1. Generalities on quantum field theory

1.1. **Classical mechanics.** In classical mechanics, we study the motion of a particle (say, of mass 1) in a Euclidean space V . This motion is described by a function of one variable, $q = q(t) \in V$, representing the position of the particle at a time t. This function must satisfy the *Newton equation of motion*,

$$\ddot{q} = -U'(q),$$

where U is the potential energy.

Another way to express this law of motion is to say that $q(t)$ must be a solution of a certain variational problem. Namely, one introduces the *Lagrangian*

$$\mathcal{L}(q) := \frac{\dot{q}^2}{2} - U(q)$$

(the difference of kinetic and potential energy), and the *action* functional

$$S(q) := \int_a^b \mathcal{L}(q)dt$$

(for some fixed $a < b$). Then the law of motion can be expressed as the *least action principle*: $q(t)$ must be a critical point of S on the space of all functions with given $q(a)$ and $q(b)$, i.e., the Newton equation is the Euler-Lagrange equation for a solution of the variational problem defined by S. Indeed, using integration by parts, for $\varepsilon \in C^1[a, b]$ with $\varepsilon(a) = \varepsilon(b) = 0$ we have

$$\tfrac{d}{ds}|_{s=0} \int_a^b \mathcal{L}(q + s\varepsilon)dt = \int_a^b (\tfrac{\partial \mathcal{L}}{\partial q}\varepsilon + \tfrac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon})dt =$$

$$\int_a^b (-U'(q)\varepsilon + \dot{q}\dot{\varepsilon})dt = -\int_a^b (U'(q) + \ddot{q})\varepsilon dt,$$

and this vanishes for all ε iff q satisfies the Newton equation $\ddot{q} = -U'(q)$.

Remark 1.1. The name "least action principle" comes from the fact that in some cases (for example when $U'' \leq 0$) the action is not only extremized but also minimized at the solution $q(t)$. In general, however, this is not the case, and the trajectory of the particle may be not a (local) minimum, but only a critical point of the action. Therefore, the law of motion is better formulated as the "extremal (or stationary) action principle"; this is the way we will think of it in the future.

Exercise 1.2. *(i) Consider the motion of a particle in a Euclidean space V. Show that if the potential is concave ($U''(q) \leq 0$) then for any $\mathbf{a}, \mathbf{b} \in V$ and $a < b \in \mathbb{R}$ there exists at most one solution of the*

Newton equation with $q(a) = \mathbf{a}$ and $q(b) = \mathbf{b}$, and it is the strict global minimum for the action with these boundary conditions (if exists).

(ii) Show that the conclusion of (i) holds if $U''(q) < \frac{\pi^2}{(b-a)^2}$ (prove and use Wirtinger's inequality: if $\varepsilon \in C^1[a,b]$ and $\varepsilon(a) = \varepsilon(b) = 0$ then $\int_a^b \varepsilon'(t)^2 dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b \varepsilon(t)^2 dt$).

(iii) Compute the unique solution in (i) if $U(q) = -\frac{1}{2}B(q,q)$, where B is a nonnegative definite symmetric bilinear form on V.

(iv) Show that the statements of (i) fail for $\dim V = 1$, $U(q) = \frac{1}{2}q^2$ and $b - a \geq \pi$.

(v) Let $\dim V = 1$ and U be a smooth potential on $\mathbb{R}$. Suppose that $\limsup_{|x|\to\infty} \frac{U(x)}{x^2} \leq 0$. Show that a solution in (i) (possibly non-unique) exists for any $a, b, \mathbf{a}, \mathbf{b}$. Give an example of a smooth potential U for which a solution in (i) does not always exist.[1]

Remark 1.3. Physicists often consider solutions of Newton's equation on the whole line rather than on a fixed interval $[a,b]$. In this case, the naive definition of an extremal does not make sense, since the action integral $S(q) = \int_{\mathbb{R}} \mathcal{L}(q)dt$ is improper and in general diverges. Instead, one makes the following "corrected" definition: a function $q(t)$ on $\mathbb{R}$ is an extremal of S if the expression

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} \mathcal{L}(q + s\varepsilon)dt := \int_{\mathbb{R}} (\frac{\partial \mathcal{L}}{\partial q}\varepsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\varepsilon})dt,$$

where $\varepsilon(t)$ is any compactly supported perturbation, is identically zero. With this definition, the extremals are exactly the solutions of Newton's equation (which, as before, is easily seen by integration by parts).

Remark 1.4. Note that this formalism also describes the motion of a system of n particles, if we combine the vectors representing their positions in a Euclidean space V into a single vector in V^n. More generally, we may consider a particle moving on a Riemannian manifold M. In this case $q(t)$ is a path on M, and the motion is described by the same equation, where $\ddot{q}$ means the covariant derivative $\nabla_{\dot{q}}\dot{q}$ of $\dot{q}$ with respect to the Levi-Civita connection. For example, if $U = 0$, this is the *geodesic flow*, whose trajectories are the geodesics on M. The same applies to a system of n particles on M, in which case $q(t)$ is a path on the configuration space M^n. Finally, a similar analysis applies to more general Lagrangians, which are arbitrary smooth functions of (finitely many) derivatives of q.

[1] One can show using calculus of variations that for any $\dim V$, if $U(q) \leq 0$ for all q then the solution always exists.

1.2. **Classical field theory.** In classical field theory, the situation is similar, but with infinitely many particles. Namely, in this case we should think not of a single particle or a finite system of particles, but rather of a "continuum of particles" (e.g. a string, a membrane, a jet of fluid); so in a $d+1$-dimensional classical field theory the motion is described by a *classical field* – a (vector-valued) function $\phi(x,t)$ depending on both space and time coordinates ($x \in \mathbb{R}^d$, $t \in \mathbb{R}$). Consequently, the equation of motion is a partial differential equation. For example, for a string or a membrane the equation of motion is the *wave equation* $\Box\phi = 0$, where $\Box$ is the *D'Alembertian* $\partial_t^2 - v^2\Delta$ (here Δ is the Laplacian with respect to the space coordinates, and v the velocity of wave propagation, e.g. for the string v^2 is proportional to the string tension).

As in classical mechanics, in classical field theory there is a Lagrangian $\mathcal{L}(\phi)$ (a smooth function of finitely many partial derivatives of ϕ), whose integral

$$S(\phi) = \int_D \mathcal{L}(\phi) dx dt$$

over a compact region D in the *spacetime* $\mathbb{R}^{d+1}$ is called the *action*. The law of motion can be expressed as the condition that the action must be extremized over any such region D with fixed boundary conditions; so the equations of motion (also called the *field equations*) are the Euler-Lagrange equations for this variational problem. For example, in the case of string or membrane, the Lagrangian is

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\phi_t^2 - v^2(\nabla\phi)^2).$$

Remark 1.5. Like in mechanics, in field theory solutions of the equations of motion on the whole space-time (rather than a compact region D) are extremals of the action in the sense that

$$\tfrac{d}{ds}|_{s=0} \int_{\mathbb{R}^{d+1}} \mathcal{L}(u + s\varepsilon) dx dt = 0,$$

where ε is a compactly supported perturbation.

1.3. **Brownian motion.** One of the main differences between classical and quantum mechanics is, roughly speaking, that quantum particles do not have to obey the classical equations of motion, but can randomly deviate from their classical trajectories. Therefore, given the position and velocity of the particle at a given time, we cannot determine its position at a later time, but can only determine the density of probability that at this later time the particle will be found at a given point.

In this sense quantum particles are similar to random (Brownian) particles. Brownian particles are a bit easier to understand conceptually, so let us begin with them.

The motion of a Brownian particle in $\mathbb{R}^k$ in a potential field

$$U : \mathbb{R}^k \to \mathbb{R}$$

is described by a stochastic process $q = q(t)$, $q = (q_1, \dots, q_k) \in \mathbb{R}^k$. That is, for each real t we have a random variable $q(t) \in \mathbb{R}^k$ (the position of the particle at a time t), such that the dependence of t is regular in some sense. Namely, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$ the random dynamics of the particle conditioned to have $q(a) = \mathbf{a}$, $q(b) = \mathbf{b}$ is "defined" as follows:[2] if $y : [a, b] \to \mathbb{R}^k$ is a continuously differentiable function with $y(a) = \mathbf{a}, y(b) = \mathbf{b}$, then the density of probability that $q(t) = y(t)$ for $t \in [a, b]$ is proportional to $e^{-S(y)/\kappa}$, where

$$S(y) := \int_a^b (\tfrac{1}{2} y'^2 + U(y)) dt$$

is the action and $\kappa > 0$ is the diffusion coefficient. Thus, the likeliest $q(t)$ is the one that minimizes S (in particular, solves the classical equations of motion $\ddot{q} = U'(q)$), while the likelihood of the other paths decays exponentially with the deviation of the action of these paths from the minimal possible.

Remark 1.6. 1. This discussion thus assumes that the extremum of S at q is actually a minimum, which we know is not always the case, but is so when U is convex, i.e., $U''(q) \geq 0$ for all q (see Exercise 1.2).

2. The reader must have noticed that compared to the discussion of classical mechanics, the sign in front of the potential U has been changed to the opposite one. This is not a misprint! It has to do with the fundamental fact discussed below that statistical mechanics is related to usual (quantum) mechanics by the *Wick rotation* $t \mapsto it$, where $i = \sqrt{-1}$. In particular, this means that Brownian motion is well defined in the physically important case of convex potential, such as the multidimensional *harmonic oscillator* potential $\frac{1}{2} B(q, q)$ where B is a positive definite bilinear form.

All the information we can hope to get about the stochastic process $q(t)$ is contained in the *correlation functions*

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle,$$

[2]We put the word "defined" in quotation marks because this definition is obviously heuristic and not rigorous; see below for more explanations.

which by definition are the expectation values of the products of random variables $q_{j_1}(t_1), \dots, q_{j_n}(t_n)$, (more specifically, by Kolmogorov's theorem the stochastic process $q(t)$ is completely determined by these functions). So such functions should be regarded as the output, or answer, of the theory of the Brownian particle.

Thus the main question is how to compute the correlation functions. Physicists write down the following "answer" motivated by the above definition: given points $t_1, \dots, t_n \in [a, b]$,

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq, \tag{1.1}$$

where integration is carried out over the space $P_{\mathbf{a},\mathbf{b}}$ of paths

$$q : [a, b] \to \mathbb{R}^n, \ q(a) = \mathbf{a}, \ q(b) = \mathbf{b},$$

and Dq is a Lebesgue measure on this space such that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}} Dq = 1.$$

Alternatively, when they do not want to normalize the Lebesgue measure, they write

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \frac{1}{Z} \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-\frac{S(q)}{\kappa}} Dq, \tag{1.2}$$

where

$$Z := \int_{P_{\mathbf{a},\mathbf{b}}} e^{-\frac{S(q)}{\kappa}} Dq$$

is the *partition function*. Such an integral is called a *path integral*, since it is an integral over the space of paths.

It is clear, however, that such definition and answer are a priori not satisfactory from the mathematical viewpoint, since the infinite dimensional integration requires justification. In the case of Brownian motion, such a justification is actually possible within the framework of the Lebesgue measure theory, and the corresponding integration theory is called the theory of *Wiener integral*. (To be more precise, one cannot define the measure Dq, but one can define the measure $e^{-\frac{S(q)}{\kappa}} Dq$ for sufficiently nice potentials $U(q)$).

Remark 1.7. As $\kappa \to 0$, the non-optimal trajectories become increasingly less likely relatively to the optimal one, so in the limit we recover the deterministic system:

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n),$$

where $\mathbf{q}(t)$ is the classical trajectory with $\mathbf{q}(a) = \mathbf{a}, \mathbf{q}(b) = \mathbf{b}$ (note that if $U \geq 0$ then this trajectory is unique by Exercise 1.2).

1.4. **Quantum mechanics.** Now let us turn to a quantum particle. Quantum mechanics is notoriously difficult to visualize, and the randomness of the behavior of a quantum particle is less intuitive and more subtle than that of a Brownian particle; nevertheless, it was pointed out by Feynman that the behavior of a quantum particle in a potential field $U(q)$ is correctly described by the same model, with the real positive parameter κ replaced by the imaginary number $-i\hbar$ where $\hbar > 0$ is the *Planck constant*, and the time t is replaced by it. In other words, the dynamics of a quantum particle can be expressed (we will discuss later how) via the *correlation functions*

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle = \int_{P_{\mathbf{a},\mathbf{b}}} q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{\frac{iS(q)}{\hbar}} Dq, \tag{1.3}$$

where Dq is normalized so that

$$\int_{P_{\mathbf{a},\mathbf{b}}} e^{\frac{iS(q)}{\hbar}} Dq = 1, \tag{1.4}$$

and $S(q)$ is now given by the same formula as in classical mechanics (and differing by sign from Brownian motion):

$$S(q) = \int_a^b (\tfrac{\dot{q}^2}{2} - U(q)) dt.$$

As before, we have to make sense of this path integral, and now the theory of Wiener integrals unfortunately does not work any more: for instance, the absolute value of the integrand in (1.4) does not decay as the path $q(t)$ deviates from the classical trajectory (in fact, it identically equals to 1!). So we will be able to make sense of (1.3) only partially, and an effective mathematically rigorous approach to quantum mechanics is, in fact, based on different techniques (Hamiltonian formalism); this is discussed in more detail below. Still, formula (1.3) is extremely helpful for motivational purposes and with appropriate care can be used for computation.

Remark 1.8. Similarly to Brownian motion (cf. Remark 1.7), in the limit $\hbar \to 0$ we are supposed to recover the classical system:

$$\langle q_{j_1}(t_1) \dots q_{j_n}(t_n) \rangle \to \mathbf{q}_{j_1}(t_1) \dots \mathbf{q}_{j_n}(t_n).$$

However, now this is achieved not because individual non-optimal trajectories become less likely, but rather due to cancellation in the oscillatory integral (1.3) which corresponds to the physical phenomenon of *quantum interference*. We will observe how this cancellation occurs in

finite-dimensional oscillatory integrals when we discuss the stationary phase formula below.

1.5. **Quantum field theory.** The situation is the same in field theory, but with infinitely many particles. Namely, a useful theory of quantum fields (used in the study of interactions of elementary particles) is obtained when one considers correlation functions

(1.5)
$$\langle \phi_{j_1}(x_1,t_1)\dots\phi_{j_n}(x_n,t_n)\rangle = \int \phi_{j_1}(x_1,t_1)\dots\phi_{j_n}(x_n,t_n)e^{\frac{iS(\phi)}{\hbar}}D\phi,$$

where $D\phi$ is normalized so that $\int e^{\frac{iS(\phi)}{\hbar}}D\phi = 1$.

Of course, from the mathematical point of view, this setting is a priori even less satisfactory than the one for a quantum particle, since it involves integration with respect to the complex-valued measure $e^{\frac{iS(q)}{\hbar}}Dq$ on functions of ≥ 2 variables which nobody knows how to define in general (even after the Wick rotation). Nevertheless, physicists imagine that certain integrals of this type exist and come to correct and interesting conclusions (both physical and mathematical). Therefore, making sense of such integrals is an interesting problem for mathematicians, and will be one of our main occupations.[3]

[3]To be more precise, we will make sense of path integrals as power series in $\hbar$.

18.238 Geometry and Quantum Field Theory

Spring 2023