2. The steepest descent and stationary phase formulas

Now, let us forget for a moment that the integrals (1.1,1.3,1.5) are infinite dimensional and hence problematic to define, and ask ourselves the following question: why should we expect to recover the usual classical mechanics or field theory when the parameter κ or $\hbar$ goes to zero? The answer is that this expectation is based on the *steepest descent* (respectively, *stationary phase*) principle from classical analysis: if $f(x)$ is a function on $\mathbb{R}^d$ then the integrals $\int g(x)e^{-\frac{f(x)}{\kappa}}dx$, $\int g(x)e^{\frac{if(x)}{\hbar}}dx$ "localize" to minima, respectively critical points, of the function f. As this classical fact is of central importance to the whole subject, let us now discuss it in some detail.

2.1. **Gaussian integrals.** We start with auxiliary facts from linear algebra and analysis. Let V be a real vector space of dimension d. Let $\mathbf{M}(V)$ be the set of non-degenerate complex-valued symmetric bilinear forms on V with non-negative definite real part. We have an open dense subset $\mathbf{M}^\circ(V) \subset \mathbf{M}(V)$ of forms with positive definite real part. If $B = P + iQ \in \mathbf{M}^\circ(V)$ where P, Q are the real and imaginary parts of B, then $P^{-1}Q : V \to V$ is a self-adjoint operator with respect to P, which therefore has real eigenvalues and diagonalizes in an orthonormal basis. In this basis $B(x, y) = \sum_{j=1}^{d} a_j x_j y_j$ where $\mathrm{Re}(a_j) = 1$. Thus $B^{-1} \in \mathbf{M}^\circ(V^*)$. It follows that the map $B \mapsto B^{-1}$ is a homeomorphism $\mathbf{M}(V) \cong \mathbf{M}(V^*)$ which restricts to a homeomorphism $\mathbf{M}^\circ(V) \cong \mathbf{M}^\circ(V^*)$.

Now fix a translation-invariant volume form dx on V. Then for every complex-valued symmetric bilinear form B on V we can define its determinant $\det B$. Thus we can define a continuous function $(\det B)^{-\frac{1}{2}}$ on $\mathbf{M}(V)$ using the branch of the square root which is positive on positive definite forms (it exists and is unique because $\mathbf{M}(V)$ is starlike with respect to any point of $\mathbf{M}^\circ(V)$, hence simply connected). Note that if $B = iQ$ where Q is a real non-degenerate form then $(\det B)^{-\frac{1}{2}} = e^{\frac{\pi i \sigma(Q)}{4}} |\det Q|^{-\frac{1}{2}}$, where σ is the signature of Q. Indeed, it suffices to check the statement for diagonal forms, hence for $d = 1$, in which case it is straighforward.

Let $\mathcal{S}(V)$ be the *Schwartz space* of V, i,.e., the space of smooth functions on V whose all derivatives are rapidly decaying at ∞ (faster than any power of $|x|$). In other words, $\mathcal{S}(V)$ is the space of smooth functions f on V such that $D(V)f \subset L^2(V)$, where $D(V)$ is the algebra of differential operators on V with polynomial coefficients. The Schwartz space has a natural Fréchet topology defined by the seminorms $||Df||_{L^2}$, $D \in D(V)$. The topological dual space $\mathcal{S}'(V)$ is the space of *tempered distributions* on V. Note that we have natural inclusions $\mathcal{S}(V) \subset L^2(V) \subset \mathcal{S}'(V)$. Recall that the Fourier transform is the operator

$$\mathcal{F} : \mathcal{S}(V) \to \mathcal{S}(V^*)$$

given by

$$\mathcal{F}(g)(p) := (2\pi)^{-\frac{d}{2}} \int_V g(x) e^{-i(p,x)} dx,$$

which defines an isometry $L^2(V) \to L^2(V^*)$ such that $(\mathcal{F}^2 g)(x) = g(-x)$. By duality, it defines an operator

$$\mathcal{F} : \mathcal{S}'(V) \to \mathcal{S}'(V^*)$$

which extends $\mathcal{F}$. For any complex symmetric bilinear form B with $\mathrm{Re}B \geq 0$ the function $e^{-\frac{1}{2}B(x,x)}$ belongs to $\mathcal{S}'(V)$, and moreover to $\mathcal{S}(V)$ iff $B \in \mathbf{M}^\circ(V)$. Furthermore, it depends continuously on B as an element of these spaces. We will call it the *complex Gaussian distribution.*

Lemma 2.1. *(Gaussian integral) For any $B \in \mathbf{M}(V)$ we have*

$$\mathcal{F}(e^{-\frac{1}{2}B(x,x)}) = (\det B)^{-\frac{1}{2}} e^{-\frac{1}{2}B^{-1}(p,p)}.$$

Proof. By continuity, it suffices to prove this when $\mathrm{Re}B > 0$. In this case B is diagonalizable, so the statement reduces to the case $d = 1$. In this case we have to show that for every $a \in \mathbb{C}$ with $\mathrm{Re}a > 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx - \frac{1}{2}ax^2} dx = \frac{1}{\sqrt{a}} e^{-\frac{1}{2a}p^2}.$$

Since both sides are holomorphic in a, it is enough to check the statement when a is real. The integral in question can be written as

$$\frac{e^{-\frac{1}{2}a^{-1}p^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx.$$

But using Cauchy's theorem,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}a(x+ia^{-1}p)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+ia^{-1}p} e^{-\frac{1}{2}ax^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx.$$

Thus the result follows from the Poisson integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

by rescaling x. $\square$

In the sense of Lemma 2.1 we can say, setting $p = 0$, that

$$(2.1) \qquad (2\pi)^{-\frac{d}{2}} \int_V e^{-\frac{1}{2}B(x,x)} dx = (\det B)^{-\frac{1}{2}}.$$

Note that this equality is also true in the sense of absolute convergence if $B \in \mathbf{M}^\circ(V)$ and conditional convergence otherwise (check it!).

2.2. **Gaussian integrals with insertions.** Now let $g \in \mathcal{S}(V)$. Consider the integral

$$I_g(\hbar) := \int_V g(\hbar^{\frac{1}{2}} x) e^{-\frac{1}{2}B(x,x)} dx, \ \hbar \geq 0,$$

where for $\hbar = 0$ we use (2.1), so

$$(2.2) \qquad I_g(0) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0).$$

Let $\Delta_B : \mathcal{S}(V) \to \mathcal{S}(V)$ be the Laplace operator corresponding to B: $\Delta_B = \sum_{j=1}^d \partial_{B^{-1}e_j^*} \partial_{e_j}$ for a basis $\{e_i\}$ of V.

Theorem 2.2. *We have*

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar), \ \hbar \geq 0.$$

Thus $I_g \in C^\infty[0, \infty)$. In particular, if g vanishes at the origin to order $2n+1$ then $I_g(0) = ... = I_g^{(n)}(0) = 0$.

The rest of the subsection is occupied by the proof of Theorem 2.2.

Lemma 2.3. *I_g is a continuous function.*

Proof. Only continuity at $\hbar = 0$ requires proof. By Plancherel's theorem and Lemma 2.1,

$$I_g(\hbar) = (g(\hbar^{\frac{1}{2}} x), e^{-\frac{1}{2}B(x,x)}) =$$

$$\hbar^{-\frac{d}{2}} (\det B)^{-\frac{1}{2}} (\widehat{g}(\hbar^{-\frac{1}{2}} p), e^{-\frac{1}{2}B^{-1}(p,p)}) = (\det B)^{-\frac{1}{2}} (\widehat{g}(p), e^{-\frac{\hbar}{2}B^{-1}(p,p)}),$$

where $\widehat{g}$ is the Fourier transform of g. But $e^{-\frac{\hbar}{2}B^{-1}(p,p)} \to 1$ in $\mathcal{S}'(V^*)$ as $\hbar \to 0$ (as the complex Gaussian distribution depends continuously of the bilinear form). Thus

$$\lim_{\hbar \to 0} I_g(\hbar) = (\det B)^{-\frac{1}{2}} (\widehat{g}(p), 1) = (2\pi)^{\frac{d}{2}} (\det B)^{-\frac{1}{2}} g(0) = I_g(0),$$

as desired. $\square$

Lemma 2.4. *If $\ell \in V^*$ and $f \in \mathcal{S}(V)$ then*

$$I_{\ell f}(\hbar) = \hbar I_{\partial_{B^{-1}\ell} f}(\hbar).$$

Proof. We have

$$I_{\ell f}(\hbar) = \hbar^{\frac{1}{2}}(\ell(x)f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \ell(x)e^{-\frac{1}{2}B(x,x)}) =$$

$$-\hbar^{\frac{1}{2}}(f(\hbar^{\frac{1}{2}}x), \partial_{B^{-1}\ell}e^{-\frac{1}{2}B(x,x)}) = \hbar^{\frac{1}{2}}(\partial_{B^{-1}\ell}f(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) =$$

$$\hbar((\partial_{B^{-1}\ell}f)(\hbar^{\frac{1}{2}}x), e^{-\frac{1}{2}B(x,x)}) = \hbar I_{\partial_{B^{-1}\ell}f}(\hbar).$$

This proves the lemma. □

Now we prove Theorem 2.2. If $\hbar > 0$ then by direct differentiation we get

$$I_g'(\hbar) = \tfrac{1}{2}\hbar^{-1}I_{Eg}(\hbar),$$

where $E := \sum_{j=1}^d e_j^*\partial_{e_j}$ is the Euler vector field on V. Thus by Lemma 2.4 we have

$$I_g'(\hbar) = I_{\frac{1}{2}\Delta_B g}(\hbar),\ \hbar > 0. \tag{2.3}$$

So, using Lemma 2.3, it suffices to show that $I_g \in C^1[0,\infty)$ (then smoothness will follow by repeated application of (2.3)). To this end, note that if C is a positive definite form on V then

$$I_{e^{-\frac{1}{2}C(x,x)}}(\hbar) = \int_V e^{-\frac{1}{2}(B+\hbar C)(x,x)}dx = (2\pi)^{\frac{d}{2}}\det(B+\hbar C)^{-\frac{1}{2}},$$

which is analytic, hence continuously differentiable on $[0,\infty)$. So subtracting from g a multiple of such function, it suffices to prove that $I_g \in C^1[0,\infty)$ when $g(0) = 0$. In this case g is well known to be a linear combination of functions of the form ℓf where $f \in \mathcal{S}(V)$ and $\ell \in V^*$. So it suffices to check that $I_g \in C^1[0,\infty)$ for $g = \ell f$. But then by Lemma 2.4 $I_g'(0) = I_{\partial_{B^{-1}\ell}f}(0) = I_{\frac{1}{2}\Delta_B g}(0)$, as

$$\tfrac{1}{2}\Delta_B g(0) = \tfrac{1}{2}\Delta_B(\ell f)(0) = \sum_j \ell(e_j)\partial_{B^{-1}e_j^*}f(0) = \partial_{B^{-1}\ell}f(0).$$

This completes the proof.

Exercise 2.5. *Let $\mathcal{S}_m(V) \subset C^m(V)$ be the subspace of functions whose derivatives of order $\le m$ are rapidly decaying. Prove that the differentiation formula of Theorem 2.2 holds for $g \in \mathcal{S}_2(V)$. Deduce that if $g \in \mathcal{S}_{2n}(V)$ then $I \in C^n[0,\infty)$, and that if moreover g vanishes at 0 to order $2n+1$ then $I_g(0) = ... = I_g^{(n)}(0) = 0$.*

2.3. **The steepest descent formula.** Let $a < b$ be real numbers and $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are smooth on (a, b).

Theorem 2.6. *(Steepest descent formula) Assume that f attains a global minimum at a unique point $c \in [a, b]$, such that $a < c < b$ and $f''(c) > 0$. Then one has*

$$\int_a^b g(x)e^{-\frac{f(x)}{\hbar}}dx = \hbar^{\frac{1}{2}}e^{-\frac{f(c)}{\hbar}}I(\hbar), \tag{2.4}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = \sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}}.$$

Proof. Without loss of generality we may put $c = 0, f(c) = 0$. Let $f''(c) = M$. Making a change of variable, we may reduce to a situation where $f(x) = \frac{M}{2}x^2$ when x is in some neighborhood U of 0. Let h be a "bump" function - a smooth function supported in U which equals 1 in a smaller neighborhood $0 \in U' \subset U$. Write $g = g_1 + g_2$, where $g_1 = hg$ and $g_2 = (1 - h)g$. Let I be defined by equation (2.4), and I_1, I_2 be defined by the same equation for g replaced by g_1, g_2, so $I = I_1 + I_2$. Since f has a unique global minimum, we see by direct differentiation that for all n, $I_2^{(n)}(\hbar)$ is rapidly decaying as $\hbar \to 0$. Thus for $g = g_2$ the result is obvious, and our job is to prove it for $g = g_1$. In other words, we may assume without loss of generality that $g = g_1$ and $g_2 = 0$. We extend g by zero to the whole real line.

Let us make a change of variables $y := \hbar^{-\frac{1}{2}}x$. Then we get

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}}y)e^{-\frac{M}{2}y^2}dy. \tag{2.5}$$

Thus the result follows from (2.2) and Theorem 2.2. □

Remark 2.7. Theorem 2.6, in fact, provides an explicit formula for the Taylor coefficients of $I(\hbar)$. Namely, as in the proof of Theorem 2.6, assume that $c = 0$ and $f(x) = \frac{1}{2}p(x)^2$ near 0, where

$$p'(0) = \sqrt{f''(0)} > 0.$$

Ignoring limits of integration (which, as we have seen, are irrelevant for the asymptotic expansion of $I(\hbar)$), we have[4]

$$I(\hbar) = \hbar^{-\frac{1}{2}}\int g(x)e^{-\frac{p(x)^2}{2\hbar}}dx \sim \int_{-\infty}^{\infty} \widetilde{g}(\hbar^{\frac{1}{2}}y)e^{-\frac{y^2}{2}}dy$$

[4]Recall that for $I \in C^\infty[0, \varepsilon)$ we write $I(\hbar) \sim \sum_{n=0}^{\infty} a_n\hbar^n$ if for every $N \geq 0$ we have $I(\hbar) = \sum_{n=0}^{N-1} a_n\hbar^n + O(\hbar^N)$ as $\hbar \to 0$.

where

$$\widetilde{g}(z) := g(p^{-1}(z))(p^{-1})'(z) = \frac{g(p^{-1}(z))}{p'(p^{-1}(z))}.$$

By Theorem 2.2, the first $n+1$ terms of the Taylor expansion of this integral are given by the integral

$$I_N(\hbar) := \int_{-\infty}^{\infty} \widetilde{g}_N(\hbar^{\frac{1}{2}} y) e^{-\frac{y^2}{2}} dy$$

where $\widetilde{g}_N$ is the $2N$-th Taylor polynomial of $\widetilde{g}$ at 0. Thus if $\widetilde{g}(z) \sim \sum_{n=0}^{\infty} b_n z^n$ then

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} \hbar^n \int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy.$$

But, setting $u = \frac{y^2}{2}$, we have

(2.6)

$$\int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} dy = 2^{n+\frac{1}{2}} \int_0^{\infty} u^{n-\frac{1}{2}} e^{-u} du = 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}} (2n-1)!!,$$

where $(2n-1)!! := \prod_{1 \le j \le n} (2j-1)$. Hence

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2}) \hbar^n.$$

2.4. **Stationary phase formula.** Theorem 2.6 has the following imaginary analog, called the *stationary phase formula.*

Theorem 2.8. *(Stationary phase formula) Let $f, g : [a,b] \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point $c \in [a,b]$, such that $a < c < b$ and $f''(c) \neq 0$, and g has vanishing derivatives of all orders at a and b. Then*

$$\int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{1}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar),$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = \sqrt{2\pi} e^{\pm \frac{\pi i}{4}} \frac{g(c)}{\sqrt{|f''(c)|}},$$

where $\pm$ is the sign of $f''(c)$.[5]

Remark 2.9. It is important to assume that g has vanishing derivatives of all orders at a and b. Otherwise we will get additional boundary contributions.

[5]This is called the stationary phase formula because the main contribution comes from the point where the phase $\frac{f(x)}{\hbar}$ is stationary.

Proof. The proof is analogous to the proof of the steepest descent formula, but slightly more subtle, as we have to keep track of cancellations. First we need the following very simple but important lemma which allows us to do so.

Lemma 2.10. *(Riemann lemma) (i) Let $f : [a, b] \to \mathbb{R}$ be a smooth function such that $f'(x) > 0$ for all $x \in [a, b]$ and $g : [a, b] \to \mathbb{R}$ a C^n-function such that*

$$g(a) = ... = g^{(n-1)}(a) = g(b) = ... = g^{(n-1)}(b) = 0.$$

Let

$$I(\hbar) := \int_a^b g(x) e^{\frac{if(x)}{\hbar}} dx.$$

Then $I(\hbar) = O(\hbar^n),\ \hbar \to 0$.

(ii) Suppose g is smooth on $[a, b]$ and all derivatives of g at a and b are zero. Then I extends (by setting $I(0) := 0$) to a smooth function on $[0, \infty)$ whose all derivatives are rapidly decaying as $\hbar \to 0$.

Proof. (i) By making a change of variables we may assume without loss of generality that $f(x) = x$. Then the proof is by induction in n. The base case $n = 0$ is obvious. For $n > 0$ note that

$$\int_a^b g(x) e^{\frac{ix}{\hbar}} dx = i\hbar \int_a^b g'(x) e^{\frac{ix}{\hbar}} dx$$

(integration by parts), which justifies the induction step.

(ii) follows from (i) by repeated differentiation. □

Now we proceed to prove the theorem. As in the proof of the steepest descent formula, we may assume that $c = 0$ and $f = \frac{M}{2}x^2$ near 0 for some $M \neq 0$, and write I as the sum $I_1 + I_2$. Moreover, by Lemma 2.10(ii)

$$I_2(\hbar) = \int_a^b g_2(x) e^{\frac{if(x)}{\hbar}} dx$$

is rapidly decaying with all derivatives, so it suffices to prove the theorem for $g = g_1$.

Again following the proof of the steepest descent formula, we have

$$I(\hbar) = \int_{-\infty}^{\infty} g(\hbar^{\frac{1}{2}} y) e^{\frac{iM}{2} y^2} dy, \tag{2.7}$$

so as before the result follows from (2.2) and Theorem 2.2. □

Remark 2.11. Since computation of the asymptotic expansion of $I(\hbar)$ is a purely algebraic procedure, the explicit formula for this expansion in the imaginary case is the same as in the real case (Remark 2.7) but with $\hbar$ replaced by $i\hbar$:

$$I(\hbar) \sim \sum_{n=0}^{\infty} b_{2n} 2^{n+\frac{1}{2}} \Gamma(n+\tfrac{1}{2})(i\hbar)^n.$$

2.5. **Non-analyticity of $I(\hbar)$ and Borel summation.** Even though $I(\hbar)$ is smooth at $\hbar = 0$, its Taylor series is usually only an asymptotic expansion which diverges for any $\hbar \neq 0$, so that this function is not analytic at 0. To illustrate this, consider the integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \hbar^{\frac{1}{2}} I(\hbar),$$

where

$$I(\hbar) = \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy. \tag{2.8}$$

Since this integral is divergent for any $\hbar < 0$, we cannot conclude its analyticity at $\hbar = 0$, and it indeed fails to be so. Namely, as in Remark 2.7, the asymptotic expansion of integral (2.8) is obtained by expanding the exponential $e^{-\frac{1}{2}\hbar y^4}$ into a Taylor series and integrating termwise using (2.6):

$$I(\hbar) \sim \sum_{n=0}^{\infty} a_n \hbar^n,$$

where

$$a_n = (-1)^n \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{y^{4n}}{2^n n!} dy =$$

$$(-1)^n \frac{2^{n+\frac{1}{2}} \Gamma(2n+\frac{1}{2})}{n!} = (-1)^n \sqrt{2\pi} \frac{(4n-1)!!}{2^n n!}.$$

It is clear that this sequence has super-exponential growth, so the radius of convergence of the series is zero.

Let us now discuss the question: to what extent does the asymptotic expansion of the function $I(\hbar)$ (which we can find using Feynman diagrams as explained below) actually determine this function?

Suppose that

$$\widetilde{I}(\hbar) = \sum_{n \geq 0} a_n \hbar^n$$

is a series with zero radius of convergence. In general, we cannot uniquely determine a function I on $[0, \varepsilon)$ whose expansion is given by such a series: it always exists (check it!) but in general there is no canonical choice. However, assume that the exponential generating function of a_n

$$g(\hbar) = \sum_{n\geq 0} a_n \frac{\hbar^n}{n!}$$

is convergent in some neighborhood of 0, analytically continues to $[0, \infty)$, and has at most exponential growth as $\hbar \to \infty$. In this case there is a "canonical" way to construct a smooth function I on $[0, \varepsilon)$ with (asymptotic) Taylor expansion $\widetilde{I}$, called the *Borel summation* of $\widetilde{I}$. Namely, the function I is defined by the formula

$$I(\hbar) = \int_0^\infty g(\hbar u)e^{-u}du = \hbar^{-1}\int_0^\infty g(u)e^{-\frac{u}{\hbar}}du,$$

i.e., $I(\hbar) = \hbar^{-1}(\mathcal{L}g)(\hbar^{-1})$, where $\mathcal{L}$ is the Laplace transform (note that since g grows at most exponentially at infinity, this is well defined for small enough $\hbar > 0$). Note that

$$I(\hbar) = \int_{-\infty}^\infty |v|g(\hbar v^2)e^{-v^2}dv = \hbar^{-\frac{1}{2}}\int_{-\infty}^\infty g_*(\hbar^{\frac{1}{2}}v)e^{-v^2}dv,$$

where $g_*(v) = |v|g(v^2)$. Thus Exercise 2.5 implies that to compute the asymptotic expansion of I, we may replace g by its Taylor polynomials at 0. Hence the identity $\int_0^\infty x^n e^{-x}dx = n!$ implies that I has the Taylor expansion $\widetilde{I}$.

For example, consider the divergent series

$$\widetilde{I} := \sum_{n\geq 0}(-1)^n n!\hbar^n.$$

Then

$$g(\hbar) = \sum_{n\geq 0}(-1)^n\hbar^n = \frac{1}{1+\hbar}.$$

Hence, the Borel summation yields

$$I(\hbar) = \int_0^\infty \frac{e^{-u}}{1+\hbar u}du = \hbar^{-1}e^{\hbar^{-1}}E_1(\hbar^{-1})$$

where $E_1(x) := \int_x^\infty \frac{e^{-u}}{u}du$ is the integral exponential.

Physicists expect that in physically interesting situations perturbation expansions in quantum field theory are Borel summable, and the actual answers are obtained from these expansions by Borel summation. The Borel summability of perturbation series has actually been established in a few nontrivial examples of QFT.

Exercise 2.12. *Show that the function given by* (2.8) *equals the Borel sum of its asymptotic expansion.*

Hint. *The function $g(z)$ in this example is a special case of the hypergeometric function ${}_2F_1$ which does not express in elementary functions. But it satisfies a hypergeometric differential equation. Write down this equation and show that the Laplace transform turns it into another second order linear differential equation, and that the function $I(\hbar)$ given by* (2.8) *satisfies this equation.*

2.6. **Application of steepest descent.** Let us give an application of Theorem 2.6. Consider the integral

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt, \ s > 0.$$

By doing a change of variable $t = sx$, we get

$$\frac{\Gamma(s+1)}{s^{s+1}} = \int_0^\infty x^s e^{-sx} dx = \int_0^\infty e^{-s(x-\log x)} dx.$$

Thus, we can apply Theorem 2.6 for $\hbar = \frac{1}{s}$, $f(x) = x - \log x$, $g(x) = 1$ (of course, the interval $[a,b]$ is now infinite, and the function f blows up on the boundary, but one can easily see that the theorem is still applicable, with the same proof). The function $f(x) = x - \log x$ has a unique critical point on $[0,\infty)$, which is $c = 1$, and we have $f''(c) = 1$. Then we get

$$\Gamma(s+1) \sim s^s e^{-s} \sqrt{2\pi s}\left(1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots\right). \tag{2.9}$$

This is the celebrated *Stirling formula.*

Moreover, we can compute the coefficients $a_1, a_2, ...$ using Remark 2.7. Namely,

$$p(x) = \sqrt{2(x - \log(1+x))} = x\sqrt{1 - \tfrac{2x}{3} + \tfrac{x^2}{2} - \ldots} = x - \tfrac{x^2}{3} + \tfrac{7x^3}{36} + ...$$

Thus

$$p^{-1}(z) = z + \tfrac{z^2}{3} + \tfrac{z^3}{36} + ...,$$

hence

$$(p^{-1})'(z) = 1 + \tfrac{2z}{3} + \tfrac{z^2}{12} + ...,$$

So for instance by Remark 2.7 $a_1 = b_2 = \frac{1}{12}$.

Remark 2.13. Another way to compute this asymptotic expansion is to use the Euler product formula for the Gamma function. Differentiating the logarithm of this formula twice, we obtain (for $z > 0$):

$$(\log \Gamma)''(z) = \sum_{n=0}^\infty \frac{1}{(z+n)^2} = \sum_{n=0}^\infty \int_0^\infty t e^{-(z+n)t} dt = \int_0^\infty \frac{te^{-zt}}{1-e^{-t}} dt.$$

Recall that the *Bernoulli numbers* are defined by the generating function

$$\sum_{n\geq 0} \frac{B_n t^n}{n!} = \frac{t}{1-e^{-t}},$$

e.g. $B_0 = 1, B_1 = \frac{1}{2}$, $B_{2n+1} = 0$ for $n \geq 1$. Thus we get for $z \to \infty$

$$(\log \Gamma)''(z) \sim \sum_{n\geq 0} B_n z^{-n-1}.$$

Integrating, we get

$$(\log \Gamma)'(z) \sim \log z + C_1 - \sum_{n\geq 1} \frac{B_n}{n} z^{-n},$$

so integrating again and adding $\log z$, we get

$$\log \Gamma(z+1) \sim z \log z - z + C_1 z + \frac{1}{2}\log z + C_2 + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}.$$

From Stirling's formula we have $C_1 = 0, C_2 = \frac{1}{2}\log(2\pi)$, so in the end we get

$$(\log \Gamma)'(z) \sim \log z - \sum_{n\geq 1} \frac{B_n}{n} z^{-n}, \tag{2.10}$$

$$\log \Gamma(z+1) \sim z \log z + \frac{1}{2}\log z + \frac{1}{2}\log(2\pi) + \sum_{n\geq 2} \frac{B_n}{n(n-1)} z^{-n+1}. \tag{2.11}$$

So

$$1 + \tfrac{a_1}{s} + \tfrac{a_2}{s^2} + \cdots = \exp\Big(\sum_{n\geq 2} \tfrac{B_n}{n(n-1)} s^{-n+1}\Big).$$

In particular, since $B_2 = \frac{1}{6}$, we get $a_1 = \frac{1}{12}$.

Exercise 2.14. *Calculate $\int_0^\pi \sin^n x dx$ for nonnegative integers n using integration by parts. Then apply steepest descent to this integral and discover a formula for π (the so called Wallis formula).*

Exercise 2.15. *The Bessel function $I_0(a)$ is defined by the formula*

$$I_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{a\cos\theta} d\theta.$$

It is an even entire function with Taylor expansion

$$I_0(a) = \sum_{n=0}^{\infty} \frac{a^{2n}}{2^{2n} n!^2}.$$

Use the steepest descent/stationary phase formulas to find the asymptotic expansion of $I_0(a)$ as $a \to +\infty$ and $a \to i\infty$. Compute the first two terms of the expansion (cf. Remark 2.22).

2.7. **Multidimensional versions of steepest descent and stationary phase.** Theorems 2.6,2.8 have multidimensional analogs. To formulate them, let V be a real vector space of dimension d with a fixed volume element dx and $D \subset V$ be a compact region with smooth boundary.[6]

Theorem 2.16. *(Multidimensional steepest descent formula) Let $f, g : D \to \mathbb{R}$ be continuous functions which are smooth in the interior of D. Assume that f achieves global minimum on D at a unique point c, such that c is an interior point and $f''(c) > 0$. Then*

$$\int_D g(x) e^{-\frac{f(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{-\frac{f(c)}{\hbar}} I(\hbar), \tag{2.12}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} \frac{g(c)}{\sqrt{\det f''(c)}}.$$

Theorem 2.17. *(Multidimensional stationary phase formula) Let $f, g : D \to \mathbb{R}$ be smooth functions. Assume that f has a unique critical point c in D, such that c is an interior point and $\det f''(c) \neq 0$, and g has vanishing derivatives of all orders on ∂D. Then*

$$\int_D g(x) e^{\frac{if(x)}{\hbar}} dx = \hbar^{\frac{d}{2}} e^{\frac{if(c)}{\hbar}} I(\hbar), \tag{2.13}$$

where $I(\hbar)$ extends to a smooth function on $[0, \infty)$ such that

$$I(0) = (2\pi)^{\frac{d}{2}} e^{\frac{\pi i \sigma}{4}} \frac{g(c)}{\sqrt{|\det f''(c)|}},$$

where σ is the signature of the symmetric bilinear form $f''(c)$.

2.8. **Morse lemma.** For the proof of these theorems it is convenient to use a fundamental result in multivariable calculus called *the Morse lemma.* This lemma easily follows by induction in dimension from the following theorem.

[6]The condition of smooth boundary is introduced for simplicity of exposition only and is not essential. The same results and proofs apply with trivial modifications to more general regions, e.g. those whose boundary is only piecewise smooth in an appropriate sense.

Theorem 2.18. *(Separation of variables) Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a non-degenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system near 0 (possibly defined in a smaller ball) in which*

$$f(x_1, ..., x_n) = f(x_1, ..., x_{d-1}) \pm x_d^2.$$

Proof. By making a linear change of variables, we can assume that the quadratic part of f has the form $Q(y) \pm u^2$, where $y := (x_1, ..., x_{d-1})$, $u := x_d$. Consider the hypersurface S defined by the equation

$$\partial_u f(y, u) = 0.$$

The linear part of $\partial_u f(y, u)$ is $\pm 2u$, so by the implicit function theorem there is a change of coordinates F near 0 (with $dF(0) = 1$) in which u is replaced by $v := \pm\frac{1}{2}\partial_u f(y, u)$ and y is kept unchanged; so $u = g(y, v)$ for some function g with $(\partial_v g)(0, 0) \neq 0$. Let

$$f_*(y, v) := f(y, u) = f(y, g(y, u)).$$

Then by the chain rule

$$\partial_v f_*(y, v) = \partial_u f_*(y, v)\tfrac{\partial u}{\partial v} = \partial_u f(y, u)\tfrac{\partial u}{\partial v} = \pm 2v\partial_v g(y, v).$$

Thus the hypersurface S in the new coordinates is defined by the equation $v = 0$. So we may assume without loss of generality that S is given by the equation $u = 0$ to start with. Then $(\partial_u f)(y, 0) = 0$, so

$$f(y, u) - f(y, 0) = h(y, u)u^2,$$

where h is a smooth function in B with $h(0, 0) = \pm 1$. By replacing u with $\widetilde{u} := \sqrt{|h(y, u)|}u$ and keeping y unchanged, we may assume that $h = \pm 1$. Then

$$f(u, y) = f(0, y) \pm u^2,$$

as claimed. □

Corollary 2.19. *(Morse lemma) Let f be a smooth function on an open ball $0 \in B \subset \mathbb{R}^d$ which has a non-degenerate critical point at 0, and suppose $f(0) = 0$. Then there is a local coordinate system $(x_1, ..., x_d)$ near 0 (possibly defined in a smaller ball) in which*

$$f = x_1^2 + ... + x_m^2 - x_{m+1}^2 - ... - x_d^2.$$

In other words, near a non-degenerate critical point a smooth function is equivalent by a change of coordinates to its quadratic part.

Proof. As mentioned above, this follows easily from Theorem 2.18 by induction in dimension. □

Exercise 2.20. *Let f be a smooth function on $\mathbb{R}^2$ which is a cubic polynomial in x:*

$$f(x,y) = a(y) + b(y)x + c(y)x^2 + d(y)x^3.$$

Assume that $a(0) = a'(0) = 0$, $b(0) = b'(0) = 0$, $a''(0) = c(0) = 2$. Find explicitly local coordinates $u = u(x,y), v = v(x,y)$ near 0 in which $f(x,y) = u^2 + v^2$.

2.9. **Proof of the multidimensional steepest descent and stationary phase formulas.** The proofs of the multidimensional steepest descent and stationary phase formulas are parallel to the proofs of their one-dimensional versions, using the Morse lemma. Namely, the Morse lemma allows us to assume without loss of generality that f is quadratic near the critical point. After this, the proof of the steepest descent formula is identical to the 1-variable case. The same applies to the stationary phase formula, using the following multivariable analog of the Riemann lemma.

Lemma 2.21. *Let $f, g : D \to \mathbb{R}$ be smooth functions such that all derivatives of g vanish on ∂D and df does not vanish anywhere on the support of g. Then the function*

$$I(\hbar) := \int_D g(x) e^{\frac{if(x)}{\hbar}} dx$$

extends to a smooth function on $[0, \infty)$ and has rapidly decaying derivatives of all orders as $\hbar \to 0$.

Proof. Since df does not vanish on supp$g$, we can cover supp$g$ by local charts U_i in which $f(x)$ is the last coordinate x_d. By compactness this cover can be chosen finite. By using a partition of unity $\{h_i\}$ on supp$g$ subordinate to this cover and replacing g with $h_i g$, we may assume without loss of generality that g is supported on a single chart. Then changing variables, we may also assume that $f(x) = x_d$. Then integrating out the variables $x_1, ..., x_{d-1}$, we reduce to the 1-dimensional case covered by Lemma 2.10. □

Remark 2.22. It is clear from the proof of the stationary phase formula that it extends to the case when f may have several critical points but all of them are interior and non-degenerate. In this case the asymptotic expansions coming from different critical points are simply added together. The same applies to the steepest descent formula if the global minimum is attained at several points all of which are interior and non-degenerate.

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