3. Feynman calculus

3.1. **Wick's theorem.** Let V be a real vector space of dimension d with volume element dx. Let $S(x)$ be a smooth function on a compact region $D \subset V$ with smooth boundary which attains its minimum at a unique point $c \in D$ in the interior of D, and let g be any smooth function on D. In the previous section we proved the steepest descent formula which implies that the function

$$I(\hbar) = \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D g(x) e^{-\frac{S(x)}{\hbar}} dx$$

admits an asymptotic power series expansion in $\hbar$:

$$I(\hbar) = a_0 + a_1\hbar + \cdots + a_m\hbar^m + \cdots \tag{3.1}$$

Our main question now will be: how to compute the coefficients a_i?

Our proof of the steepest descent formula shows that although the problem of computing $I(\hbar)$ is transcendental, the problem of computing the coefficients a_i is, in fact, purely algebraic, and involves only differentiation of the functions S and g at the point c. Indeed, recalling the proof of equation (3.1), we see that the calculation of a_i reduces to calculation of integrals of the form

$$\int_V P(x) e^{-\frac{B(x,x)}{2}} dx,$$

where P is a polynomial and B is a positive definite bilinear form (in fact, $B(v,u) = (\partial_v \partial_u S)(c)$). But such integrals can be exactly evaluated. Namely, it is sufficient to consider the case when P is a product of linear functions, in which case the answer is given by the following elementary formula, known to physicists as *Wick's theorem.*

For a positive integer k, consider the set $\{1, \dots, 2k\}$. By a *matching* σ on this set we will mean its partition into k disjoint two-element subsets (pairs). A matching can be visualized by drawing $2k$ points and connecting two points with an edge if they belong to the same pair (see Fig. 1). This will give k edges which are not connected to each other.

Let us denote the set of matchings on a set T by $\Pi(T)$ and the set $\Pi(\{1, \dots, 2k\})$ by Π_k. It is clear that $|\Pi_k| = \frac{(2k)!}{2^k \cdot k!} = (2k-1)!!$. For any $\sigma \in \Pi_k$, we can think of σ as a permutation of $\{1, \dots, 2k\}$, such that $\sigma^2 = 1$ and σ has no fixed points. Namely, σ maps any element i to the second element $\sigma(i)$ of the pair containing i.

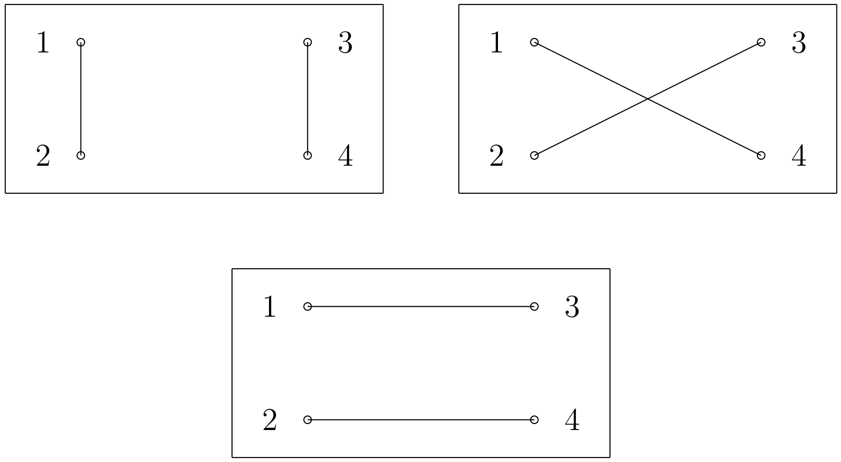


FIGURE 1. Matchings of the set $\{1, 2, 3, 4\}$

Theorem 3.1. *(Wick's theorem) Let B^{-1} denote the inverse form to B on V^*, and $\ell_1, \dots, \ell_N \in V^*$. Then, if N is even, we have*

$$\int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\sigma \in \Pi_{N/2}} \prod_{i \in \{1,\dots,N\}/\sigma} B^{-1}(\ell_i, \ell_{\sigma(i)})$$

If N is odd, the integral is zero.

Proof. If N is odd, the statement is obvious, because the integrand is an odd function. So consider the even case $N = 2k$. Since both sides of the equation are symmetric polylinear forms in $\ell_1, \dots, \ell_N$, it suffices to prove the result when $\ell_1 = \dots = \ell_N = \ell$. Further, it is clear that the formula in question is stable under linear changes of variable, so we can choose a coordinate system in such a way that $B(x, x) = x_1^2 + \dots + x_d^2$, and $\ell(x) = x_1$. Therefore, it is sufficient to assume that $d = 1$ and $\ell(x) = x$. In this case, the theorem says that

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}} dx = (2\pi)^{\frac{1}{2}} (2k-1)!!,$$

which is formula (2.6). □

Example 3.2. We have

$$\int_V \ell_1(x)\ell_2(x) e^{-\frac{B(x,x)}{2}} dx = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} B^{-1}(\ell_1, \ell_2),$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x) e^{-\frac{B(x,x)}{2}} dx =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} (B^{-1}(\ell_1, \ell_2)B^{-1}(\ell_3, \ell_4) + B^{-1}(\ell_1, \ell_3)B^{-1}(\ell_2, \ell_4) + B^{-1}(\ell_1, \ell_4)B^{-1}(\ell_2, \ell_3)).$$

Wick's theorem shows that the problem of computing a_i is of combinatorial nature. In fact, the central role in this computation is played by certain finite graphs, which are called *Feynman diagrams*. They are the main subject of the remainder of this section.

3.2. **Feynman diagrams and Feynman's theorem.** We come back to the problem of computing the coefficients a_i. Since each particular a_i depends only on a finite number of derivatives of g at c, it suffices to assume that g is a polynomial, or, more specifically, a product of linear functions: $g = \ell_1 \dots \ell_N$, $\ell_i \in V^*$. Thus, it suffices to be able to compute the series expansion of the integral

$$\langle \ell_1 \dots \ell_N \rangle := \hbar^{-\frac{d}{2}} e^{\frac{S(c)}{\hbar}} \int_D \ell_1(x) \dots \ell_N(x) e^{-\frac{S(x)}{\hbar}} dx. \tag{3.2}$$

Without loss of generality we may assume that $c = 0$ and $S(c) = 0$. Then the (asymptotic) Taylor expansion of S at c is

$$S(x) = \frac{B(x,x)}{2} - \sum_{i \geq 3} \frac{B_i(x, \dots, x)}{i!},$$

where $B_i := d^i f(0)$. Therefore, regarding the left hand side of (3.2) as a power series in $\hbar$ and making a change of variable $x \mapsto \hbar^{\frac{1}{2}} x$ (like in the last section), we get

$$\langle \ell_1 \dots \ell_N \rangle = \hbar^{\frac{N}{2}} \int_V \ell_1(x) \dots \ell_N(x) e^{-\frac{B(x,x)}{2} + \sum_{i \geq 3} \hbar^{\frac{i}{2}-1} \frac{B_i(x,\dots,x)}{i!}} dx.$$

Note that this is only an identity of asymptotic expansions in $\hbar$, as we ignored the rapidly decaying error which comes from replacing the region D by the whole space. But it implies in particular that $\langle \ell_1 \dots \ell_N \rangle = O(\hbar^{\lceil \frac{N}{2} \rceil})$ as $\hbar \to 0$ (as the expansion contains only integer powers of $\hbar$).

The theorem below, due to Feynman, gives the value of this integral in terms of Feynman diagrams. This theorem is easy to prove but is central in quantum field theory, and will be one of our main theorems. Before formulating Feynman's theorem, let us introduce some notation.

Let $G_{\geq 3}(N)$ be the set of isomorphism classes of graphs with N 1-valent "external" vertices, labeled by $1, \dots, N$, and a finite number of unlabeled "internal" vertices, of any valency ≥ 3. Note that here and below graphs are allowed to have multiple edges between two vertices and loops from a vertex to itself (see Fig. 2).

For each graph $\Gamma \in G_{\geq 3}(N)$, we define the *Feynman amplitude* of Γ as follows.

1. Put the covector ℓ_j at the j-th external vertex.

2. Put the tensor B_i at each i-valent internal vertex.

3. Take the contraction of the tensors along edges of Γ, using the bilinear form B^{-1}. This will produce a number, called the *(Feynman) amplitude* of Γ and denoted $F_\Gamma(\ell_1, \ldots, \ell_N)$.

Remark 3.3. If Γ is not connected, then F_Γ is defined to be the product of numbers obtained from the connected components. Also, the amplitude of the empty diagram is defined to be 1.

Example 3.4. Let

$$B_3 := \sum_i b_i^{13} \otimes b_i^{23} \otimes b_i^{33}, \; B_4 := \sum_j b_j^{14} \otimes b_j^{24} \otimes b_j^{34} \otimes b_j^{44},$$

where $b_i^{jk} \in V^*$. Then for the graph Γ_3 in Fig. 2 the amplitude equals

$$F_{\Gamma_3}(\ell_1, \ell_2) =$$

$$\sum_i B^{-1}(\ell_1, b_i^{13}) B^{-1}(b_i^{23}, b_i^{33}) \cdot \sum_{i,j} B^{-1}(b_i^{13}, b_j^{14}) B^{-1}(b_i^{23}, b_j^{24}) B^{-1}(b_i^{33}, b_j^{34}) B^{-1}(b_j^{44}, \ell_2).$$

Theorem 3.5. *(Feynman) One has*

$$\langle \ell_1 \ldots \ell_N \rangle = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \ldots, \ell_N), \tag{3.3}$$

where $b(\Gamma)$ is the number of edges minus the number of internal vertices of Γ.

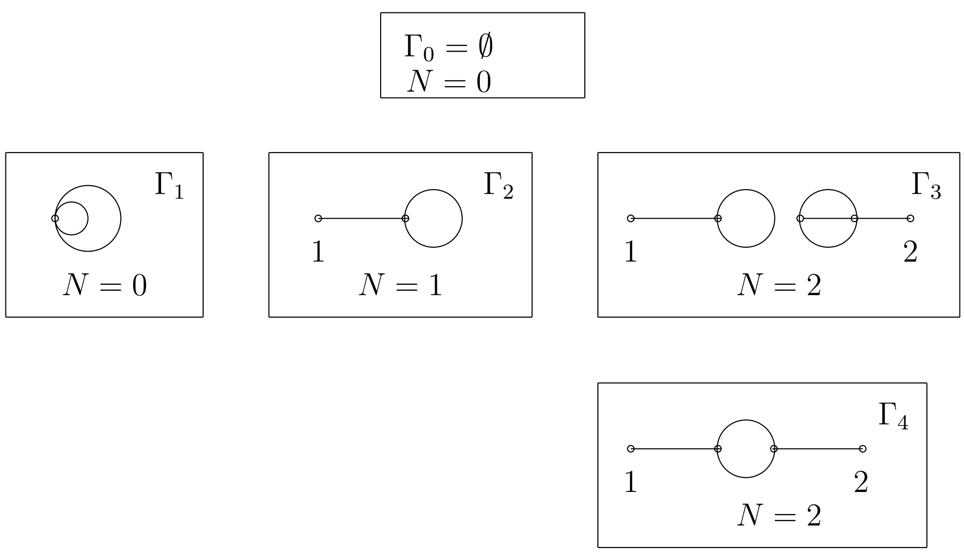


FIGURE 2. Examples of elements of $G_{\geq 3}(N)$.

Here $\mathrm{Aut}(\Gamma)$ denotes the group of automorphisms of Γ, and by an automorphism of Γ we mean a permutation of vertices **and** edges (possibly flipping the self-loops) which fixes each external vertex and preserves the graph structure, see Fig. 3. Thus there can exist nontrivial automorphisms which act trivially on the set of vertices and even ones also acting trivially on the set of edges. For example, there is an automorphism of Γ_4 that flips the upper and lower arc, and an automorphism of Γ_2 that flips the self-loop.

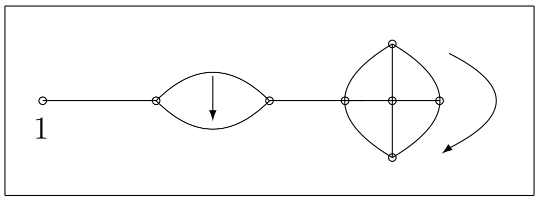


FIGURE 3. An automorphism of a graph

Remark 3.6. 1. Note that this sum is infinite, but $\hbar$-adically convergent.

2. Theorem 3.5 is a generalization of Wick's theorem: the latter is obtained if $S(x) = \frac{B(x,x)}{2}$. Indeed, in this case graphs which give nonzero amplitudes do not have internal vertices, and thus reduce to graphs corresponding to matchings σ.

Let us now make some comments about the terminology. In quantum field theory, the function $\langle \ell_1 \dots \ell_N \rangle$ is called the *N-point correlation function*, and graphs Γ are called *Feynman diagrams*. The form B^{-1} which is put on the edges is called the *propagator.*. The cubic and higher terms $\frac{B_i}{i!}$ in the expansion of the function S are called *interaction terms*, since such terms (in the action functional) describe interaction between particles. The situation in which S is quadratic (i.e., there is no interaction) is called a *free theory*; i.e. for the free theory the correlation functions are determined by Wick's formula.

Remark 3.7. Sometimes it is convenient to consider normalized correlation functions

$$\langle \ell_1 \dots \ell_N \rangle_{\mathrm{norm}} := \frac{\langle \ell_1 \dots \ell_N \rangle}{\langle \emptyset \rangle}$$

where $\langle\emptyset\rangle$ denotes the integral without insertions. Feynman's theorem implies that they are given by the formula

$$\langle \ell_1 \dots \ell_N \rangle_{\text{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $G^*_{\geq 3}(N)$ is the subset of all graphs in $G_{\geq 3}(N)$ which have no components without external vertices.

3.3. **A weighted version of Feynman's theorem.** Before proving Theorem 3.5, we would like to slightly modify and generalize it. Namely, in quantum field theory it is often useful to consider an interacting theory as a deformation of a free theory. This means that $S(x) = \frac{B(x,x)}{2} + \widetilde{S}(x)$, where $\widetilde{S}(x)$ is a perturbation

$$\widetilde{S}(x) := -\sum_{i \geq 0} g_i \frac{B_i(x, \dots, x)}{i!}$$

in which $g_r, r \geq 0$ are (formal) parameters. One benefit of these parameters is that they will allow us to group the amplitudes of Feynman diagrams in the sum (3.3) by the numbers of vertices of each valency. Namely, consider the *partition function*

$$Z = \hbar^{-\frac{d}{2}} \int_V e^{-\frac{S(x)}{\hbar}} dx$$

as a series in g_i. Let $\mathbf{n} = (n_0, n_1, n_2, \dots)$ be a sequence of nonnegative integers, almost all zero. Let $G(\mathbf{n})$ denote the set of isomorphism classes of graphs with n_0 0-valent vertices, n_1 1-valent vertices, n_2 2-valent vertices, etc. (thus, now we are considering graphs without external vertices). For $\Gamma \in G(\mathbf{n})$, let F_Γ is the amplitude of Γ defined as before. Thus

$$F_\Gamma = \prod_i g_i^{n_i} \cdot \mathbb{F}_\Gamma,$$

where $\mathbb{F}_\Gamma$ is the Feynman amplitude computed without the factors g_j.

Theorem 3.8. *One has*

$$Z = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma =$$

$$\frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

where $b(\Gamma) = \sum_i n_i(\frac{i}{2} - 1)$ *is the number of edges minus the number of vertices of* Γ.

Note that we may view Z as an element of the algebra

$$\mathbb{C}[g_0\hbar^{-\frac{3}{2}}, g_1\hbar^{-1}, g_2\hbar^{-\frac{1}{2}}; g_j, j \geq 3][[\hbar^{\frac{1}{2}}]],$$

i.e., it can be specialized to numerical values of

$$g_0\hbar^{-\frac{3}{2}}, g_1\hbar^{-1}, g_2\hbar^{-\frac{1}{2}}, g_3, g_4, ...,$$

giving an element of $\mathbb{C}[[\hbar^{\frac{1}{2}}]]$. Also Z can be specialized to $\hbar = 1$, giving an element of $\mathbb{C}[[g_j, j \geq 0]]$, and the theorem is, in fact, equivalent to this specialization. Still we choose to keep $\hbar$ to be able to take the classical limit $\hbar \to 0$.

We will prove Theorem 3.8 in the next subsection. Meanwhile, let us show that Theorem 3.5 is in fact a special case of Theorem 3.8. Indeed, because of symmetry of the correlation functions with respect to $\ell_1, \dots, \ell_N$, it is sufficient to consider the case $\ell_1 = \cdots = \ell_N = \ell$. In this case, denote the correlation function $\langle \ell^N \rangle$ (expectation value of ℓ^N). Clearly, to compute $\langle \ell^N \rangle$ for all N, it is sufficient to compute the generating function

$$\langle e^{\ell} \rangle = \hbar^{-\frac{d}{2}} \int_V e^{\ell(x) - \frac{S(x)}{\hbar}} dx := \sum_{N=0}^{\infty} \frac{\langle \ell^N \rangle}{N!},$$

which up to scaling and multiplication of ℓ by i is the Fourier transform of the Feynman density $e^{-\frac{S(x)}{\hbar}} dx$. But this expectation value is exactly the one given by Theorem 3.8 for $g_i = 1$, $i \geq 3$, $g_0 = g_2 = 0$, $g_1 = \hbar$, $B_1 = \ell$, $B_0 = 0$, $B_2 = 0$. Thus, Theorem 3.8 implies Theorem 3.5 (the factor $N!$ in the denominator is accounted for by the fact that in Theorem 3.8 we consider unlabeled, rather than labeled, 1-valent vertices).

3.4. **Proof of Feynman's theorem.** Now we will prove Theorem 3.8. Let us make a change of variable $y = \hbar^{-\frac{1}{2}} x$. Expanding the exponential in a Taylor series, we obtain

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

where

$$Z_{\mathbf{n}} = \int_V e^{-\frac{B(y,y)}{2}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} (\hbar^{\frac{i}{2}-1} B_i(y, \dots, y))^{n_i} dy.$$

Writing B_i as a sum of products of linear functions, and using Wick's theorem, we find that the value of the integral for each $\mathbf{n}$ can be expressed combinatorially as follows.

1. Attach to each factor B_i a "flower" — a vertex with i outgoing edges (see Fig. 4).

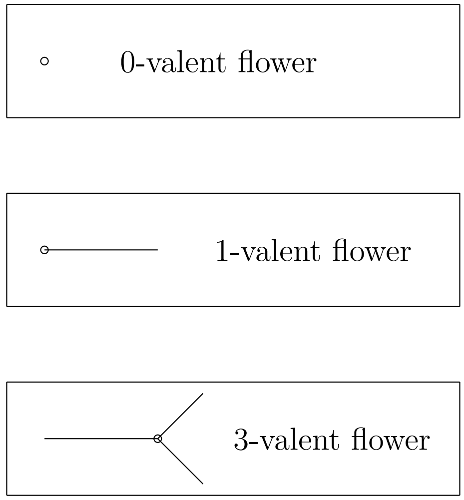


FIGURE 4.

2. Consider the set $T_{\mathbf{n}}$ of ends of these outgoing edges (see Fig. 5), and for any matching σ of this set, consider the corresponding contraction of the tensors B_i using the form B^{-1}. This will produce a scalar $\mathbb{F}(\sigma)$.

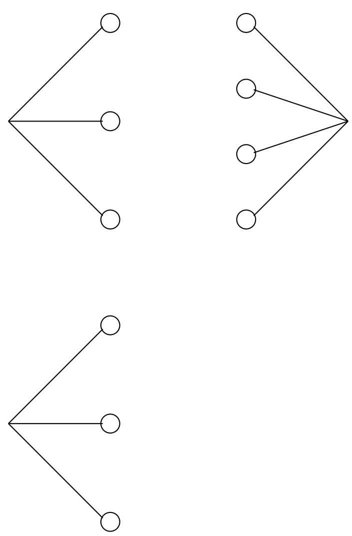


FIGURE 5. The set $T_{\mathbf{n}}$ for $\mathbf{n} = (0, 0, 0, 2, 1, 0, 0, \ldots)$ (the set of white circles)

3. The integral $Z_{\mathbf{n}}$ is given by

$$Z_{\mathbf{n}} = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{i!^{n_i} n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma). \tag{3.4}$$

Now, recall that matchings on a set can be visualized by drawing its elements as points and connecting them with edges. If we do this with the set $T_{\mathbf{n}}$, all ends of outgoing edges will become connected with each other in some way, i.e. we will obtain a certain (unoriented) graph $\Gamma = \Gamma_\sigma$ (see Fig. 6). Moreover, it is easy to see that the scalar $\mathbb{F}(\sigma)$ is nothing but the amplitude $\mathbb{F}_\Gamma$.

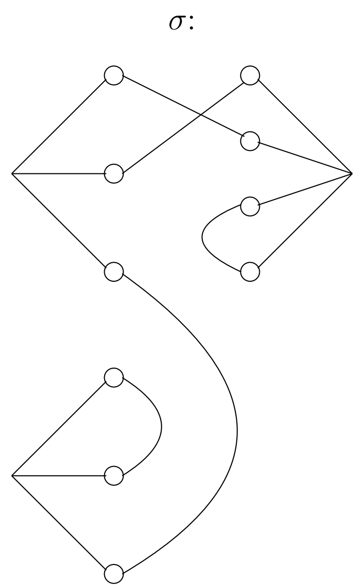


FIGURE 6. A matching σ of $T_{\mathbf{n}}$ and the corresponding graph Γ.

It is clear that any graph Γ with n_i i-valent vertices for each i can be obtained in this way. However, the same graph can be obtained in many different ways, so if we want to collect identical terms in the sum over σ, and turn it into a sum over Γ, we must find the number of σ which yield a given Γ.

For this purpose, we will consider the group $\mathbb{G}_{\mathbf{n}}$ of permutations of $T_{\mathbf{n}}$, which preserves "flowers" (i.e. endpoints of any two edges outgoing from the same flower end up again in the same flower). This group involves

1) permutations of "flowers" with a given valency;
2) permutation of the i edges inside each i-valent "flower".

More precisely, the group $\mathbb{G}_{\mathbf{n}}$ is the semidirect product of symmetric groups

$$\mathbb{G}_{\mathbf{n}} = \prod_i (S_{n_i} \ltimes S_i^{n_i}).$$

Note that $|\mathbb{G}_{\mathbf{n}}| = \prod_i i!^{n_i} n_i!$, which is the product of the numbers in the denominator of formula (3.4).

The group $\mathbb{G}_{\mathbf{n}}$ acts on the set $\Pi(T_{\mathbf{n}})$ of all matchings σ of $T_{\mathbf{n}}$. Moreover, it acts transitively on the set $\Pi_\Gamma(T_{\mathbf{n}})$ of matchings of $T_{\mathbf{n}}$ which yield a given graph Γ. Furthermore, it is easy to see that the stabilizer of a given matching is $\mathrm{Aut}(\Gamma)$. Thus, the number of matchings giving Γ is

$$N_\Gamma = \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|}.$$

Hence,

$$\sum_{\sigma \in \Pi(T_{\mathbf{n}})} \mathbb{F}(\sigma) = \sum_\Gamma \frac{\prod_i i!^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|} \mathbb{F}_\Gamma.$$

Finally, note that the exponent of $\hbar$ in equation (3.4) is $\sum_i n_i(\frac{i}{2} - 1)$, which is the number of edges of Γ minus the number of vertices, i.e. $b(\Gamma)$. Substituting this into (3.4), we get the result.

Example 3.9. Let $d = 1$, $V = \mathbb{R}$, $g_i = g$, $B_i = z^i$ for all $i \geq 0$ (where z is a formal variable), $\hbar = 1$. Then we find the asymptotic expansion

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + g e^{zx}} = \sum_{n \geq 0} g^n \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|},$$

where $G(n, k)$ is the set of isomorphism classes of graphs with n vertices and k edges.[7] Expanding the left hand side, we get

$$\sum_k \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{\frac{z^2 n^2}{2}}}{n!},$$

and hence

$$\sum_{\Gamma \in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!}.$$

Exercise 3.10. *Check this by direct combinatorics.*

[7]This integral converges for $g < 0$, $z \in \mathbb{R}$, but this is not important for us here, since we consider the integral formally.

3.5. **Sum over connected diagrams.** Now we will show that the logarithm of the partition function Z is also given by summation over diagrams, but with only connected diagrams taken into account. This significantly simplifies the analysis of Z in the first few orders of perturbation theory, since the number of connected diagrams with a given number of vertices and edges is significantly smaller than the number of all diagrams.

Theorem 3.11. *Let $Z_0 = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det B}}$. Then one has*

$$\log \frac{Z}{Z_0} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_c(\mathbf{n})} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|}$$

where $G_c(\mathbf{n})$ is the set of connected graphs in $G(\mathbf{n})$.[8]

Proof. For any graphs Γ_1, Γ_2, let $\Gamma_1\Gamma_2$ stand for the disjoint union of Γ_1 and Γ_2, and for any graph Γ let Γ^n denote the disjoint union of n copies of Γ. Then every graph can be uniquely written as $\Gamma_1^{k_1} \dots \Gamma_l^{k_l}$, where Γ_j are connected non-isomorphic graphs. Moreover, it is clear that $\mathbb{F}_{\Gamma_1\Gamma_2} = \mathbb{F}_{\Gamma_1}\mathbb{F}_{\Gamma_2}$, $b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$, and

$$|\mathrm{Aut}(\Gamma_1^{k_1} \dots \Gamma_l^{k_l})| = \prod_j |\mathrm{Aut}(\Gamma_j)|^{k_j} k_j!.$$

Thus, exponentiating the equation of Theorem 3.11, and using the above facts together with the Taylor series for the function e^x, we arrive at Theorem 3.8. As Theorem 3.8 has been proved, so is Theorem 3.11 □

3.6. **The loop expansion.** Note that since summation in Theorem 3.11 is over connected Feynman diagrams, the number $b(\Gamma)$ is the number of loops in Γ minus 1. In particular, the lowest coefficient in $\hbar$ is that of $\hbar^{-1}$, and it is the sum over all trees; the next coefficient is to $\hbar^0$, and it is the sum over all diagrams with one loop (cycle); the next coefficient to $\hbar$ is the sum over two-loop diagrams, and so on. Therefore, physicists refer to the expansion of Theorem 3.11 as the *loop expansion.*

Let us study the two most singular terms in this expansion (with respect to $\hbar$), i.e. the terms given by the sum over trees and 1-loop graphs.

Let x_0 be the critical point of the function S. It exists and is unique, since g_i are assumed to be formal parameters. Let $G^{(j)}(\mathbf{n})$ denote the

[8]We define a connected graph as a graph with exactly one connected component. So the empty graph, which has zero connected components, is not considered connected.

set of classes of graphs in $G_c(\mathbf{n})$ with j loops. Let

$$\left(\log\frac{Z}{Z_0}\right)_j := \sum_{\mathbf{n}}\prod_i g_i^{n_i}\sum_{\Gamma\in G^{(j)}(\mathbf{n})}\frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

so that

$$\log\frac{Z}{Z_0} = \sum_{j=0}^{\infty}\left(\log\frac{Z}{Z_0}\right)_j \hbar^{j-1}.$$

Theorem 3.12.

$$\left(\log\frac{Z}{Z_0}\right)_0 = -S(x_0), \tag{3.5}$$

and

$$\left(\log\frac{Z}{Z_0}\right)_1 = \frac{1}{2}\log\frac{\det B}{\det S''(x_0)}. \tag{3.6}$$

Proof. First note that the statement is purely combinatorial. This means, in particular, that it is sufficient to check that the statement yields the correct asymptotic expansion of the right hand sides of equations (3.5),(3.6) in the case when S is a polynomial with real coefficients of the form $\frac{B(x,x)}{2} - \sum_{i=0}^{N} g_i\frac{B_i(x,\ldots,x)}{i!}$ and $\hbar > 0$. To do so, let $Z := \hbar^{-\frac{d}{2}}\int_{\mathbf{B}} e^{-\frac{S(x)}{\hbar}}dx$, where $\mathbf{B}$ is a ball centered at 0. For sufficiently small g_i, the function S has a unique global minimum point x_0 in $\mathbf{B}$, which is non-degenerate. Thus, by the steepest descent formula, we have

$$\frac{Z}{Z_0} = e^{-\frac{S(x_0)}{\hbar}}I(\hbar),$$

where $I(\hbar) \sim \sqrt{\frac{\det B}{\det S''(x_0)}}(1 + a_1\hbar + a_2\hbar^2 + \cdots)$ (asymptotically). Thus,

$$\log\frac{Z}{Z_0} = -S(x_0)\hbar^{-1} + \frac{1}{2}\log\frac{\det B}{\det S''(x_0)} + O(\hbar).$$

This implies the result. □

Physicists call the expression $(\log\frac{Z}{Z_0})_0$ the *classical (or tree) approximation* to the quantum mechanical quantity $\hbar\log\frac{Z}{Z_0}$, and the sum $(\log\frac{Z}{Z_0})_0 + \hbar(\log\frac{Z}{Z_0})_1$ the *one-loop approximation*. Similarly one defines higher loop approximations. Note that the classical approximation is obtained by finding the critical point and value of the classical action $S(x)$, which in the classical mechanics and field theory situation corresponds to solving the classical equations of motion.

3.7. **Nonlinear equations and trees.** As we have noted, Theorem 3.12 does not involve integrals and is purely combinatorial. Therefore, there should exist a purely combinatorial proof of this theorem. Such a proof indeed exists. Here we will give a combinatorial proof of the first statement of the Theorem (formula (3.5)).

Consider the equation $S'(x) = 0$, defining the critical point x_0. This equation can be written as $x = \beta(x)$, where

$$\beta(x) := \sum_{i \geq 1} g_i \frac{B^{-1} B_i(x, \ldots, x, -)}{(i-1)!},$$

where $B^{-1} : V^* \to V$ is the operator corresponding to the form B^{-1}.

In the sense of power series norm, β is a contracting mapping. Thus, $x_0 = \lim_{N \to \infty} \beta^N(x)$, for any initial vector, for example $0 \in V$. In other words, we will obtain x_0 if we keep substituting the series $\beta(x)$ into itself. This leads to summation over trees (explain why!). More precisely, we get the following expression for x_0:

$$x_0 = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G^{(0)}(\mathbf{n},1)} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

where $G^{(0)}(\mathbf{n}, 1)$ is the set of trees with one external vertex and n_i internal vertices of degree i. Now, since $S(x) = \frac{B(x,x)}{2} - \sum_i g_i \frac{B_i(x,\ldots,x)}{i!}$, the expression $-S(x_0)$ equals the sum of expressions $\prod_i g_i^{n_i} \frac{\mathbb{F}_\Gamma}{|\mathrm{Aut}(\Gamma)|}$ over all trees (without external vertices). Indeed, the term $\frac{B(x_0,x_0)}{2}$ corresponds to gluing two trees with external vertices (identifying the two external vertices, so that they disappear); so it corresponds to summing over trees with a marked edge, i.e. counting each tree as many times as it has edges. On the other hand, the term $g_i \frac{B_i(x_0,\ldots,x_0)}{i!}$ corresponds to gluing i trees with external vertices together at these vertices (making a tree with a marked vertex). So $\sum_i g_i \frac{B_i(x_0,\ldots,x_0)}{i!}$ corresponds to summing over trees with a marked vertex, i.e. counting each tree as many times as it has vertices. But the number of vertices of a tree exceeds the number of edges by 1. Thus, the difference $-S(x_0)$ of the above two contributions corresponds to summing over trees, counting each exactly once. This implies formula (3.5).

3.8. **The case $d = 1$.** In the case $d = 1$ we can compute the tree sum $-S(x_0)$ even more explicitly. Namely, let

$$S(x) := \frac{x^2}{2} - gh(x)$$

where $h(x) = \sum_{n\geq 0} c_n x^n$ with $c_1 \neq 0$. Then x_0 is the solution of the equation $x = gh'(x)$, i.e., $x_0 = f(g)$ where $x = f(y)$ is the inverse function to $y = \frac{x}{h'(x)}$. So the tree approximation takes the form $-S(x_0) = F(g)$ where

$$F(g) = -\frac{f(g)^2}{2} + gh(f(g)).$$

Thus

$$F'(g) = -f(g)f'(g) + h(f(g)) + gh'(f(g))f'(g).$$

But $h'(f(g)) = \frac{f(g)}{g}$, so the first and third summands cancel and we get

$$F'(g) = h(f(g)),$$

hence

$$-S(x_0) = \int_0^g h(f(a))da. \tag{3.7}$$

3.9. **Counting trees and Cayley's theorem.** In this section we will apply Theorem 3.12 to tree counting problems, in particular will prove a classical theorem due to Cayley that the number of labeled trees with n vertices is n^{n-2}.

We consider essentially the same situation as we considered above in Example 3.9: $d = 1$, $B_i = 1$, $g_i = g$. Thus, we have $S(x) = \frac{x^2}{2} - ge^x$. By Theorem 3.12, we have

$$\sum_{n\geq 0} g^n \sum_{\Gamma \in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = -S(x_0),$$

where $T(n)$ is the set of isomorphism classes of trees with n vertices, and x_0 is the root of the equation $S'(x) = 0$, i.e. $x = ge^x$.

In other words, let $x = f(y)$ be the function inverse to the function $y = xe^{-x}$ near $x = 0$, then $x_0 = f(g)$. The function $f(y)$ is related to (the principal branch of) the *Lambert function* $W(y)$ by the formula $f(y) = -W(-y)$. By (3.7)

$$-S(x_0) = \int_0^g e^{f(a)}da = \int_0^g \frac{f(a)}{a}da.$$

Thus it remains to find the Taylor expansion of f. This expansion is given by the following classical result.

Proposition 3.13. *One has*

$$f(g) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} g^n.$$

Proof. Let $f(g) = \sum_{n\geq 1} a_n g^n$. Then

$$a_n = \frac{1}{2\pi i}\oint \frac{f(g)}{g^{n+1}}dg = \frac{1}{2\pi i}\oint \frac{x}{(xe^{-x})^{n+1}}d(xe^{-x}) =$$

$$\frac{1}{2\pi i}\oint e^{nx}\frac{1-x}{x^n}dx = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}.$$

□

So we get

$$-S(x_0) = \int_0^g \frac{f(a)}{a}da = \sum_{n\geq 1}\frac{n^{n-2}}{n!}g^n.$$

This shows that

$$\sum_{\Gamma\in T(n)}\frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}.$$

But each isomorphism class of unlabeled trees with n vertices has $\frac{n!}{|\mathrm{Aut}(\Gamma)|}$ nonisomorphic labelings. Thus we obtain

Corollary 3.14. *(A. Cayley) The number of labeled trees with n vertices is n^{n-2}.*

3.10. **Counting trees with conditions.** In a similar way we can count labeled trees with conditions on vertices. For example, let us compute the number of labeled trivalent trees with m vertices (i.e. trees that have only 1-valent and 3-valent vertices). Clearly, $m = 2k$, otherwise there is no such trees. The relevant action functional is

$$S(x) = \tfrac{x^2}{2} - g(x + \tfrac{x^3}{6}).$$

Then the critical point x_0 is obtained from the equation

$$x = g(1 + \tfrac{x^2}{2}),$$

which yields

$$x_0 = \frac{1-\sqrt{1-2g^2}}{g}.$$

Thus, by (3.7) the tree sum equals

$$-S(x_0) = \int_0^g \left(\tfrac{1-\sqrt{1-2a^2}}{a} + \tfrac{(1-\sqrt{1-2a^2})^3}{6a^3}\right)da =$$

$$\frac{2}{3}\int_0^g \tfrac{1-(1+a^2)\sqrt{1-2a^2}}{a^3}da = \frac{(1-2g^2)^{\frac{3}{2}} - (1-3g^2)}{3g^2}.$$

Expanding this in a Taylor series, we find

$$-S(x_0) = \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{(n+1)!}g^{2n}.$$

Hence, we get

Corollary 3.15. *The number N_k of trivalent labeled trees with $2n$ vertices is $(2k-3)!!\frac{(2k)!}{(k+1)!}$.*

For example, $N_1 = 1$ (a single edge), $N_2 = 4$ (a single tree with $4!$ labelings modulo a group of order 6), $N_3 = 90$ (a single tree with $6!$ labelings modulo a group of order 8), etc.

3.11. **Counting oriented trees.** Feynman calculus can be used to count not only non-oriented, but also oriented graphs. For example, suppose we want to count labeled oriented trees, whose vertices are either sources or sinks (see Fig. 7). In this case, it is easy to see (check it!) that the relevant integration problem is in two dimensions, with the action $S = xy - be^x - ae^y$ (the form xy is not positive definite, but this is immaterial since our computations are purely formal). So the critical point is found from the equations

$$xe^{-y} = a,\ ye^{-x} = b.$$

Like before, look for a solution $(x, y) = (x_0, y_0)$ in the form

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq}a^p b^q,\ y = b + \sum_{p\geq 1, q\geq 1} d_{pq}a^p b^q.$$

A calculation with residues similar to the one we did for unoriented trees yields

$$c_{pq} = \frac{1}{(2\pi i)^2}\oint\oint \frac{x}{a^{p+1}b^{q+1}} da \wedge db =$$

$$\frac{1}{(2\pi i)^2}\oint\oint \frac{e^{qx+py}}{x^p y^{q+1}}(1 - xy)dx \wedge dy = \frac{q^{p-1}p^{q-1}}{(p-1)!q!}.$$

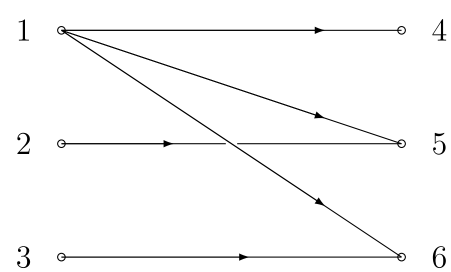


Figure 7. A labeled oriented tree with 3 sources and 3 sinks.

Similarly, $d_{pq} = \frac{q^{p-1}p^{q-1}}{p!(q-1)!}$. Now, similarly to the unoriented case, we find that $-a\partial_a S(x,y) = x$, $-b\partial_b S(x,y) = y$, so

$$-S(x,y) = b + \int_0^a \frac{x}{u} du = a + b + \sum_{p,q\geq 1} \frac{p^{q-1}q^{p-1}}{p!q!} a^p b^q$$

This implies that the number of labeled trees with p sources and q sinks is $p^{q-1}q^{p-1}\frac{(p+q)!}{p!q!}$. In particular, if we specify which vertices are sources and which are sinks, the number of labeled trees is $p^{q-1}q^{p-1}$.

Exercise 3.16. *Do this calculation in detail.*

3.12. **The matrix-tree theorem.** These calculations can be generalized to compute the number of *colored* labeled trees. For this we first need to define the *Kirchhoff polynomial* $K_m(\mathbf{u})$. Namely, for a collection of variables $\mathbf{u} := (u_{ik})$, $1 \leq i \neq k \leq m$, $u_{ik} = u_{ki}$ consider the quadratic form

$$U(\mathbf{y}) := \sum_{1\leq i<k\leq m} u_{ik}(y_i - y_k)^2.$$

Generically it has a 1-dimensional kernel spanned by $\mathbf{1} = (1, ..., 1)$, so it is nondegenerate on the subspace defined by the equation $\sum_i y_i = 0$. This subspace carries a volume form $\omega_0(v_1, ..., v_{m-1}) := \omega(v_1, ..., v_{m-1}, \mathbf{1})$, where ω is the standard volume form on $\mathbb{R}^m$, and with respect to this form we have

$$K_m(\mathbf{u}) := \det U = \det(\delta_{i\ell} \sum_{k\neq \ell} u_{k\ell} - u_{i\ell})_{(j)}$$

for any $1 \leq j \leq m$, where the subscript (j) means that the j-th row and column are removed. The polynomial K_m is called the *Kirchhoff polynomial.* For instance, $K_2 = u_{12}$, $K_3 = u_{12}u_{13} + u_{13}u_{23} + u_{12}u_{23}$, etc.

Now let $\mathbf{p} = (p_1, ..., p_m)$ be a m-tuple of positive integers and $\mathbf{r} = (r_{ij}, 1 \leq i \leq j \leq m)$ be a collection of nonnegative integers with $|\mathbf{r}| = |\mathbf{p}| - 1$, where $|\mathbf{r}| := \sum_{i\leq j} r_{ij}$, $|\mathbf{p}| := \sum_k p_k$. Suppose vertices of the tree are given colors $1, ..., m$, and we want to compute the number $N(\mathbf{p}, \mathbf{r})$ of labeled trees with the first p_1 vertices colored with 1, the next p_2 with 2,..., the last p_m with m, and r_{ij} edges going between vertices of color i and vertices of color j.

It suffices to compute the polynomial

$$Q_{\mathbf{p}}(\mathbf{z}) := \sum_{\mathbf{r}:|\mathbf{r}|=|\mathbf{p}|-1} N(\mathbf{p}, \mathbf{r}) \prod_{i\leq j} z_{ij}^{r_{ij}}.$$

Theorem 3.17. *We have*

$$Q_{\mathbf{p}}(\mathbf{z})=(p_1...p_m)^{-1}K(p_kz_{k\ell}p_\ell,k\neq\ell)\prod_\ell(\sum_k p_kz_{k\ell})^{p_\ell-1}.$$

Note that for $m=1$ and $\mathbf{z}=1$ this recovers Cayley's theorem, while for $m=2$ and $\mathbf{z}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ it recovers our count of oriented trees.

Proof. We attach to each color j a real variable x_j. Then the corresponding action is

$$S(x,y)=\frac{1}{2}x^TBx-\sum_{j=1}^m a_je^{x_j},$$

where $B=(b_{ij})$ is inverse to the matrix $\mathbf{z}:=(z_{ij})$ with $z_{ij}=z_{ji}$. Then by Theorem 3.12, $Q_{\mathbf{p}}(\mathbf{z})$ is the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$, where x is the critical point of S.

The equation for the critical point of S is

$$\sum_i x_ib_{ij}e^{-x_j}=a_j.$$

Let $X_j:=\sum_i x_ib_{ij}$, then $x_i=\sum_j z_{ij}X_j$, $a_i=X_ie^{-x_i}$, and

$$-S(x)=\int X_j\frac{da_j}{a_j}$$

for all j. In other words, the coefficient to $\prod_k a_k^{p_k}$ in $-S(x)$ equals the coefficient to the same monomial in $X_j(\mathbf{z},\mathbf{a})$ divided by p_j. Thus, denoting by $D_T(\mathbf{z})$ the principal minor of $\mathbf{z}$ corresponding to a subset $T\subset\{1,...,m\}$, we get

$$Q_{\mathbf{p}}(\mathbf{z})=\tfrac{p_j^{-1}}{(2\pi i)^m}\oint X_j(\prod_k a_k^{-p_k-1})d\mathbf{a}=$$

$$\tfrac{p_j^{-1}}{(2\pi i)^m}\oint X_j(\prod_k(X_ke^{-x_k})^{-p_k-1})d(X_1e^{-x_1})\wedge...\wedge d(X_me^{-x_m})=$$

$$\tfrac{p_j^{-1}}{(2\pi i)^m}\oint\sum_{T\subset\{1,...,m\}}(-1)^{|T|}D_T(\mathbf{z})X_j(\prod_{\ell\notin T}X_\ell^{-1})(\prod_\ell X_\ell^{-p_\ell})e^{\sum_{k,\ell}p_kz_{k\ell}X_\ell}dX_1\wedge...\wedge dX_m$$

$$=p_j^{-1}\sum_{T\subset\{1,...,m\}}(-1)^{|T|}\frac{p_j-1+\delta_{jT^c}}{\sum_k p_kz_{kj}}D_T(\mathbf{z})\prod_\ell\frac{(\sum_k p_kz_{k\ell})^{p_\ell-\delta_{\ell T}}}{(p_\ell-\delta_{\ell T})!}=$$

$$p_j^{-1}\left(\frac{p_j-1}{\sum_k p_kz_{kj}}\det(\delta_{i\ell}\sum_k p_kz_{k\ell}-z_{i\ell}p_\ell)+\det(\delta_{i\ell}\sum_k p_kz_{k\ell}-z_{i\ell}p_\ell)_{(j)}\right)\prod_\ell\frac{(\sum_k p_kz_{k\ell})^{p_\ell-1}}{p_\ell!},$$

where $\delta_{\ell T} = 1$ if $\ell \in T$ and 0 otherwise. The first determinant is zero, so we get

$$Q_{\mathbf{p}}(\mathbf{z}) = (p_1 ... p_m)^{-1} \det(\delta_{i\ell} \sum_k p_k z_{k\ell} p_\ell - p_i z_{i\ell} p_\ell)_{(j)} \prod_\ell \frac{(\sum_k p_k z_{k\ell})^{p_\ell - 1}}{p_\ell !}.$$

This implies the theorem. □

Theorem 3.17 is a weighted version of *Kirchhoff's matrix-tree theorem*, which is a generalization of Cayley's theorem. More precisely, take $\mathbf{z} = A_\Gamma$ to be the adjacency matrix of a graph Γ (without self-loops), m the number of vertices of Γ, and $p_i = 1$ for all i. Then $Q_{\mathbf{p}}(\mathbf{z}) = N_\Gamma$ is the number of spanning trees of Γ, and Theorem 3.17 says that

$$N_\Gamma = \det U,$$

where U is the quadratic form

$$U(\mathbf{y}) = \sum_{i<j} (A_\Gamma)_{ij} (y_i - y_j)^2 = (\Delta_\Gamma \mathbf{y}, \mathbf{y}),$$

where $\Delta_\Gamma = D_\Gamma - A_\Gamma$ is the Laplace operator of Γ (D_Γ being the diagonal matrix of vertex degrees). Thus we get

Corollary 3.18. *(The matrix-tree theorem)*

$$N_\Gamma = \frac{1}{m} \lambda_1 ... \lambda_{m-1},$$

where λ_i are the non-zero eigenvalues of Δ_Γ.

Cayley's theorem is obtained from this result when Γ is a complete graph, in which case $\lambda_i = m$ for all i, so we get $N_\Gamma = m^{m-2}$.

3.13. **1-particle irreducible diagrams and the effective action.** Let $Z = Z_S$ be the partition function corresponding to the action S. In the previous subsections we have seen that the "classical" (or "tree") part $(\log \frac{Z_S}{Z_0})_0$ of the quantity $\hbar \log \frac{Z_S}{Z_0}$ is quite elementary to compute – it is just minus the critical value of the action $S(x)$. Thus, if we could find a new "effective" action S_{eff} (a "deformation" of S) such that

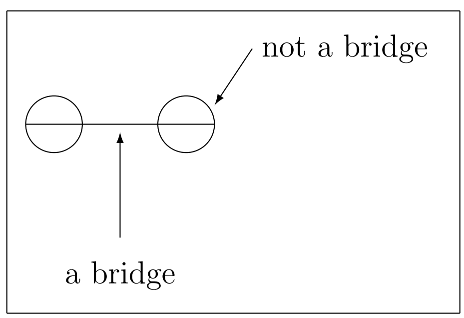
$$\hbar^{-1} (\log \frac{Z_{S_{\text{eff}}}}{Z_0})_0 = \log \frac{Z_S}{Z_0}$$

(i.e. the classical answer for the effective action is the quantum answer for the original one), then we can consider the quantum theory for the action S solved. In other words, the problem of solving the quantum theory attached to S (i.e. finding the corresponding integrals) essentially reduces to the problem of computing the effective action S_{eff}.

We will now give a recipe of computing the effective action in terms of amplitudes of Feynman diagrams, and see that it is computationally easier than computing the sum over connected diagrams.

Definition 3.19. An edge e of a connected graph Γ is said to be a *bridge* if the graph $\Gamma \setminus e$ is disconnected. A connected graph without bridges is called *1-particle irreducible* (1PI).[9]

To compute the effective action, we will need to consider graphs with external edges (but having at least one internal vertex). Such a graph Γ (with N external edges) will be called 1-particle irreducible if so is the corresponding "amputated" graph (i.e. the graph obtained from Γ by removal of the external edges). In particular, a graph with one internal vertex is always 1-particle irreducible, while a single edge graph without internal vertices is defined *not* to be 1-particle irreducible. The notions of a bridge and a 1-particle irreducible graph are illustrated by Fig. 8.



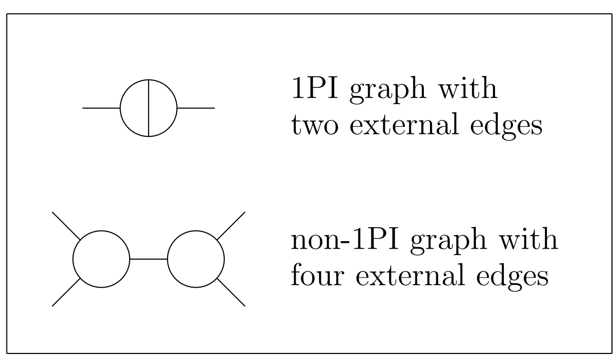


FIGURE 8.

[9]This is the physical terminology. The mathematical term is "2-connected".

Denote by $G_{1\mathrm{PI}}(\mathbf{n}, N)$ the set of isomorphism classes of 1-particle irreducible graphs with N external edges and n_i i-valent internal vertices for each i (where isomorphisms are not allowed to move external edges).

Theorem 3.20. *The effective action S_{eff} is given by the formula*

$$S_{\mathrm{eff}}(x) = \frac{B(x,x)}{2} - \sum_{i \geq 0} \frac{\mathcal{B}_i(x, ..., x)}{i!},$$

where

$$\mathcal{B}_N(x, \dots, x) = \hbar \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\Gamma \in G_{1\mathrm{PI}}(\mathbf{n}, N)} \frac{\mathbb{F}_\Gamma(Bx, \dots, Bx)}{|\mathrm{Aut}(\Gamma)|}.$$

Thus, $S_{\mathrm{eff}} = S + \hbar S_1 + \hbar^2 S_2 + ..$ The expressions $\hbar^j S_j$ are called the *j-loop corrections* to the effective action.

This theorem allows physicists to worry only about 1-particle irreducible diagrams, and is the reason why you will rarely see other diagrams in a QFT textbook. As before, it is very useful in doing low order computations, since the number of 1-particle irreducible diagrams with a given number of loops is much smaller than the number of connected diagrams with the same number of loops.

Proof. The proof is based on the following lemma from graph theory.

Lemma 3.21. *Any connected graph Γ can be uniquely represented as a tree whose vertices are 1-particle irreducible subgraphs (with external edges), and edges are the bridges of Γ.*

The lemma is obvious. Namely, let us remove all bridges from Γ. Then Γ will turn into a disjoint union of 1-particle irreducible graphs which should be taken to be the vertices of the said tree.

The tree corresponding to the graph Γ is called the *skeleton* of Γ (see Fig. 9).

It is easy to see that Lemma 3.21 implies Theorem 3.20. Indeed, it implies that the sum over all connected graphs occuring in the expression of $\log \frac{Z_S}{Z_0}$ can be written as a sum over skeleton trees, so that the contribution from each tree is (proportional to) the contraction of tensors $\mathcal{B}_i$ put in its vertices, and $\mathcal{B}_i$ is the (weighted) sum of amplitudes of all 1-particle irreducible graphs with i external edges. □

3.14. **1-particle irreducible diagrams and the Legendre transform.** Recall the notion of *Legendre transform.* Let f be a smooth function on a vector space Y, such that the map $Y \to Y^*$ given by $x \to df(x)$ is a diffeomorphism. Then one can define the Legendre

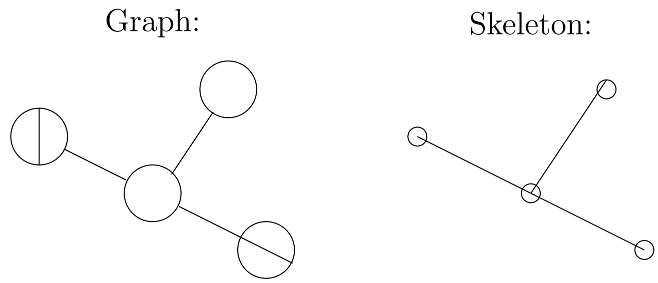


FIGURE 9. The skeleton of a graph.

transform of f as follows. For $p \in Y^*$, let $x_0 = x_0(p)$ be the critical point of the function $(p, x) - f(x)$ (i.e. the unique solution of the equation $df(x) = p$). Then the Legendre transform of f is the function on Y^* defined by

$$L(f)(p) = (p, x_0) - f(x_0).$$

It is easy to see that the differential of $L(f)$ is also a diffeomorphism $Y^* \to Y$ (in fact, inverse to $df(x)$), and that $L^2(f) = f$.

Example 3.22. Let $f(x) = \frac{ax^2}{2}$, $a \neq 0$. Then $px - f = px - \frac{x^2}{2}$ has a critical point at $p = \frac{x}{a}$, and the critical value is $\frac{p^2}{2a}$. Thus $L(\frac{ax^2}{2}) = \frac{p^2}{2a}$. More generally, if $f(x) = \frac{B(x,x)}{2}$ where B is a non-degenerate symmetric form on Y then $L(f)(p) = \frac{B^{-1}(p,p)}{2}$. E.g., the Legendre transform of a Lagrangian $\frac{mv^2}{2} - U(x)$ of a particle of mass m with respect to velocity $v = \dot{x}$ is its Hamiltonian (energy) $\frac{p^2}{2m} + U(x)$, and vice versa. This is, in fact, so in complete generality, which is why Legendre transform plays an important role in classical mechanics and field theory.

Note that the stationary phase formula implies that the Legendre transform is the classical analog of the Fourier transform. Indeed, the leading term of the asymptotics as $\hbar \to 0$ of the logarithm of the (suitably normalized) Fourier transform $\hbar^{-\frac{d}{2}} \int_V e^{\frac{i(-(p,x)+S(x))}{\hbar}} dx$ of the Feynman density $e^{\frac{iS(x)}{\hbar}} dx$ (where the integral is understood in the sense of distributions) is $-\frac{iL(S)(p)}{\hbar}$.

Now let us consider Theorem 3.20 in the situation of Theorem 3.5. Thus, $S(x) = \frac{B(x,x)}{2} + O(x^3)$, and we look at

$$Z(p) = \hbar^{-\frac{d}{2}} \int_V e^{\frac{(p,x)-S(x)}{\hbar}} dx.$$

By Theorem 3.20, one has

$$\log \frac{Z(p)}{Z_0} = -\hbar^{-1} S_{\text{eff}}(x_0, p),$$

where the effective action $S_{\text{eff}}(x,p)$ is the sum over 1-particle irreducible graphs and $x_0 = x_0(p)$ is its critical point.

Now, we must have $S_{\text{eff}}(x,p) = -p \cdot x + S_{\text{eff}}(x)$, since the only 1PI graph which contains 1-valent internal vertices (corresponding to p) is the graph with one edge, connecting an internal vertex with an external one (so it yields the term $-p\cdot x$, and other graphs contain no p-vertices). This shows that $\hbar \log \frac{Z(p)}{Z_0}$ is the critical value of $p \cdot x - S_{\text{eff}}(x)$. Thus we have proved the following.

Proposition 3.23. *We have*

$$S_{\text{eff}}(x) = L(\hbar \log \tfrac{Z(p)}{Z_0}),\ \ \hbar \log \tfrac{Z(p)}{Z_0} = L(S_{\text{eff}}(x)).$$

Physicists formulate this result as follows: the effective action is the Legendre transform of $\hbar$ times the logarithm of the generating function for quantum correlators (and vice versa).

Exercise 3.24. *Compute the 1-loop contribution to* $\log \frac{Z}{Z_0}$ *for*

$$S(x) = \tfrac{x^2}{2} - g(x + \tfrac{x^3}{6}).$$

Using this, compute the number of labeled n-vertex 1-loop graphs with 1-valent and 3-valent vertices only (be careful with double edges and self-loops!). Check your answer by directly enumerating such graphs with small number of vertices.

Exercise 3.25. *Find the exponential generating function* $\sum_n a_n \frac{z^n}{n!}$ *for the numbers a_n of labeled n-vertex trees with 1-valent and 4-valent vertices. You may express the answer via inverse functions to polynomials.*

Exercise 3.26. *Find the one-loop contribution to the effective action for* $S(x) = \frac{x^2}{2} - \frac{gx^3}{6}$*. That is, one has* $S_{\text{eff}} = S + \hbar S_1 + O(\hbar^2)$*, and you need to find S_1. Which Feynman diagrams need to be considered?*

Exercise 3.27. *Consider the heat equation* $u_t = \frac{1}{2}\Delta_B u$*, where Δ_B is the Laplace operator attached to B defined in Subsection 2.2. It is solved by the heat flow* $u(x,t) = e^{\frac{t\Delta_B}{2}} u(x,0)$*. Show that the effective action S_{eff} for the action* $S(x) = \frac{B(x,x)}{2} - \widetilde{S}(x)$ *can be computed as the sum of contributions of 1PI Feynman diagrams without self-loops for the action* $S^\circ(x) := \frac{B(x,x)}{2} - \widetilde{S}^\circ(x)$ *where* $\widetilde{S}^\circ(x) := e^{\frac{\hbar\Delta_B}{2}} \widetilde{S}(x)$ *obtained by transforming $\widetilde{S}$ by the heat flow for time $\hbar$.*