive invariant inner product. Gauge fields are connections A on principal G-bundles E on M, so in local trivialization A is a 1-form on M with values in $\mathfrak{g}$ and the covariant derivative with respect to A looks like $\nabla_A = d + A$. The connection A has curvature F_A (called *field strength* in physical terminology), which is a 2-form on M with values in the adjoint bundle $\text{ad}E$. In local trivialization the curvature of A is the Maurer-Cartan form

$$F_A = dA + \frac{1}{2}[A, A].$$

In particular, if G is abelian then we just have $F_A = dA$. The kinetic term for a gauge field A is $|F_A|^2$, where the squared norm is taken with respect to the inner product on $(\wedge^2 T^*M \otimes \text{ad}E)_x$ induced by the inner products on T_xM and $\mathfrak{g}$ (note that this does not depend on the identification of Lie algebras $(\text{ad}E)_x \cong \mathfrak{g}$ since the form on $\mathfrak{g}$ is invariant).

It makes sense to fix the topological type of the C^∞-bundle E (which does not change under deformations) and consider the space $\text{Conn}(E)$ of all connections A on E. If $A_1, A_2 \in \text{Conn}E$ then $\nabla_{A_1} - \nabla_{A_2} \in \Omega^1(M) \otimes \text{ad}E$, so $\text{Conn}(E)$ is an affine space with underlying vector space $\Omega^1(M) \otimes \text{ad}E$. Moreover, this space carries a natural right affine linear action of the gauge group $\mathcal{G}_E = C^\infty(M, E)$, which in local trivialization looks like

$$A^g = g^{-1}dg + g^{-1}Ag.$$

The configuration space of a classical gauge theory is then

$$\mathcal{M} := \sqcup_{\text{topological types } E} \text{Conn}(E)/\mathcal{G}_E,$$

so the phase space is the cotangent bundle $T^*\mathcal{M}$.

12. Perturbative expansion for interacting QFT

12.1. **General strategy of quantization.** We now pass to non-free field theories defined by the action $S(\phi) := \int \mathcal{L}(\phi)dx$ in Minkowski space $V \cong \mathbb{R}^d$, where $\mathcal{L}(\phi)$ is a local Poincaré-invariant Lagrangian. The general strategy of quantization of such theories is as follows.

Step 1. Write down the Euclidean path integral correlators for the theory:

$$\langle\, \phi(x_1)...\phi(x_n)\,\rangle = \int \phi(x_1)...\phi(x_n)e^{-\frac{S_E(\phi)}{\hbar}}D\phi.$$

Compute the corresponding formal expansion in $\hbar$ using the Feynman rules (as we have done in the case of quantum mechanics, $d = 1$).

Step 2. Perform Borel summation of this formal series, to obtain actual functions defined for small enough $\hbar > 0$.

Step 3. Perform the Wick rotation of these functions to Minkowski space to obtain Wightman correlation functions W_n.

Step 4. Use the functions W_n to define a Wightman QFT, i.e., extract the Hilbert space $\mathcal{H}$, the representation π of the (double cover of the) Poincaré group on $\mathcal{H}$, the vacuum vector Ω and the field map ϕ.

All these steps are non-trivial, and while Step 1 can be performed fully rigorously, starting from Step 2 a rigorous implementation is only known for a handful of theories treated in constructive field theory (and for many Lagrangians the ultimate Wightman QFT, in fact, does not exist). For most physically interesting theories, doing these steps rigorously is still an open problem. In this section, we will only discuss Step 1.

12.2. **The ϕ^3 theory.** As a running example, we will use the theory of a scalar boson ϕ with Euclidean Lagrangian

$$\mathcal{L}_E(\phi) := \frac{1}{2}((d\phi)^2 + m^2\phi^2) + \frac{g}{6}\phi^3,$$

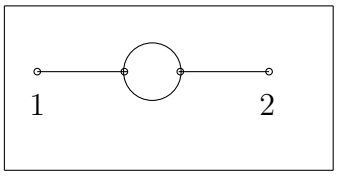
which we will call the *ϕ^3-theory*. This theory is a deformation of the theory of free scalar boson obtained by adding a single interaction term $\frac{g}{6}\phi^3$, which in Feynman calculus corresponds to a 3-valent vertex. Physically this vertex corresponds to an interaction in which two particles collide and transform into a third one.

We will set $\hbar = 1$ and consider the formal expansion in powers of g (which is equivalent to Step 1 by rescaling ϕ).

Let us compute the 1-loop correction to the 2-point correlation function of the free theory

$$\widehat{G}_0(p) = \frac{1}{p^2 + m^2}$$

in the momentum space presentation. It is easy to see that this correction is given by a single Feynman diagram



The amplitude of this Feynman diagram is

$$A(p) = \frac{g^2}{2(p^2 + m^2)^2} \int_V \frac{dq}{(q^2 + m^2)((p - q)^2 + m^2)}.$$

If $d < 4$, this integral is convergent and can be computed explicitly. To this end, we may use the following lemma from multivariable calculus, which is known in physics literature as the *Feynman famous formula*:

Lemma 12.1. *Let Δ_n be the $n-1$-dimensional simplex defined in $\mathbb{R}^n$ by the equation*

$$y_1 + ... + y_n = 1,$$

and dy be the Lebesgue measure on Δ_n of volume 1. Then for positive numbers $a_1, ..., a_n$ we have

$$\int_{\Delta_n} \frac{dy}{(a_1 y_1 + ... + a_n y_n)^n} = \frac{1}{a_1 ... a_n}.$$

Proof. We have

$$\frac{1}{(a_1 y_1 + ... + a_n y_n)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(a_1 y_1 + ... + a_n y_n)t} dt.$$

So we get

$$\int_{\Delta_n} \frac{dy}{(a_1 y_1 + ... + a_n y_n)^n} = \frac{1}{(n-1)!} \int_{\Delta_n} \int_0^\infty t^{n-1} e^{-(a_1 y_1 + ... + a_n y_n)t} dt dy$$

$$= \frac{1}{(n-1)!} \int_{t\Delta_n} \int_0^\infty e^{-a_1 z_1 + ... + a_n z_n} dt dz$$

$$= \int_{z_1,\ldots,z_n \geq 0} e^{-a_1 z_1 + \ldots + a_n z_n} dz = \prod_{j=1}^{n} \int_0^\infty e^{-a_j z_j} dz_j = \frac{1}{a_1 \ldots a_n}.$$

□

Applying the Feynman famous formula to our integral and making a change of variable $q \mapsto q + (1-y)p$, we have

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \int_0^1 \int_V \frac{dq}{((1-y)q^2 + y(p-q)^2 + m^2)^2} dy =$$

$$\int_0^1 \int_V \frac{dq}{(q^2 + M^2(y,p))^2} dy,$$

where

$$M^2(y,p) := y(1-y)p^2 + m^2.$$

Now, using spherical coordinates

$$\int_V \frac{dq}{(q^2+M^2)^2} = C_d \int_0^\infty \frac{r^{d-1} dr}{(r^2+M^2)^2},$$

where C_d is the area of the unit sphere in $\mathbb{R}^d$. Thus for $d = 2$

$$\int_V \frac{dq}{(q^2+M^2)^2} = 2\pi \int_0^\infty \frac{r dr}{(r^2+M^2)^2} = \pi \int_0^\infty \frac{ds}{(s+M^2)^2} = \frac{\pi}{M^2}.$$

It follows that

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \pi \int_0^1 \frac{dy}{y(1-y)p^2 + m^2}$$

$$= \frac{2\pi}{p^2 \sqrt{\frac{4m^2}{p^2} + 1}} \operatorname{arccotanh} \sqrt{\frac{4m^2}{p^2} + 1}.$$

The case $d = 3$ can be computed similarly.

However, for $d \geq 4$ we encounter our first difficulty: the integral diverges (as the integrand behaves at infinity as $|q|^{-4}$). More specifically, for a cutoff $\Lambda > 0$, define

$$A_\Lambda(p) := \frac{g^2}{2(p^2+m^2)^2} \int_{|q| \leq \Lambda} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)},$$

the integral over the ball in V of radius Λ. Then

$$A_\Lambda(p) \sim \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}),\ \Lambda \to \infty$$

for $d = 4$ and

$$A_\Lambda(p) \sim C_d \frac{g^2}{2(d-4)(p^2+m^2)^2} \Lambda^{d-4},\ \Lambda \to \infty$$

if $d > 4$. A way to remedy this difficulty is to add a Λ-dependent term in the Lagrangian, called a *counterterm*, which blows up as $\Lambda \to \infty$ but which will cancel this divergence, in the sense that when integration is performed over the ball $|q| \leq \Lambda$ then the integral has a finite limit as $\Lambda \to \infty$.

For example, consider $d = 4$. In this case modulo g^3 the momentum space 2-point function computed with cutoff Λ looks like

$$\widehat{G}_{\Lambda,m^2}(p) = \frac{1}{p^2+m^2} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

(here we explicitly indicate dependence of $\widehat{G}$ on m^2 since we are about to vary it). Let us try to fix the divergence by replacing the parameter m^2 by $m^2 + Kg^2 \log(\frac{\Lambda}{m})$ for a constant K. So we have

$$\widehat{G}_{\Lambda,m^2+Kg^2\log(\frac{\Lambda}{m})}(p) = \frac{1}{p^2+m^2+Kg^2\log(\frac{\Lambda}{m})} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

$$= \frac{1}{p^2+m^2} + (\pi^2 - K) \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

where we ignore terms of order higher than g^2. Thus to cancel the divergence, we should take $K = \pi^2$, i.e., replace the Lagrangian with

$$\mathcal{L}_{E,\Lambda} := \frac{1}{2}((d\phi)^2 + (m^2 + \pi^2 g^2 \log(\tfrac{\Lambda}{m}))\phi^2) + \frac{g}{6}\phi^3.$$

For this Lagrangian, if integration is performed with cutoff Λ, then the 2-point function modulo g^2 will have a finite limit as $\Lambda \to \infty$, given by

$$\widehat{G}(p) = \frac{1}{p^2+m^2} + \frac{g^2}{2(p^2+m^2)^2} I(p),$$

where

$$I(p) = \lim_{\Lambda\to\infty} \left(\int_{\mathbb{R}^4} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} - 2\pi^2 \log(\tfrac{\Lambda}{m}) \right).$$

This limit is easy to compute using the Feynman famous formula. Namely, computing similarly to the $d < 4$ case, we get

$$I(p) = \int_0^1 I(p,y)dy,\ I(p,y) := \lim_{\Lambda\to\infty} \left(\int_0^\Lambda \frac{r^3 dr}{(r^2 + M^2(y,p))^2} - 2\pi^2 \log(\tfrac{\Lambda}{m}) \right).$$

So

$$I(p,y) = 2\pi^2(\log m - \frac{1}{2}(1 + \log(y(1-y)p^2 + m^2))).$$

Thus

$$I(p) = 2\pi^2 \left(\frac{1}{2} + \sqrt{\frac{4m^2}{p^2} + 1} \cdot \operatorname{arccotanh} \sqrt{\frac{4m^2}{p^2} + 1} \right).$$

For $d > 4$ the calculation becomes more elaborate. Namely, while for $d = 5$ we have

$$A_\Lambda(p) \sim C_5 \frac{g^2}{2(p^2+m^2)^2}\Lambda + O(1), \lambda \to \infty,$$

so the procedure is the same, with mass parameter modification $m^2 \mapsto m^2 + K\Lambda$, already for $d = 6$ we will have to take a deeper expansion of the divergent integral:

$$A_\Lambda(p) \sim \frac{g^2}{4(p^2+m^2)^2}(C_6\Lambda^2 + Cp^2\log(\tfrac{\Lambda}{m}) + O(1)),\ \Lambda \to \infty$$

We can cancel the most singular term $C_6\Lambda^2$ by mass modification $m^2 \mapsto m^2 + K\Lambda^2$, but after that we will still have logarithmic divergence, $Cp^2\log(\frac{\Lambda}{m})$, which depends on p. To kill this divergence, we must modify the coefficient of $\frac{1}{2}(d\phi)^2$ in the Lagrangian by a counterterm, changing it from 1 to $1 + C'g^2\log(\frac{\Lambda}{m})$ for an appropriate constant C'. Also we find that the 1-loop correction to the 3-point function is logarithmically divergent: the corresponding contribution in momentum presentation is, up to scaling,

$$\frac{g^3}{\prod_{j=1}^3(p_j^2+m^2)}J(p_1,p_2,p_3)\delta(p_1+p_2+p_3),$$

where

$$J(p_1,p_2,p_3) = \int_V \frac{dq}{(q^2+m^2)((q-p_1)^2+m^2)((q-p_1-p_2)^2+m^2)}$$

for $p_1 + p_2 + p_3 = 0$, which is divergent and behaves like $\log\Lambda$ when computed over the ball of radius Λ. So to kill this divergence, we must change the coefficient of $\frac{1}{6}\phi^3$ in the Lagrangian by a counterterm, changing it from g to $g + C''g^3\log(\frac{\Lambda}{m})$.

We are starting to see the main idea of *renormalization theory*, which allows us to regularize divergent integrals coming from Feynman diagrams in all orders of perturbation series. This idea is that the coefficients of the Lagrangian are actually **not** meaningful physical quantities, — they are just mathematical parameters depending on the scale (cutoff) Λ at which we are doing the computation, and may blow up when $\Lambda \to \infty$ (called the *ultraviolet limit*, as Λ has the meaning of frequency of oscillation). Rather, the meaningful quantity is the answer, the correlation functions $\langle\, \phi(x_1)...\phi(x_n)\,\rangle$ (or their Fourier transforms, if we work in the momentum realization). This answer depends on some parameters, which are the actual parameters of the theory. So the coefficients in the Lagrangian must be adjusted in such a way that the answer has a finite limit as $\Lambda \to \infty$. The specific answer we will

get will depend on the adjustment procedure, but in good cases (called renormalizable) will lie in a nice universal family (often, but not always depending on finitely many parameters).

12.3. **Super-renormalizable, renormalizable, and non-renormalizable theories.** Let us discuss this more systematically. Consider a theory of a scalar boson with a general Lagrangian. Given a Feynman diagram Γ, we have the corresponding Feynman integral I_Γ in momentum space realization, which is an integral of a rational volume form over a real vector space. We can define the *superficial degree of divergence* $D(\Gamma)$ to be the degree of the numerator of this form (where the differentials of coordinates have degree 1) minus the degree of its denominator. It is clear that if $D(\Gamma) \geq 0$ then the integral diverges. Note that the converse is false: if $d(\Gamma) < 0$, the integral may still diverge.

Let us compute $D(\Gamma)$. The degree of the denominator is easy to compute: it is just $2e(\Gamma)$ where $e(\Gamma)$ is the number of internal edges of Γ (indeed, every edge contributes a propagator, which is the inverse of a quadratic function). On the other hand, the number of integrations over V is the number of loops, i.e, $d(e(\Gamma) - v(\Gamma) + 1)$, where $v(\Gamma)$ is the number of internal vertices. Finally, the terms in the Lagrangian containing derivatives of ϕ contribute the number of such derivatives to the degree of the numerator. It follows that

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d + N,$$

where N is the total number of derivatives in vertex monomials. In particular, when there are no derivatives, we have

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d.$$

This shows that we may compute $D(\Gamma)$ as a sum of contributions over vertices, defining the degree $D(\Phi)$ of a differential monomial Φ standing at a fully internal vertex (one whose all edges are internal) as the contribution of this vertex to $D(\Gamma)$. Indeed, every Φ contributes

$$D(\Phi) = \frac{d-2}{2}e(\Phi) - d + N_\Phi,$$

where $e(\Phi)$ is the number of edges of Φ (i.e., its degree with respect to ϕ) and N_Φ is the number of derivatives in Φ.

We see that a more natural invariant is

$$[\Phi] := D(\Phi) + d,$$

as it is multiplicative:

$$[\Phi_1\Phi_2] = [\Phi_1][\Phi_2].$$

This is not surprising since Φ comes with a volume factor dx, so $D(\Phi)$ is actually the scaling dimension of Φdx; thus to get the scaling dimension of Φ, we need to add d (as the scaling dimension of dx is $-d$). This motivates

Definition 12.2. The number $[\Phi]$ is called the *classical scaling dimension* of the differential monomial Φ.

Thus for a Feynman diagram Γ we have

$$D(\Gamma) = d - \frac{k(d-2)}{2} + \sum_{\Phi} D(\Phi), \tag{12.1}$$

where k is the number of external vertices of Γ.

For example, for $\Phi = \phi^n$ we get

$$D(\phi^n) = \tfrac{n}{2}(d-2) - d = (\tfrac{n}{2} - 1)d - n,$$

Each derivative adds a 1 to the degree, so for instance

$$D(\phi^{n-2}(d\phi)^2) = (\tfrac{n}{2} - 1)(d-2).$$

So for the 1-loop Feynman diagram Γ for the k-point function (a cycle with k legs), we have

$$D(\Gamma) = d - \frac{k}{2}(d-2) + kD(\phi^3) = d - \frac{k}{2}(d-2) + \frac{k}{2}(d-6) = d - 2k.$$

Definition 12.3. Let Φ be a differential monomial in ϕ. We will say that Φ is *super-renormalizable* if $D(\Phi) < 0$, *renormalizable* (or critical) if $D(\Phi) = 0$, and *non-renormalizable* if $D(\Phi) > 0$.

Thus super-renormalizable terms improve convergence, renormalizable ones do not affect it, and non-renormalizable ones worsen it.

Example 12.4. 1. The kinetic term $(d\phi)^2$ has $D = 0$, so is renormalizable; in fact, this is so by definition in any QFT. Note that this can be used to easily compute the classical scaling dimensions of monomials. Namely, we have $[(d\phi)^2] = D((d\phi)^2) + d = d$, so $2[\phi] + 2 = d$, i.e. $[\phi] = \frac{d-2}{2}$. Using multiplicativity, we now immediately compute $[\Phi]$ for any Φ.

2. The mass term ϕ^2 has $D = -2$, so it is super-renormalizable. The term ϕ^3 has $D = \frac{1}{2}d - 3$, so it is super-renormalizable for $d < 6$, renormalizable for $d = 6$ and non-renormalizable for $d > 6$.

Definition 12.5. A Lagrangian is called

• super-renormalizable if all its terms except the kinetic term are super-renormalizable;

• renormalizable (or critical) if all its terms are at worst renormalizable and there is at least one renormalizable non-kinetic (i.e., interacting) term;

• non-renormalizable if it contains non-renormalizable terms.

Clearly, every Lagrangian is of exactly one of these three types.

Proposition 12.6. *(i) If a Lagrangian is super-renormalizable then the degree of superficial divergence of the corresponding Feynman diagrams is bounded above, and there are finitely many superficially divergent diagrams with any given number of external edges; moreover, if $d > 2$ then there are finitely many superficially divergent diagrams altogether.*

(ii) If a Lagrangian is renormalizable, then there are infinitely many superficially divergent diagrams with a fixed number of external edges, but the degree of superficial divergence of these diagrams is still bounded above.

(iii) If a Lagrangian is non-renormalizable, then the degree of superficial divergence of diagrams with a fixed number of external edges is unbounded above.

Proof. This is clear from formula (12.1). □

This means that for a non-renormalizable Lagrangian, regularization of divergent integrals will definitely get out of control. Namely, if we want to regularize diagrams with unbounded above degree of superficial divergence, then we will have to introduce counterterms with unlimited number of derivatives, and our renormalized Lagrangian will no longer depend on a finite number of derivatives of ϕ.

On the other hand, if the Lagrangian is renormalizable, then for $d > 2$ there are only finitely many terms that we will need to modify in the renormalization procedure; namely, these are the possible super-renormalizable and renormalizable terms in the Lagrangian. The fact that this procedure works to all orders of perturbation theory is a rather non-trivial fact which we will not prove here; but the result is a finite-parametric family of perturbative QFT.

In two dimensions, there is an additional feature - there are infinitely many (super)renormalizable terms in the Lagrangian; but they all have at most two derivatives.

Finally, in the super-renormalizable case the renormalization procedure is completed in finitely many steps.

12.4. **Critical dimensions of some important QFT.** For interacting QFT defined by Lagrangians, the theory is only (super-)renormalizable in small dimensions, and becomes non-renormalizable when dimension

grows. If a theory is renormalizable in some dimension d and non-renormalizable for bigger dimensions, we say that d is the *critical dimension* of the theory.

12.4.1. *Scalar bosons.* For example, since $D(\phi^n) = (\frac{n}{2} - 1)d - n$, for a scalar boson, a term ϕ^n is (super-)renormalizable iff $d \leq \frac{2n}{n-2}$. So in a (super-)renormalizable theory, the term ϕ^3 can be present only for $d \leq 6$, ϕ^4 only for $d \leq 4$, ϕ^5 and ϕ^6 only for $d \leq 3$. Also, since $D(\phi^{n-2}(d\phi)^2) = (\frac{n}{2} - 1)(d-2)$, such terms with $n > 2$ cannot be present in a (super-)renormalizable theory unless $d = 2$. With more derivatives things get even worse. So we obtain

Proposition 12.7. *For the scalar bosonic field ϕ, the most general (super-)renormalizable non-quadratic Poincaré-invariant Lagrangian is (up to scaling):*

- *$d > 6$: none;*
- *: $d = 5, 6$: $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_3(\phi)$;*
- *: $d = 4$: $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_4(\phi)$;*
- *: $d = 3$: $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_6(\phi)$;*
- *: $d = 2$: $\mathcal{L} = \frac{1}{2}g(\phi)(d\phi)^2 + U(\phi)$,*

where P_m is a polynomial of degree m, and U and g are arbitrary (real analytic) functions.

Note that without loss of generality, one may assume that P_m are missing the constant and linear terms. Thus the number of parameters for the theory with Lagrangian $\frac{1}{2}(d\phi)^2 + P_m(\phi)$ is $m-1$ (the coefficients of P_m).

12.4.2. *Fermions.* Recall that for a fermionic field ψ the kinetic term looks like $(\psi, \mathbf{D}\psi)$. This implies that

$$2[\psi] + 1 = d,$$

i.e.,

$$[\psi] = \frac{d-1}{2},$$

which is always positive. So for mass terms $(\psi, M\psi)$ we have $D = -1$ and they are super-renomalizable. Beyond quadratic, we see that the only possibly (super-)renormalizable terms in ψ for $d \geq 2$ are of the general shape ψ^{2k}, and

$$D(\psi^{2k}) = 2k[\psi] - d = k(d-1) - d = (k-1)(d-1) - 1.$$

The only case when this is (super-)renormalizable is $d = 2$ and $k = 2$, i.e., the term ψ^4, in which case $D = 0$ (critical). Such terms indeed occur in the so-called *Gross-Neveu model*.

For $d > 2$, any fermionic term in a renormalizable Lagrangian must therefore be quadratic in the fermions. But it can contain other (bosonic) fields as factors. For example, $[\phi^n \psi^2] = n\frac{d-2}{2} + d - 1$, so

$$D(\phi^n \psi^2) = n\frac{d-2}{2} - 1.$$

This shows that in 3 dimensions we can have a term $\phi\psi^2$ (Yukawa interaction) and $\phi^2\psi^2$, while in 4 dimensions we can have only the Yukawa term $\phi\psi^2$, and for $d > 4$ there are no possible (super-)renromalizable terms.

12.4.3. *Gauge theory.* A similar result holds when ϕ is vector-valued, i.e., has any number of components. This allows us to treat another important example, which is *gauge theory*.

Recall from Subsection 11.13 that to define a gauge theory, we fix a compact Lie group G (for example, $U(n)$) and the field is a connection ∇ on a principal G-bundle P on V. Since all such bundles are trivial, we may think of ∇ as a 1-form A with values in $\mathfrak{g} = \mathrm{Lie}G$; i.e. $\nabla_A = d + A$. The curvature of ∇_A is given by the formula

$$F_A = dA + \frac{1}{2}[A, A],$$

and the Lagrangian of the pure gauge theory is

$$\mathcal{L} := \int_V |F_A|^2 dx.$$

As mentioned in Subsection 11.13, he subtlety here is that A is only considered up to gauge transformations $\nabla_A \mapsto g^{-1}\nabla_A g$, i.e., $A \mapsto g^{-1}dg + g^{-1}Ag$, where $g : V \to G$ is a smooth function with prescribed behavior at infinity, but this is irrelevant for the discussion of critical dimension.

If G is abelian (e.g. $G = U(1)$) then the Lagrangian is quadratic and this theory is free (this is the quantum electrodynamics without matter, i.e., quantization of Maxwell equations). This theory satisfies Wightman axioms in all dimensions, and its Wightman functions can be explicitly computed similarly to the case of scalar boson.

However, if G is non-abelian (e.g. $G = SU(2)$ for weak interactions and $G = SU(3)$ for strong interactions in the standard model) then the Lagrangian is not quadratic and the equations of motion are not linear (they are the *Yang-Mills equations*). Treating A as a (vector-valued) boson, we see that the non-quadratic terms in the Lagrangian are of schematic form $A^2 dA$ and A^4. The degrees of these terms are $\frac{1}{2}(d-4)$ and $d-4$, so we see that this theory is critical in dimension 4 (the physical case!) and super-renormalizable in lower dimensions, but

non-renormalizable for $d > 4$. Note that the fact that we have a vector boson rather than a collection of scalar bosons (under the action of $\mathbf{P}$) does not matter for the dimension count.

Note also that in $d \leq 4$ dimensions we can also consider renormalizable Lagrangians with terms $(\nabla_A \phi)^2$ or $(\psi, \mathbf{D}_A \psi)$, where ϕ is a scalar and ψ a spinor with values in the associated bundle $P \times_G \rho$, where ρ is a finite dimensional representation of G (it is easy to check that all occurring monomials have $D \leq 0$). Such terms do occur in the standard model; the simplest case is $(\psi, \mathbf{D}_A \psi)$ where A is a $U(1)$-connection and ψ is a spinor valued in the tautological representation of $U(1)$, corresponding to an electron.

12.4.4. *σ-model.* The *σ-model* is a theory of a scalar boson taking values in a Riemannian manifold M. Thus the field is a map $\phi : V \to M$, and the Lagrangian is $\mathcal{L} = \frac{1}{2}(d\phi)^2$, which in local coordinates has the form

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{\dim M} g_{ij}(\phi) d\phi^i d\phi^j,$$

where g_{ij} is the Riemannian metric on M. We may also add a potential $U(\phi)$, where U is a smooth function on M. By the above computations, this Lagrangian for a non-constant metric is renormalizable only in dimension $d = 2$, but in this case g_{ij} and U can be arbitrary.

12.4.5. *Gravity.* The theory of gravity (general relativity) is a theory of a bosonic field $h(x)$ taking values in symmetric tensors $S^2 V^*$; i.e., the Minkowskian metric on V is perturbed by setting $g = g_0 + h$, where g_0 is the standard Minkowskian metric. The Lagrangian of general relativity is

$$\mathcal{L} = R(g)$$

where R is the scalar curvature of the metric g. Since curvature is expressed in terms of second derivatives of the metric, up to scaling this can be schematically written in terms of h as

$$\mathcal{L} = (dh)^2 + ...$$

where the dots stand for terms having at most two derivatives in h. Thus the general shape of this Lagrangian (for the purposes of computing classical scaling dimensions) is the same as for the σ-model; so this theory is only renormalizable in two dimensions. This is one of the main reasons why it has not yet been possible to incorporate gravity into the standard model, which lives in 4 spacetime dimensions.

Remark 12.8. We have seen in Subsection 11.11 that even in a free quantum field theory, the composite operators like $\phi^2(x)$ are not automatically defined, and require a normal ordering procedure to regularize them. This is all the more so in an interacting QFT.

It turns out that the normal ordering procedure, composite operators, and operator product expansion in a critical perturbative QFT can be defined analogously to the free case, using renormalization theory. We will not discuss it here and refer the reader to [QFS], vol. 1, p. 452.

13. Two-dimensional conformal field theory

13.1. **Classical free massless scalar in two dimensions.** Consider a free massless scalar boson ϕ on $\mathbb{R}^2$ with Lagrangian $\mathcal{L} = \frac{1}{2}(d\phi)^2$. In this case the local functional $\phi(t,x)$ satisfies the 2-dimensional wave (=string) equation

$$\phi_{tt} - \phi_{xx} = 0,$$

so it splits into a sum of two functionals

$$\phi = \tfrac{1}{\sqrt{2}}\phi_L + \tfrac{1}{\sqrt{2}}\phi_R,$$

where

$$\phi_L(t,x) = \psi_L(x+t),\ \phi_R(t,x) = \psi_R(x-t),$$

which for obvious reasons are called the *left-mover and right-mover*. In other words, we have

$$(\partial_t - \partial_x)\phi_L = 0,\ (\partial_t + \partial_x)\phi_R = 0.$$

So we get

$$\phi_x + \phi_t = \sqrt{2}\psi_L'(x+t),\ \phi_x - \phi_t = \sqrt{2}\psi_R'(x-t).$$

So the Poisson bracket of ψ_L', ψ_R' is given by

$$\{\psi_L'(x), \psi_L'(y)\} = \delta'(x-y),\ \{\psi_R'(x), \psi_R'(y)\} = -\delta'(x-y),$$

$$\{\psi_L'(x), \psi_R'(y)\} = 0.$$

Thus upon Wick rotation, which replaces t with it and makes ϕ complex-valued, setting $u := x + it$, we have

$$\overline{\partial}_u \phi_L = 0,\ \partial_u \phi_R = 0,$$

i.e., $\phi_L = \psi_L(u)$ is holomorphic and $\phi_R = \psi_R(\overline{u})$ is antiholomorphic.

Now consider the case when x runs over the circle $\mathbb{R}/2\pi\mathbb{Z}$, with Lebesgue measure normalized to have volume 1. Then, if we still want to have a decomposition of ϕ into a left-mover and a right-mover, we should "kill the zero mode" by requiring that $\int_0^{2\pi} \phi(t,x)dx = 0$ (otherwise we have a solution $\phi(t,x) = t$ of the string equation which cannot be written as a sum of a left-moving and right-moving periodic wave). Then we may introduce the coordinate $z = e^{iu}$ which takes values in $\mathbb{C}^\times$, and ϕ_L, ϕ_R become holomorphic, respectively antiholomorphic fields on $\mathbb{C}^\times$, which we'll denote by φ, φ^*. So we have Laurent expansions

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n},\ \varphi^*(\overline{z}) = \sum_{n\in\mathbb{Z}} \varphi_n^* \overline{z}^{-n},$$

with $\varphi_0 = \varphi_0^* = 0$. When z is on the unit circle, these are just the Fourier expansions of $\phi_L(0,x)$, $\phi_R(0,x)$, and for

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} := i\partial_z\varphi(z),\ a^*(\overline{z}) = \sum_{n\in\mathbb{Z}} a_n^* \overline{z}^{-n-1} := -i\overline{\partial}_z\varphi^*(\overline{z}),$$

where $a_0 = a_0^* = 0$, we have

$$za = \partial_u\phi_L = \psi_L'(u),\ \overline{z}a^* = \overline{\partial}_u\phi_R = \psi_R'(\overline{u}).$$

Thus for $z = e^{iu}, w = e^{iv}$ we get

$$\{za(z), wa(w)\} = \delta'(u-v). \tag{13.1}$$

Note that

$$\delta'(u-v) = i\sum_{n\in\mathbb{Z}} nz^n w^{-n}.$$

So setting

$$\delta(w-z) := \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$$

(Fourier expansion of the distribution $\delta(w-z)$ on $(S^1)^2$, where $|z| = |w| = 1$), we can write (13.1) as

$$\{a(z), a(w)\} = -i\delta'(w-z). \tag{13.2}$$

In components, this takes the form

$$\{\sum_{m\in\mathbb{Z}} a_m z^{-m}, \sum_{n\in\mathbb{Z}} a_{-n} w^n\} = -i\sum_{n\in\mathbb{Z}} nz^{-n}w^n.$$

Thus we get

$$\{a_n, a_m\} = -in\delta_{n,-m}. \tag{13.3}$$

Similarly,

$$\{a_n^*, a_m^*\} = in\delta_{n,-m}, \tag{13.4}$$

and

$$\{a_n, a_m^*\} = 0, \tag{13.5}$$

which in terms of generating functions can be written as

$$\{a^*(z), a^*(w)\} = i\delta'(w-z), \{a(z), a^*(w)\} = 0. \tag{13.6}$$

Finally, let us write down the hamiltonian of the theory in terms of the Fourier (=Laurent) modes a_n. Recall that in the original notation it has the form

$$H = \frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}} (\phi_t^2 + \phi_x^2)dx.$$

Thus we have

$$H = \frac{1}{4}\int_{\mathbb{R}/2\pi\mathbb{Z}}((\overline{z}a^* - za)^2 + (\overline{z}a^* + za)^2)dx = \frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}}(\overline{z}^2 a^{*2} + z^2 a^2)dx,$$

i.e.,

$$H = \sum_{n>0}(a_{-n}a_n + a^*_{-n}a^*_n). \tag{13.7}$$

It satisfies the relations

$$\{a_m, H\} = -ima_m,\ \{a^*_m, H\} = ima^*_m.$$

13.2. **Free quantum massless scalar on $\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}$ with killed zero mode.** Consider now the free QFT of a massless scalar boson ϕ on $\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}$ with Minkowskian metric $dt^2 - dx^2$, with killed zero mode, i.e., a quantization of the classical field theory described in Subsection 13.1. Since this is not a theory on a vector space, it won't satisfy Wightman axioms. However, we can naturally quantize the commutation relations (13.3),(13.4),(13.5) (with $\hbar = 1$), by replacing them with

$$[a_n, a_m] = n\delta_{n,-m},\ [a^*_n, a^*_m] = -n\delta_{n,-m},\ [a_n, a^*_m] = 0.$$

In other words, for $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $a^*(\overline{z}) = \sum_{n\in\mathbb{Z}} a^*_n \overline{z}^{-n-1}$ we have

$$[a(z), a(w)] = \delta'(w-z),\ [a^*(z), a^*(w)] = -\delta'(w-z),\ [a(z), a^*(w)] = 0,$$

which quantize equations (13.2),(13.6) (this is a field-theoretic generalization of the analysis of Subsection 8.5, with an infinite sequence of harmonic oscillators labeled by positive integers). Thus we see that the Euclidean space-locality property is satisfied.

This shows that we have an infinite system of independent harmonic oscillators. To restate this algebraically, consider the infinite dimensional Heisenberg Lie algebra $\mathcal{A}$ with basis $a_n, n \neq 0$ and K (central) with commutation relations

$$[a_n, a_m] = n\delta_{n,-m}K.$$

Then we see that some dense subspace of the Hilbert space $\mathcal{H}$ of our theory should carry a pair of commuting actions of $\mathcal{A}$ (by left-movers and right-movers), with K acting by 1 and -1, respectively (we'll denote the second copy of $\mathcal{A}$ by $\mathcal{A}^*$).

Let us now describe the Hilbert space $\mathcal{H}$. Note that the Lie algebra $\mathcal{A}$ has an irreducible *Fock representation* $\mathcal{F}$ generated by Ω with defining relations

$$a_n\Omega = 0,\ n > 0, \quad K\Omega = \Omega.$$

As a vector space, $\mathcal{F}$ is the *Fock space*

$$\mathcal{F} = \mathbb{C}[X_1, X_2, ...]$$

(with $\Omega = 1$), on which the operators a_{-n} for $n > 0$ act by multiplication by X_n and a_n act by $n\frac{\partial}{\partial X_n}$.

Now, the hamiltonian of the system (which we rescale for convenience by a factor of 2) should satisfy the commutation relations

$$[\widehat{H}, a_n] = -na_n,\ [\widehat{H}, a_n^*] = na_n^*.$$

Thus we see that if we want the spectrum of $\widehat{H}$ to be bounded below and if $\Omega \in \mathcal{H}$ is the lowest eigenvector of $\widehat{H}$ then we must have

$$a_n\Omega = 0,\ a_{-n}^*\Omega = 0$$

for $n > 0$. But in this case the space $\mathcal{D}$ generated from Ω by the action of a_n, a_n^* has to be the irreducible representation $\mathcal{F} \otimes \mathcal{F}^*$ of the Lie algebra $\mathcal{A} \oplus \mathcal{A}^*$, where $\mathcal{F}^* := \mathbb{C}[X_1^*, X_2^*, ...]$ with a_n^* acting by multiplication by X_n^* and $a_{-n}^* \mapsto n\frac{\partial}{\partial X_n^*}$ for $n > 0$.

Thus the space $\mathcal{D}$ is the tensor product of polynomial algebras $\mathbb{C}[X_j]$ and $\mathbb{C}[X_j^*]$. Each of the algebras $\mathbb{C}[X_j]$ carries a positive inner product with X_j^n being an orthogonal basis and $||X_j^n||^2 = j^n n!$, and similarly for $\mathbb{C}[X_j^*]$. This yields a positive inner product $\langle\ ,\ \rangle$ on $\mathcal{F}, \mathcal{F}^*$ and $\mathcal{D}$, with respect to which $a_i^\dagger = a_{-i}$ and $a_i^{*\dagger} = a_{-i}^*$. The Hilbert space $\mathcal{H}$ is the completion of $\mathcal{D}$ with respect to $\langle\ ,\ \rangle$.

This implies that the quantum Hamiltonian has to be given by the formula

$$\widehat{H} = \sum_{n>0}(a_{-n}a_n + a_n^*a_{-n}^*) + C$$

obtained by quantizing the classical hamiltonian (13.7) (note that the annihilation operators are written on the right to make sure the infinite sum makes sense). We may write $\widehat{H}$ as the sum of left-moving and right-moving parts:

$$\widehat{H} = \widehat{H}_L + \widehat{H}_R,$$

where

$$\widehat{H}_L := \sum_{n>0} a_{-n}a_n + \tfrac{C}{2},\ \widehat{H}_L := \sum_{n>0} a_n^*a_{-n}^* + \tfrac{C}{2}.$$

13.3. **ζ-function regularization.** At the moment it is not clear what the right value of C should be. To answer this question, recall that the

hamiltonian of a single harmonic oscillator is $z\partial_z + \frac{1}{2}$ acting on $\mathbb{C}[z]$. This suggests that the formula for $\widehat{H}$ should be

$$\widehat{H} = \sum_{n>0}(a_{-n}a_n + a_n^* a_{-n}^* + n) = \frac{1}{2}\sum_{n\neq 0}(a_{-n}a_n + a_n^* a_{-n}^*),$$

which is a more symmetric and natural formula for quantization of H. This formula, however, does not make sense, since the series

$$1 + 2 + 3 + ...$$

is divergent. We may, however, regularize it using *ζ-function regularization.*

Namely, recall that the Riemann ζ-function is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

It is well known that this function extends meromorphically to the entire complex plane with a unique (simple) pole at $s = 1$ and satisfies the functional equation, which says that the function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is symmetric under the change $s \mapsto 1 - s$:

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\zeta(s).$$

Now, it is natural to define

$$C = 1 + 2 + 3 + ... := \zeta(-1).$$

But the functional equation for $s = 2$ implies that

$$\zeta(-1) = \frac{\pi^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}\zeta(2) = -\frac{\pi^{-\frac{3}{2}}}{2\pi^{\frac{1}{2}}}\frac{\pi^2}{6} = -\frac{1}{12}.$$

So from this point of view it is natural to set

$$C := -\frac{1}{12}$$

Remark 13.1. Recall that for integer $\mathrm{g} \geq 1$

$$\zeta(2\mathrm{g}) = (-1)^{\mathrm{g}+1}2^{2\mathrm{g}-1}\frac{B_{2\mathrm{g}}}{(2\mathrm{g})!}\pi^{2\mathrm{g}}.$$

So the functional equation for ζ implies that

$$\zeta(1-2\mathrm{g}) = \pi^{\frac{1}{2}-2\mathrm{g}}\frac{\Gamma(\mathrm{g})}{\Gamma(\frac{1}{2}-\mathrm{g})}\cdot(-1)^{\mathrm{g}+1}2^{2\mathrm{g}-1}\frac{B_{2\mathrm{g}}}{(2\mathrm{g})!}\pi^{2\mathrm{g}} = -\frac{B_{2\mathrm{g}}}{2\mathrm{g}}.$$

Thus the Harer-Zagier theorem can be interpreted as the statement that the Euler characteristic of the moduli space of curves of genus g

is $\zeta(1-2\mathrm{g})$. In particular, for $\mathrm{g}=1$ we get the Euler characteristic of $SL_2(\mathbb{Z})$, which is $-\frac{1}{12}$.

13.4. **Modularity of the partition function.** The value $-1/12$ turns out indeed to be the most natural value of C. To see this, let us return to Euclidean signature $|du|^2 = dt^2 + dx^2$ and put our theory on the complex torus $E = E_\tau = \mathbb{C}^\times/q^{\mathbb{Z}} \cong \mathbb{R}/2\pi T\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, where

$$T > 0,\ \tau = iT,\ q = e^{2\pi i\tau} = e^{-2\pi T} \in (0,1).$$

In this case, as we know from quantum mechanics, we should consider the partition function

$$Z(\tau) := \mathrm{Tr}(e^{2\pi i\tau \widehat{H}}).$$

Note that

$$\widehat{H}(P \otimes Q) = (\deg P + \deg Q + C)P \otimes Q,$$

where $P \in \mathcal{F}$ and $Q \in \mathcal{F}^*$, and the degree is given by

$$\deg(X_n) = \deg(X_n^*) = n.$$

Thus we have

$$Z(\tau) = \frac{e^{2\pi i\tau(C+\frac{1}{12})}}{\eta(\tau)^2},$$

where

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind η-function. Now recall that $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$, namely,

$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau} \cdot \eta(\tau).$$

So the partition function Z has a nice modular property for a unique value of C, which is exactly $-\frac{1}{12}$.

Let us explain why we should expect $Z(\tau)$ to have a modular property.

For this, note that the Lagrangian of the theory

$$\mathcal{L}(\phi) = \frac{1}{4\pi}\int_E (d\phi)^2 = \frac{1}{4\pi}\int_E d\phi \wedge *d\phi$$

is *conformally invariant*, as it is written purely in terms of the Hodge *-operator which depends only on the conformal structure on E. The same is true for the equation of motion, which is the Laplace's equation $\Delta\phi = 0$. In other words, our classical field theory is *conformal*. Thus we could hope that the corresponding quantum theory is conformal as well. This should mean that $Z(-\frac{1}{\tau}) = Z(\tau)$, since the complex tori $E_{-\frac{1}{\tau}}$ and E_τ are conformally equivalent.

This said, we note that this modular property is only satisfied up to a linear factor in τ: in fact, we have

$$Z(-\tfrac{1}{\tau}) = -i\tau Z(\tau).$$

This is because we have killed the zero mode, which we should, in fact, have included (after all, the space cycle in the torus E_τ is not $SL_2(\mathbb{Z})$-invariant, hence neither is the condition that the integral of ϕ over this cycle vanishes). This is done in the next subsection.

13.5. **Including the zero mode.** The zero mode corresponds to the periodic solutions $\phi(t,x) = \alpha + \mu t$ of the string equation ($\alpha, \mu \in \mathbb{R}$), which for nonzero μ cannot be split into a left-moving and right-moving periodic wave. So putting back the zero mode corresponds to replacing the Hilbert space $\mathcal{H}$ with $\mathcal{H}_{\text{full}} := \mathcal{H} \otimes L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is the Hilbert space of a quantum-mechanical free massless particle, and the Hamiltonian $\widehat{H}$ by

$$\widehat{H}_{\text{full}} := \widehat{H} + \widehat{\mu}^2,$$

where $\widehat{\mu}$ is the quantum momentum operator for this quantum mechanical particle, acting on $L^2(\mathbb{R})$ by multiplication by the momentum μ. Thus we may write

$$\mathcal{H}_{\text{full}} = \int_{\mathbb{R}} \mathcal{H}_\mu d\mu,$$

with $\mathcal{H}_\mu = \mathcal{F}_\mu \otimes \mathcal{F}_\mu^*$ where $\mathcal{F}_\mu = \mathcal{F}$ but with $a_0 = \mu$ instead of $a_0 = 0$, and similarly $\mathcal{F}_\mu^* = \mathcal{F}^*$ but with $a_0^* = \mu$ instead of $a_0^* = 0$. Then we still have

$$\widehat{H} = \widehat{H}_L + \widehat{H}_R,$$

where

$$\widehat{H}_L = \tfrac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \tfrac{1}{24},\ \widehat{H}_R = \tfrac{1}{2}a_0^{*2} + \sum_{n>0} a_n^* a_{-n}^* - \tfrac{1}{24}.$$

According to Remark 8.29, the partition function of such a particle when time runs over $\mathbb{R}/L\mathbb{Z}$ is, up to scaling, $L^{-\frac{1}{2}}$. Thus the full partition function should be

$$\mathcal{Z}(\tau) = (-i\tau)^{-\frac{1}{2}} Z(\tau).$$

And then we have the genuine modular property:

$$\mathcal{Z}(-\tfrac{1}{\tau}) = \mathcal{Z}(\tau).$$

We note that the function $\mathcal{Z}(\tau)$ has a natural extension to arbitrary $\tau \in \mathbb{C}_+$ (not necessarily purely imaginary), which is just the path integral over a "non-rectangular" complex torus E_τ. To explain this, note that we have a natural action of the translation group $\mathbb{R}/2\pi\mathbb{Z}$ on our spacetime, hence we should expect its action on the Hilbert

space $\mathcal{H}$. The infinitesimal generator D of this group should satisfy the commutation relations

$$[D, a_n] = na_n,\ [D, a_n^*] = na_n^*$$

(which differs from the corresponding relations for $\widehat{H}$ by the sign in the first relation). As $D\Omega = 0$, it follows that

$$D(P \otimes Q) = (\deg P - \deg Q)P \otimes Q,$$

i.e.,

$$D = \widehat{H}_L - \widehat{H}_R.$$

Let $s \in \mathbb{R}$ and $\tau := iT + s$. Then a twisted version of the Feynman-Kac formula implies that given $s \in \mathbb{R}$, we have

$$Z(\tau) = \mathrm{Tr}(e^{-2\pi T\widehat{H}} e^{2\pi isD}) = |q|^{-\frac{1}{12}} \mathrm{Tr}(q^{\widehat{H}_L} \overline{q}^{\widehat{H}_R}),$$

where $q = e^{-2\pi(T+is)} = e^{2\pi i\tau}$. Thus we still have

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}.$$

Hence

$$\mathcal{Z}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2},$$

which is a (real analytic) modular function for $SL(2, \mathbb{Z})$, i.e., invariant under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. Thus here we have a genuine quantum conformal symmetry (as the moduli of complex tori E_τ is exactly $\mathbb{C}_+/SL_2(\mathbb{Z})$). Indeed, this function is obviously symmetric under $\tau \mapsto \tau + 1$, and we've seen that it is invariant under $\tau \mapsto -1/\tau$, but these two transformations generate $SL_2(\mathbb{Z})$.

13.6. **Correlation functions on the cylinder and torus.** We may also consider correlation functions of the quantum fields a and a^*. They are computed separately in $\mathcal{F}$ and $\mathcal{F}^*$ and can be easily found using representation theory. For example, we have $a_n a_{-n}\Omega = n\Omega$ for $n > 0$, so the 2-point function is given by

$$\langle\, \Omega, a(z)a(w)\Omega \,\rangle = \sum_{n=1}^{\infty} nz^{-n-1}w^{n-1} = \frac{1}{(z-w)^2}.$$

More precisely, the series converges only for $|w| < |z|$, but the function analytically continues to all $z \neq w$. Since our theory is free, the higher correlation functions are given by Wick's formula:

Proposition 13.2. *We have*

$$\langle\, \Omega, a(z_1)....a(z_{2k})\Omega \,\rangle = \sum_{\sigma\in\Pi_{2k}} \frac{1}{\prod_{j\in[1,2k]/\sigma}(z_j - z_{\sigma(j)})^2},$$

and the $2k+1$*-point correlation functions are zero.*

We note that since $\mathcal{F}$ is generated by Ω as an $\mathcal{A}$-module, these functions determine $a(z)$ as a local operator (=quantum field). More generally, they determine the operators $a(z_1)...a(z_r)$ when $z_i \neq z_j$, which are symmetric in $z_1, .., z_r$ due to space locality. However, these operators are not well defined (have poles) on the diagonals $z_i = z_j$.

Exercise 13.3. *Give a direct algebraic proof of Proposition 13.2.*

Exercise 13.4. *Compute the normalized 2-point correlation function of the quantum field* $\widetilde{a}(z) := za(z)$ *on the torus* $E := \mathbb{R}/2\pi T\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$ *in terms of theta functions.*

Hint. *This correlation function is given by*

$$\frac{\langle\, \widetilde{a}(z)\widetilde{a}(w)\,\rangle_E}{\langle\, \emptyset\,\rangle_E} = \mathrm{Tr}_{\mathcal{F}}(\widetilde{a}(z)\widetilde{a}(w)e^{-2\pi T\widehat{H}_L}).$$

13.7. **Infinitesimal conformal symmetry: the Virasoro algebra.** We have already pointed out that the theory of a free massless scalar in two dimensions is classically conformally invariant and saw some manifestations of the fact that this invariance survives at the quantum level (modular invariance of the partition function on the torus). However, to study conformal symmetry systematically, we need to consider *infinitesimal conformal symmetry*, given by "infinitesimal conformal mappings", i.e., holomorphic vector fields on $\mathbb{C}^\times$.

For simplicity we consider polynomial vector fields $P(z)\partial_z$ where P is a Laurent polynomial (this is sufficient since polynomial fields are dense in all holomorphic vector fields in an appropriate topology). Such vector fields form a Lie algebra called the *Witt algebra* (or *centerless Virasoro algebra* in the physics literature), and we'll denote it by W. A convenient basis of W is $\{L_n = -z^{n+1}\partial_z, n\in\mathbb{Z}\}$ which satisfies the commutation relations

$$[L_n, L_m] = (n-m)L_{m+n},\ m, n\in\mathbb{Z}.$$

The Lie algebra W acts by symmetries of the classical field theory of a free massless scalar, since its Lagrangian is conformally invariant. In fact, importantly, this action is only $\mathbb{R}$-linear and not $\mathbb{C}$-linear, which is a good thing - this means that we have an action of the complexification $W_{\mathbb{C}} = W\oplus W^*$, where W^* is the Lie algebra of antiholomorphic vector fields; in other words, we have two commuting actions of W.

If our theory is quantum-mechanically conformally invariant, then the Lie algebra $W \oplus W^*$ should act on the space $\mathcal{D}$ in a way compatible with the action of $\mathcal{A} \oplus \mathcal{A}^*$, i.e., so that

$$[L_n, a(z)] = z^{n+1}a'(z) + (n+1)z^n a(z),$$

$$[L_n^*, a^*(\overline{z})] = \overline{z}^{n+1}a^{*\prime}(\overline{z}) + (n+1)\overline{z}^n a_*(\overline{z}),$$

$$[L_n^*, a(z)] = [L_n, a^*(\overline{z})] = 0,$$

or in components

$$[L_n, a_m] = -ma_{m+n},\ [L_m^*, a_n^*] = -ma_{m+n}^*,\ [L_n, a_m^*] = [L_n^*, a_m] = 0.$$

Is there such an action? To figure this out, first note that the operators L_0, L_0^* satisfy the same commutation relations with a, a^* as $\widehat{H}_L, -\widehat{H}_R$ respectively. Since $\mathcal{D}$ is an irreducible $\mathcal{A} \oplus \mathcal{A}^*$-module, this means that by Schur's lemma we must have

$$L_0 = \widehat{H}_L + C_L,\ L_0^* = -\widehat{H}_R + C_R$$

for some constants C_L, C_R. This shows that L_n has to shift the grading in $\mathcal{F}$ by n, and similarly for L_n^* and $\mathcal{F}^*$.

Now by analogy with the formula

$$L_0 = \sum_{k \geq 1} a_{-k}a_k + \text{const},$$

define for $n \neq 0$

$$L_n := \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{-k}a_{k+n}. \tag{13.8}$$

It is easy to check that this operator on $\mathcal{F}$ (and hence on $\mathcal{D} = \mathcal{F} \otimes \mathcal{F}^*$) is well defined, and satisfies the desired commutation relations

$$[L_n, a(z)] = -z^{n+1}a'(z) + (n+1)z^n a(z),\ [L_n, a^*(z)] = 0.$$

Again using irreducibility of $\mathcal{D}$ and Schur's lemma, we see that if the desired action of W exists at all, then L_n **must** be given by formula (13.8) (note that here we can't add a constant since L_n must shift the degree). So it remains to check if the constructed operators satisfy the commutation relations of W.

First assume $n \neq -m$. In this case using the Jacobi identity, we see that the operator $[L_n, L_m] - (n-m)L_{m+n}$ commutes with a, a^*, so again by Schur's lemma it must be a constant; however, since it shifts degree, we get the desired relation

$$[L_n, L_m] - (n-m)L_{m+n} = 0.$$

So it remains to consider the case $n = -m > 0$. In this case the same argument shows that

$$[L_n, L_{-n}] - 2nL_0 = C(n),$$

where $C(n) \in \mathbb{C}$, and we have an action of W if $C(n) = 0$ for all n. So let us compute $C(n)$. To this end, note that the eigenvalue by which $[L_n, L_{-n}]$ acts on Ω is $2nC_L + C(n)$. So it suffices to compute this eigenvalue, i.e., the vector $L_n L_{-n}\Omega$.

In terms of the polynomial realization, we have

$$L_{-n}\Omega = \tfrac{1}{2} \sum_{0<j<n} X_j X_{n-j}.$$

Thus

$$L_n L_{-n}\Omega = \tfrac{1}{4} \sum_{0<j<n} j(n-j) \tfrac{\partial^2}{\partial X_j \partial X_{n-j}} \sum_{0<j<n} X_j X_{n-j} = \tfrac{1}{2} \sum_{0<j<n} j(n-j) = \frac{n^3 - n}{12}.$$

So

$$C(n) = \frac{n^3 - n}{12}.$$

Thus we see that we almost have an action of W, but not quite - no matter how we choose C_L, the cubic term in n will be present (a quantum anomaly)! Instead, we have a *projective* representation of W, which is, in fact, a representation of a *central extension* of W. Such projective actions are, in fact, common in quantum mechanics, since quantum states correspond not to actual unit vectors in the space of states, but rather to vectors up to a phase factor, on which (as well as on quantum observables) there is a genuine action of the symmetry group. Prototypical examples of this are the *Heisenberg uncertainty relation* $[\widehat{p}, \widehat{x}] = -i\hbar$, when the classical 2-dimensional group (or Lie algebra) of translations of the phase plane is replaced in quantum theory by the 3-dimensional Heisenberg group (Lie algebra), and the *phenomenon of spin*, when the classical rotational symmetry group $SO(3)$ is replaced in quantum theory by its double cover $SU(2)$.

This motivates the following definition.

Definition 13.5. The *Virasoro algebra* is the 1-dimensional central extension of the Witt algebra W with basis $L_n, n \in \mathbb{Z}$ and C (a central element) with commutation relations

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{n^3 - n}{12}\delta_{n,-m}C.$$

Thus we have a 1-dimensional central ideal $\mathbb{C}C \subset \mathrm{Vir}$ spanned by C, and $\mathrm{Vir}/\mathbb{C}C \cong W$.

So we obtain

Theorem 13.6. *The formulas*

$$L_0 = \sum_{k\geq 1} a_{-k}a_k,\ \ L_n = \tfrac{1}{2}\sum_{k\in\mathbb{Z}} a_{-k}a_{k+n}, n \neq 0$$

define an action of Vir *on* $\mathcal{F}$ *with* C *acting by* 1.

It is easy to check that the same theorem holds more generally on the space $\mathcal{F}_\mu$ where $a_0 = \mu$. The only change is that L_0 acquires an additional summand $\frac{1}{2}\mu^2$:

$$L_0 = \tfrac{1}{2}\mu^2 + \sum_{k\geq 1} a_{-k}a_k.$$

If C acts on a representation $\mathbb{V}$ of Vir by a scalar c (as it will, for instance, on every irreducible representation) then one says that $\mathbb{V}$ has *central charge* c. Thus $\mathcal{F}_\mu$ is a representation of Vir of central charge $c = 1$.

Similarly, the formulas

$$L_0^* = -\tfrac{1}{2}\mu^2 - \sum_{k\geq 1} a_k^* a_{-k}^*,\ \ L_n^* = -\tfrac{1}{2}\sum_{k\in\mathbb{Z}} a_k^* a_{-k+n}^*, n \neq 0$$

define an action of Vir on $\mathcal{F}_\mu^*$ with the central element C^* acting by -1 (i.e., of central charge $c = -1$).

Thus we obtain two commuting projective actions of W on the space $\mathcal{D} = \mathcal{F} \otimes \mathcal{F}^*$ which define usual linear actions only for the central extension Vir of W. Still, the corresponding adjoint action of W on quantum observables is a genuine linear action, so this quantum field theory is *conformal*.

We note that the Virasoro action preserves the positive Hermitian form on $\mathcal{F}_\mu$ in the sense that

$$L_n^\dagger = L_{-n}.$$

Thus $\mathcal{F}_\mu$ is a *positive energy unitary representation of* Vir (positive energy means that L_0 is diagonalizable with spectrum bounded below).

More generally, we may consider the theory of ℓ massless scalars $\phi_1, ..., \phi_\ell$. In this case $\mathcal{D} = \mathcal{F}^{\otimes\ell} \otimes \mathcal{F}^{*\otimes\ell}$, and $\mathcal{F}^{\otimes\ell}$ is a positive energy unitary Vir-module with central charge $c = \ell$ (the tensor product of ℓ copies of $\mathcal{F}$).

Exercise 13.7. *1. Show that* Vir *is a non-trivial central extension of* W *(i.e., not isomorphic to* $W \oplus \mathbb{C}$ *as a Lie algebra).*

2. Show that Vir *is a universal central extension of* W*, i.e., every non-trivial central extension of* W *by* $\mathbb{C}$ *is isomorphic to* Vir.

13.8. **Normal ordering, composite operators and operator product expansion in conformal field theory.** Let us now summarize the theory of normal ordering, composite operators and operator product expansion from Subsection 11.11 in the case of conformal field theory, for the running example of a quantum massless scalar boson. We have seen that the operator product $a(z)a(w)$ is well defined only if $w \neq z$ and has a pole when $w = z$, leading to the local operator $a(z)^2$ not being well defined. So let us expand this operator product in a Laurent series near $w = z$ and identify the singular part involving negative powers of $w - z$. For this purpose consider the difference

$$: a(z)a(w) := a(z)a(w) - \frac{1}{(z-w)^2}.$$

The formula for the correlation functions for $a(z)$ implies that

$$\langle \Omega, a(z_1)...a(z_{i-1}) : a(z_i)a(z_{i+1}) : a(z_{i+2})...a(z_n)\Omega \rangle =$$

$$\sum_{\sigma \in \Pi_{2k}: \sigma(i) \neq i+1} \frac{1}{\prod_{j \in \Pi_{2k}/\sigma} (z_j - z_{\sigma(j)})^2}.$$

Note that this function is regular at $z_i = z_{i+1}$, hence the operator $: a(z)a(w) :$ is regular at $z = w$, i.e., defined for all $z, w \in \mathbb{C}^\times$. This operator is called the *normally ordered product* of $a(z)$ and $a(w)$. In particular, although the square $a(z)^2$ is not defined, we have a well defined normally ordered square $: a(z)^2 :$.

In terms of Laurent coefficients,

$$: a(z)a(w) := \sum_{m,n \in \mathbb{Z}} : a_n a_m : z^{-n-1} w^{-m-1},$$

where $: a_n a_m := a_n a_m$ if $m \geq n$ and $: a_n a_m := a_m a_n$ if $m < n$ (normal ordering of modes). Of course, this ordering only matters if $m + n = 0$. In particular, we see that

$$\tfrac{1}{2} : a(z)^2 := T(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

the generating function of the Virasoro modes L_n. This operator is called the *(quantum) energy-momentum tensor*.

Thus we see that the Virasoro modes L_n may be viewed as Noether charges for the corresponding infinitesimal conformal symmetries, in the holomorphic sector of the theory. The corresponding Noether currents are $z^{n+1}T(z)$, as

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) dz.$$

The Noether charges for the full theory are then $L_n + \overline{L}_n$, with currents $z^{n+1}T(z) + \overline{z^{n+1}T(z)}$. In particular, the Hamiltonian H, up to adding a constant, is $L_0 + \overline{L_0}$, which agrees with formula (11.5).

Similarly, we may define the normal ordered products of more than two factors, $: a(z_1)....a(z_n) :$. This can be done by induction in n. Namely, we have

(13.9)

$$: a(z_0)a(z_1)...a(z_n) := a(z_0) : a(z_1)...a(z_n) : - \sum_{k\in[1,n]} \frac{: \prod_{j\neq k} a(z_j) :}{(z_0 - z_k)^2}$$

It is easy to see that the operator $: a(z_1)....a(z_n) :$ has no singularities and is well defined for all values $z_1, ..., z_n \in \mathbb{C}^\times$. Thus for every $r_1, .., r_n$ we have the operator

$$: a^{(r_1)}(z_1)...a^{(r_n)}(z_n) := \partial_{z_1}^{r_1}...\partial_{z_n}^{r_n} : a(z_1)....a(z_n) :$$

Setting $z_1 = ... = z_n$, we can then define the local operator $: P(a)(z) :$ for any differential polynomial P in $a(z)$. This local operator, called a *composite operator*, is a quantization of the corresponding local functional $P(a)(z)$ in classical field theory.

Exercise 13.8. *(The state-operator correspondence) Show that the map $P \mapsto P(a)(z)\Omega|_{z=0}$ is well defined and gives an isomorphism between the space $\mathcal{V}$ of (polynomial) local operators and the Fock space $\mathcal{F}$.*

More generally, repeatedly using (13.9), we have

$$: a(z_1)...a(z_n) : \cdot : a(w_1)...a(w_m) := \sum_{I\subset[1,n], J\subset[1,m], s: I\cong J} \frac{: \prod_{i\notin I} a(z_i) \prod_{j\notin J} a(w_j) :}{\prod_{i\in I}(z_i - w_{s(i)})^2}.$$

So setting $z_i = z, w_j = w$, we obtain

$$: a(z)^n :: a(w)^m := \sum_{k=0}^{\min(m,n)} k! \binom{n}{k}\binom{m}{k} \frac{: a(z)^{n-k}a(w)^{m-k} :}{(z-w)^{2k}}.$$

E.g. for $n = m = 1$ we get the familiar identity

$$a(z)a(w) = \frac{1}{(z-w)^2} + : a(z)a(w) := \frac{1}{(z-w)^2} + \text{regular terms.}$$

More generally, for $n = 1$ and any m we get

$$a(z) : a(w)^m := \frac{m : a^{m-1}(w) :}{(z-w)^2} + : a(z)a(w)^m :$$

$$= \frac{m : a^{m-1}(w) :}{(z-w)^2} + \text{regular terms.}$$

For $m = 2$ this can be written as

$$a(z)T(w) = \frac{a(w)}{(z-w)^2} + \text{regular terms},$$

which encodes the commutation relations between a_i and L_j.

For $n = 2$, $m = 2$ we get

$$:a(z)^2::a(w)^2:= \frac{2}{(z-w)^4} + \frac{4:a(z)a(w):}{(z-w)^2} + :a(z)^2a(w)^2:=$$

$$\frac{2}{(z-w)^4} + \frac{4:a(w)^2:}{(z-w)^2} + \frac{4:a(w)a'(w)}{z-w} + \text{regular terms}.$$

This can also be written as

$$T(z)T(w) = \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms},$$

which encodes the commutation relations between L_i. More generally, at central charge c this relation would look like

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms}.$$

These are the simplest examples of the *operator product expansion*. In fact, we have the following theorem, whose proof we will leave to the reader:

Theorem 13.9. *For any local operators $P, Q \in \mathcal{V}$, there exist a unique finite sequence of local operators $R_1, ..., R_N \in \mathcal{V}$ such that*

$$P(a)(z)Q(a)(w) = \sum_{j=1}^{N} R_j(a)(w)(z-w)^{-j} + \text{regular terms},$$

where $(z-w)^{-j} := \sum_{k\geq 0} \binom{k+j-1}{j-1} z^{-j-k}w^k$.

Note that the space locality property implies that $Q(a)(w)P(a)(z)$ is given by the same formula, but with $(z-w)^{-j}$ expanded in the opposite direction, i.e., $(z-w)^{-j} := -\sum_{k<0} \binom{k+j-1}{j-1} z^{-j-k}w^k$. Thus. we have

$$[P(a)(z), Q(a)(w)] = \sum_{j=1}^{N} \frac{1}{(j-1)!} R_j(a)(w)\delta^{(j-1)}(w-z).$$

Thus Theorem 13.9 gives us information about commutators between the modes of P and Q. For example, as we have seen above,

$$[a(z), a(w)] = \delta'(w-z),$$

and also

$$[a(z), T(w)] = a(w)\delta'(w-z),$$

$$[T(z),T(w)] = \frac{c}{12}\delta'''(w-z) + 2T(w)\delta'(w-z) + T'(w)\delta(w-z),$$

where in our example $c = 1$.

Moreover, it is clear that one can uniquely continue the expansion of Theorem 13.9 to also include terms of nonnegative degree; namely, we simply need to expand the regular terms into a Taylor series with respect to $z - w$ for fixed w. For example, we have an asymptotic expansion

$$a(z)a(w) \sim \frac{1}{(z-w)^2} + \sum_{k=0}^{\infty} :a^{(k)}(w)a(w): \frac{(z-w)^k}{k!}$$

So in general we have

$$P(a)(z)Q(a)(w) \sim \sum_{j=-\infty}^{N} R_j(a)(w)(z-w)^{-j}.$$

This formula is called the *operator product expansion* of the product of P and Q. The operator product expansion satisfies certain axioms, which means that it defines on the space $\mathcal{V} \cong \mathcal{F}$ an algebraic structure called a *vertex algebra* (which we will not discuss here, however).

13.9. **Vertex operators.** *Vertex operators* are obtained by quantizing the local functional $e^{i\lambda\varphi(z)}$, where

$$\varphi(z) = -i\int a(z)dz = -i(a_0 \log z + \sum_{n\neq 0} \tfrac{a_{-n}}{n} z^n + a_0^\vee)$$

and $a_0^\vee$ is a constant of integration (dual variable to a_0). In other words, we have

$$e^{i\lambda\varphi(z)} = e^{\lambda\int a(z)dz} = e^{\lambda(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} e^{\lambda a_0^\vee}.$$

A natural quantization of this functional is the operator

$$X(\lambda,z) :=: e^{\lambda(a_0\log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} : e^{\lambda a_0^\vee} =$$

$$= e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}} z^{\lambda\mu} e^{\lambda\partial_\mu},$$

which, due to the last factor, acts from $\mathcal{F}_\mu$ to $\mathcal{F}_{\mu+\lambda}$ by $X_0(\lambda,z)z^{\lambda\mu}$, where

$$X_0(\lambda,z) := e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}}.$$

Here we work over the group algebra of $\mathbb{C}$ with basis $z^\alpha, \alpha \in \mathbb{C}$.

Now note that if $[A,B]$ commutes with A, B then by the Campbell-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]},$$

and that

$$[\sum_{n>0} \frac{a_n}{n} z^{-n}, \sum_{n>0} \frac{a_{-n}}{n} w^n] = \sum_{n>0} \frac{z^{-n} w^n}{n} = -\log(1 - \tfrac{w}{z}).$$

Thus

$$X_0(\lambda, z) X_0(\nu, w) = (1 - \tfrac{w}{z})^{\lambda\nu} : X_0(\lambda, z) X_0(\nu, w) :$$

for $|w| < |z|$. So we get

$$X(\lambda, z) X(\nu, w) = (z - w)^{\lambda\nu} : X(\lambda, z) X(\nu, w) :$$

for $|w| < |z|$, where the normal ordering puts ∂_μ to the right of μ. More generally, we see that

$$X(\lambda_1, z_1)...X(\lambda_n, z_n) = \prod_{1 \le j < k \le n} (z_j - z_k)^{\lambda_j \lambda_k} : X(\lambda_1, z_1)...X(\lambda_n, z_n) :$$

for $|z_1| > ... > |z_n|$. In particular, denoting the highest weight vector of $\mathcal{F}_\mu$ by Ω_μ, we have

$$\langle\, \Omega_{\mu+\lambda}, X(\lambda_1, z_1)...X(\lambda_n, z_n)\Omega_\mu \,\rangle = \prod_{j=1}^{n} z_j^{\lambda_j \mu} \prod_{1 \le j < k \le n} (z_j - z_k)^{\lambda_j \lambda_k}.$$

for $|z_1| > ... > |z_n|$.

We see that this correlation function admits analytic continuation to the complement of the diagonals $z_i \neq z_j$, but this continuation is not, in general, single valued. In other words, the fields $X(\lambda, z)$ in general do not satisfy space locality. Instead, we have

$$X(\lambda, z) X(\nu, w) = e^{\pi i \lambda\nu} X(\nu, w) X(\lambda, z), \tag{13.10}$$

which is understood in the sense of analytic continuation along a path where $v := w/z$ passes from the region $|v| < 1$ to the region $|v| > 1$ along positive reals, avoiding the point $v = 1$ from above. In particular,

$$X(\lambda, z) X(\lambda, w) = e^{\pi i \lambda^2} X(\lambda, w) X(\lambda, z),$$

i.e., $X(\lambda, z)$ has "statistics $\lambda^2/2$" (where statistics $\alpha \in \mathbb{R}/\mathbb{Z}$ means that switching the order produces a phase factor $e^{2\pi i \alpha}$; e.g. statistics 0 corresponds to bosons and statistics 1/2 to fermions).

Note that if we apply commutation relation (13.10) twice, we obtain a multiplier $e^{2\pi i \lambda\nu}$, which corresponds to the fact that the operator product $X(\lambda, z) X(\nu, w)$ is multivalued in general.

This is an example of appearance of a *braiding* in conformal field theory. Namely, relation (13.10) is called *braided space-locality* (or *braided commutativity*), since it can be viewed as commutativity in a suitable braided monoidal category.

Note also that

$$X'(\lambda, z) = \lambda : a(z)X(\lambda, z) :$$

where $X' := \partial_z X$, and

$$[a_n, X(\lambda, z)] = \lambda z^n X(\lambda, z).$$

Hence

$$[L_n, X(\lambda, z)] = z^{n+1}X'(\lambda, z) + \frac{\lambda^2}{2}(n+1)z^n X(\lambda, z),$$

which implies that $X(\lambda, z)$ has spin $\lambda^2/2$. Thus we have the spin-statistics property for $X(\lambda, z)$, which generalizes the usual one: spin modulo $\mathbb{Z}$ equals statistics.

As noted in Remark 11.2, such quantum fields are called "anyons" (as they can have any spin and statistics) and can exist only in two dimensions. The most general spin-statistics property for these anyons says that if X, Y are anyons of spins $s_X, s_Y \geq 0$ then

$$X(z)Y(w) = e^{2\pi i\sqrt{s_X s_Y}}Y(w)X(z).$$

In particular, we see that if $\lambda^2 \in \mathbb{Z}$ is odd then $X(n\lambda, z)$ behave like fermions for odd n and like bosons for even n with respect to each other (i.e., the corresponding operators $X(n_1\lambda, z)$ and $X(n_2\lambda, z)$ commute if $n_1 n_2$ is even and anticommute if $n_1 n_2$ is odd), while for even λ^2 they all behave like bosons (i.e., the operators commute).

13.10. **The circle-valued theory.** Now consider the theory of a massless scalar on $\mathbb{C}^\times$ with values in the circle $\mathbb{R}/2\pi r\mathbb{Z}$. This theory is the same as the line-valued one, except for the zero mode, which entails the following circle-valued solutions of the string equation:

$$\phi(t, x) = \alpha + \mu t + Nrx,$$

where $\alpha \in \mathbb{R}/2\pi r\mathbb{Z}$, $\mu \in \mathbb{R}$, and N is an integer (the winding number). The space of such solutions is a disjoint union of cylinders T^*S^1 labeled by values of N. So in quantum theory we get the Hilbert space

$$\mathcal{H}_r^\circ = \bigoplus_{N,\ell\in\mathbb{Z}} \mathcal{H}_r^\circ(N, \ell),$$

where $\mathcal{H}_r^\circ(N, \ell)$ is the completion of $\mathcal{F}_{\frac{1}{\sqrt{2}}(\ell r^{-1}+Nr)} \otimes \mathcal{F}^*_{\frac{1}{\sqrt{2}}(\ell r^{-1}-Nr)}$. Thus we obtain the following formula for the partition function on the torus E_τ:

$$\mathcal{Z}_r^\circ(\tau) = |\eta(\tau)|^{-2}\vartheta_r(\tau, \overline{\tau}),$$

where

$$\vartheta_r(\tau, \overline{\tau}) := \sum_{\ell,N\in\mathbb{Z}} e^{\frac{1}{2}\pi i\tau(\ell r^{-1}+Nr)^2 - \frac{1}{2}\pi i\overline{\tau}(\ell r^{-1}-Nr)^2} =$$

$$\sum_{\ell,N\in\mathbb{Z}} e^{-\pi(\ell^2 r^{-2}+N^2 r^2)\mathrm{Im}\tau+2\pi i\ell N\mathrm{Re}\tau}.$$

This shows an interesting duality $\mathcal{Z}_r^\circ(\tau)=\mathcal{Z}_{r^{-1}}^\circ(\tau)$; in fact, we see that the whole theory with parameter r is equivalent to the one with parameter r^{-1}. This duality is called *T-duality*, and it plays an important role in string theory.

Also we note that ϑ_r is a real modular form of weight 1:

$$\vartheta_r(-\tfrac{1}{\tau},-\tfrac{1}{\overline{\tau}})=|\tau|\vartheta_r(\tau,\overline{\tau}),$$

which leads to modular invariance of the function $\mathcal{Z}_r(\tau)$, as expected in a conformal field theory. To see this, it is enough to note that in the exponential we have a quadratic form on $\mathbb{Z}^2$ with matrix

$$Q(\tau)=\begin{pmatrix} r^2\mathrm{Im}\tau & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & r^{-2}\mathrm{Im}\tau \end{pmatrix}$$

So

$$Q(\tau)^{-1}=|\tau|^{-2}\begin{pmatrix} r^{-2}\mathrm{Im}\tau & i\mathrm{Re}\tau \\ i\mathrm{Re}\tau & r^2\mathrm{Im}\tau \end{pmatrix}=\begin{pmatrix} r^{-2}\mathrm{Im}\tau' & -i\mathrm{Re}\tau' \\ -i\mathrm{Re}\tau' & r^2\mathrm{Im}\tau' \end{pmatrix}=SQ(\tau')S,$$

where $\tau':=-\frac{1}{\tau}$ and $S=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus the result follows from the Poisson summation formula.

We see that if $r^2=\frac{p}{q}\in\mathbb{Q}$ (in lowest terms), this conformal field theory has a special property called *rationality*: the Hilbert space $\mathcal{H}_r^\circ$ is the completion of a finite sum of "sectors" $\oplus_{i=1}^n \mathcal{V}_i\otimes\mathcal{V}_i^*$, where the left-moving fields act on $\mathcal{V}_i$ and right-moving ones in $\mathcal{V}_i^*$, so that $\vartheta_r(\tau,\overline{\tau})$ and hence $\mathcal{Z}_r^\circ(\tau)$ are finite sums of products of a holomorphic and an antiholomorphic function (in fact, it is easy to see that $n=2pq$). For example, the vacuum vector Ω is contained in the tensor product $\mathcal{V}(pq)\otimes\mathcal{V}(pq)^*$ where for $s\in\mathbb{Z}_{>0}$ we defined $\mathcal{V}(s):=\oplus_{m\in\mathbb{Z}}\mathcal{F}_{m\sqrt{2s}}$. The space $\mathcal{V}(s)$ is a vertex algebra called the *lattice vertex algebra* attached to the even lattice $\sqrt{2s}\mathbb{Z}$. This algebra is generated by the vertex operators $X(m\sqrt{2s},z)$ (which, as we know, satisfy the bosonic version of space locality).

Example 13.10. Consider the case $r=1$. In this case we have two sectors, the vacuum sector $\mathcal{V}(2)\otimes\mathcal{V}(2)^*$ and another one, $\mathcal{W}\otimes\mathcal{W}^*$, where $\mathcal{W}=\oplus_{n\in 2\mathbb{Z}+1}\mathcal{F}_{\frac{n}{\sqrt{2}}}$. The particles corresponding to $\mathcal{F}_{\frac{n}{\sqrt{2}}}$ for odd n are anyons with statistics $\frac{1}{4}$, so they satisfy the braided commutativity relation of the form $X(z)Y(w)=iY(w)X(z)$.

It is not difficult to show that the Fourier modes of the vertex operators $X(\sqrt{2},z)$ and $X(-\sqrt{2},z)$ generate a projective action of the Lie

algebra $\mathfrak{sl}_2[z,z^{-1}]$ on $\mathcal{V}(2)=\mathbb{L}_0$ and on $\mathcal{W}=\mathbb{L}_1$, which are exactly the irreducible integrable representations of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2=\mathfrak{sl}_2[t,t^{-1}]\oplus\mathbb{C}K$ (the universal central extension of $\mathfrak{sl}_2[t,t^{-1}]$ at level $k=1$ (i.e., K acts by 1), namely $X(\sqrt{2},z)$, $X(-\sqrt{2},z)$, $\sqrt{2}a(z)$ give the currents $e(z)$, $f(z)$ and $h(z)$, where for $b\in\mathfrak{sl}_2$

$$b(z):=\sum_n (b\otimes t^n)z^{-n-1}.$$

This is the so called *Frenkel-Kac vertex operator construction* of level 1 irreducible integrable modules (defined for any finite dimensional simply-laced simple Lie algebra) in the simplest special case $\mathfrak{g}=\mathfrak{sl}_2$. Thus the circle-valued theory of a free boson for $r=1$ is the so-called *Wess-Zumino-Witten model* in the simplest example of the Lie algebra $\mathfrak{sl}_2$ and level 1.

Example 13.11. Let $r=\sqrt{2}$. In this case we have four sectors: $\mathcal{V}_j\otimes\mathcal{V}_{-j}^*$, $j=0,1,2,3$, where $\mathcal{V}_j=\oplus_{n\in 4\mathbb{Z}+j}\mathcal{F}_{\frac{n}{2}}$. In particular, $\mathcal{V}_0=\mathcal{V}(4)$ and particles in $\mathcal{V}_2=\mathcal{F}_1$ are fermions arising in the boson-fermion correspondence.

13.11. **Free massless fermions.** In a similar way to free massless bosons, one can describe the theory of a free massless fermion $\xi(z)$. As explained in Subsection 11.4, in two dimensions it makes sense to consider chiral spinors taking values in the tautological representation of $\mathrm{Spin}(2)=U(1)$ with kinetic term $(\xi,\mathbf{D}\xi)$. So we have a single quantum field

$$\xi(z)=\sum_{n\in\mathbb{Z}+\frac{1}{2}}\xi_n z^{-n-\frac{1}{2}}$$

and the conjugate quantum field $\xi_*(\overline{z})$. The modes of $\xi(z)$ satisfy the relation

$$[\xi(z),\xi(w)]_+=\delta(z-w),$$

where $[,]_+$ is the supercommutator. This yields the Clifford algebra relations

$$\xi_n\xi_m+\xi_m\xi_n=\delta_{m,-n}$$

for $m,n\in\mathbb{Z}$. This algebra has a unique irreducible positive energy representation $\Lambda=\wedge(\xi_{-1/2},\xi_{-3/2},...)$ on which ξ_j acts by multiplications for $j<0$ and by differentiations for $j>0$. There is an invariant positive Hermitian inner product on Λ in which the Clifford monomials in ξ_j, $j>0$ form an orthonormal basis (invariance means that $\xi_j^\dagger=\xi_{-j}$). Thus the Hilbert space of the theory is the completion of $\mathcal{D}:=\Lambda\otimes\Lambda^*$, where Λ^* is the dual of Λ corresponding to antiholomorphic fields.

The hamiltonian H is supposed to satisfy commutation relations

$$[H, \xi_n] = -\xi_n, \ [H, \xi_n^*] = \xi_n^*,$$

So we have

$$H = H_L + H_R,$$

where

$$H_L = \sum_{n>0} n\xi_{-n}\xi_n$$

and similarly for H_R. The Virasoro algebra is defined by

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}+\frac{1}{2}} n : \xi_n \xi_{-n+m} :,$$

i.e., $H_L = L_0$.

Exercise 13.12. *Show that these operators L_n satisfy the Virasoro commutation relations with central charge $c = \frac{1}{2}$.*

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References

[A] V. Arnold, Mathematical methods of classical mechanics, Graduate Texts in Mathematics, Springer, 1989.

[Co] S. Coleman, Lectures on quantum field theory, `https://arxiv.org/pdf/1110.5013`

[BIPZ] E. Brézin, C. Itzykson, G. Parisi, J.-B. Zuber, Planar diagrams, Comm. Math. Phys., v. 59, p.35–51, 1978.

[De] P. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach, Courant lecture notes, v. 3, AMS, 2000.

[QFS] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, Quantum fields and strings: a course for mathematicians, v. 1 and 2, AMS, Providence, 1999.

[HZ] J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, Inv. Math.,v.85, p.457– 485.

[IZ] C. Itzykson, J.-B. Zuber, Quantum Field theory, Dover, 2006.

[PS] M. Peskin, D. Schroeder, An introduction to quantum field theory, Taylor and Francis, 1995.

[W] S. Weinberg, Quantum theory of fields, volumes I-III, Cambridge U. Press, 2005.

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