4. Matrix integrals

Let $\mathfrak{h}_N$ be the space of Hermitian matrices of size N. The inner product on $\mathfrak{h}_N$ is given by $B(A_1, A_2) = \mathrm{Tr}(A_1 A_2)$. In this section we will consider integrals of the form

$$Z_N := \hbar^{-\frac{N^2}{2}} \int_{\mathfrak{h}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where the Lebesgue measure dA is normalized by the condition

$$\int_{\mathfrak{h}_N} e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = 1$$

(so we don't have to drag around the $\sqrt{2\pi}$-factors), and

$$S(A) := \frac{\mathrm{Tr}(A^2)}{2} - \sum_{m \geq 1} g_m \frac{\mathrm{Tr}(A^m)}{m}$$

is the action functional.[10] We will be interested in the behavior of the coefficients of the expansion of Z_N in g_i for large N. The study of this behavior will lead us to considering not simply Feynman graphs, but actually fat (or ribbon) graphs, which are in fact 2-dimensional surfaces. Thus, before we proceed further, we need to do some 2-dimensional combinatorial topology.

4.1. **Fat graphs.** Recall from the proof of Feynman's theorem that given a finite collection of flowers and a matching σ on the set T of endpoints of their edges, we can obtain a graph Γ_σ by connecting (or gluing) the points which fall into the same pair.

Now, given an i-flower, let us inscribe it in a closed disk D (so that the ends of the edges are on the boundary). Then take its small tubular neighborhood in D. This produces a region with piecewise smooth boundary. We will equip this region and its boundary with the standard orientation, and call it a *fat i-valent flower*. The boundary of a fat i-valent flower has the form $P_1Q_1P_2Q_2 \dots P_iQ_iP_1$, where P_i, Q_i are the angle points, the intervals P_jQ_j are arcs on ∂D, and Q_jP_{j+1} are (smooth) arcs lying inside D (see Fig. 10).

Now, given a collection of usual flowers and a matching σ as above, we can consider the corresponding fat flowers, and glue them, respecting the orientation, along intervals P_jQ_j according to σ. This will produce a compact oriented surface with boundary (the boundary is glued from intervals Q_jP_{j+1}). We will denote this surface by $\widetilde{\Gamma}_\sigma$, and

[10] Note that we divide by m and not by $m!$. We will see below why such normalization will be more convenient.

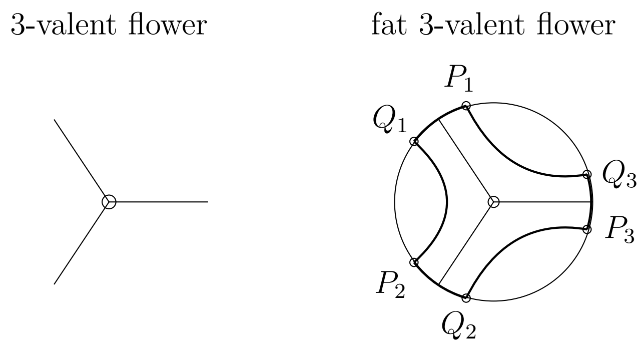


FIGURE 10.

call it the *fattening* of Γ with respect to σ. A fattening of a graph will be called a *fat (or ribbon) graph.*

Thus, a fat graph is not just an oriented surface with boundary, but such a surface together with a partition of this surface into fat flowers.

Note that the same graph Γ can have many fattenings which are non-homeomorphic (albeit homotopy equivalent) surfaces, and in particular the genus g of the fattening is *not* determined by Γ (see Fig. 11).

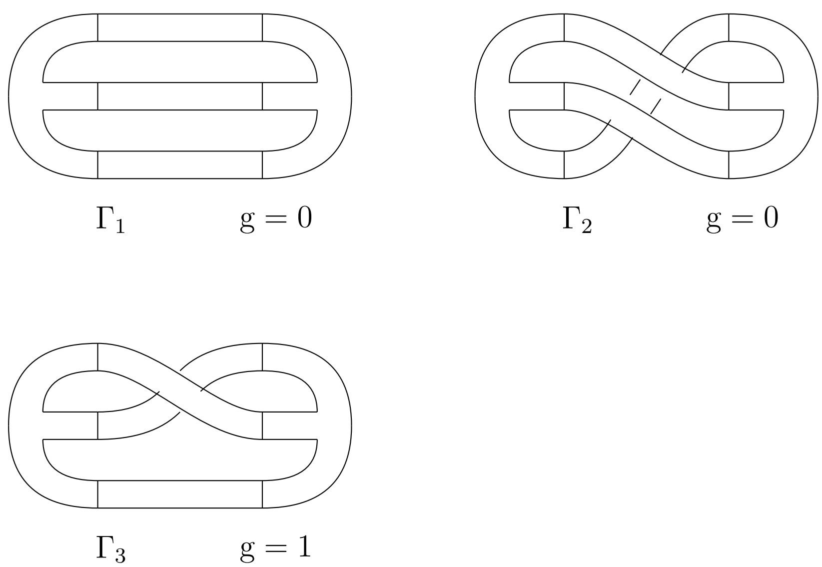


FIGURE 11. Gluing a fat graph from fat flowers

4.2. **Matrix integrals in large N limit, planar graphs, and the genus expansion.** Let us now return to the study of the integral Z_N. We have

$$B_m(A, ..., A) = (m-1)!\mathrm{Tr}(A^m).$$

Thus by Feynman's theorem,

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{(g_i \hbar^{\frac{i}{2}-1})^{n_i}}{i!^{n_i} n_i!} \sum_{\sigma \in \Pi_c(T_{\mathbf{n}})} \mathbb{F}(\sigma),$$

where the summation is taken over the set $\Pi_c(T_{\mathbf{n}})$ of all matchings of $T = T_{\mathbf{n}}$ that produce a connected graph Γ_σ, and $\mathbb{F}(\sigma)$ denotes the contraction of the tensors $(m-1)!\mathrm{Tr}(A^m)$ using σ. So let us compute $\mathbb{F}(\sigma)$.

Let $\{e_i\}$ be the standard basis of $\mathbb{C}^N$, and $\{e_i^*\}$ the dual basis of the dual space. Then the tensor $\mathrm{Tr}(A^m)$ can be written as

$$\mathrm{Tr}(A^m) = \sum_{i_1,\ldots,i_m=1}^{N} (e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}).$$

Thus

$$B_m = \sum_{s \in S_{m-1}} \sum_{i_1,\ldots,i_m=1}^{N} s(e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*)$$

(sum over all possible cyclic orderings of edges of an m-valent flower). Hence

$$\mathbb{F}(\sigma) = \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma),$$

where $\widetilde{\mathbb{F}}(\sigma)$ is obtained by contracting the tensors

$$\sum_{i_1,\ldots,i_m=1}^{N} e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^* \tag{4.1}$$

according to the fat graph $\widetilde{\Gamma}_\sigma$. It follows that

$$\log Z_N = \sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i!^{n_i} n_i!} \sum_{\sigma \in \Pi(T_{\mathbf{n}})} \sum_{s \in \prod_i S_{i-1}^{n_i}} \widetilde{\mathbb{F}}(s\sigma) =$$

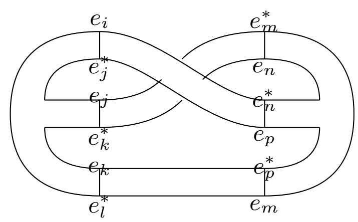
$$\sum_{\mathbf{n}} \prod_i \frac{g_i^{n_i} \hbar^{n_i(\frac{i}{2}-1)}}{i^{n_i} n_i!} \sum_{\sigma} \widetilde{\mathbb{F}}(\sigma)$$

(the product $\prod_i i!^{n_i}$ in the denominator got replaced by $\prod_i i^{n_i}$ since in the sum $\sum_{s,\sigma} \widetilde{\mathbb{F}}(s\sigma)$ every term $\widetilde{\mathbb{F}}(\sigma)$ occurs $|\prod_i S_{i-1}^{n_i}| = \prod_i (i-1)!^{n_i}$ times).

For a surface Σ with boundary, let $\nu(\Sigma)$ denote the number of connected components of the boundary.

Proposition 4.1. $\widetilde{\mathbb{F}}(\sigma) = N^{\nu(\widetilde{\Gamma}_\sigma)}$.

Proof. One can visualize each summand in the sum (4.1) as a labeling of the angle points $P_1, Q_1, \ldots, P_m, Q_m$ on the boundary of a fat m-valent flower by $i_1, i_2, i_2, i_3, \ldots, i_m, i_1$. Now, the contraction using σ of some set of such monomials is nonzero iff the subscript is constant along each boundary component of $\widetilde{\Gamma}_\sigma$ (see Fig. 12). This implies the result. $\square$



Contraction nonzero iff
$i = r$, $j = p$, $j = m$, $k = r$,
$k = p$, $i = m$,
that is
$i = r = k = p = j = m$.

FIGURE 12. Contraction defined by a fat graph.

Let $\widetilde{G}_c(\mathbf{n})$ be the set of isomorphism classes of connected fat graphs with n_i i-valent vertices for $i \geq 1$. For $\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})$, let $b(\widetilde{\Gamma})$ be the number of edges minus the number of vertices of the underlying usual graph Γ.

Corollary 4.2.

$$\log Z_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|} =$$

$$\sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\widetilde{\Gamma})} \hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Proof. Let $\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}} := \prod_i (S_{n_i} \ltimes (\mathbb{Z}/i\mathbb{Z})^{n_i})$. This group acts on $T_{\mathbf{n}}$, so that $\widetilde{\Gamma}_\sigma = \widetilde{\Gamma}_{g\sigma}$, for any $g \in \mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}$. Moreover, the group acts transitively on the set of σ giving a fixed fat graph $\widetilde{\Gamma}_\sigma$, and the stabilizer of any σ is $\mathrm{Aut}(\widetilde{\Gamma}_\sigma)$. This implies the result, as $|\mathbb{G}_{\mathbf{n}}^{\mathrm{cyc}}| = \prod_i i^{n_i} n_i!$ which cancels the denominators. $\square$

Now for any compact connected surface Σ with boundary, let $\mathrm{g}(\Sigma)$ be the genus of Σ. Then for a connected fat graph $\widetilde{\Gamma}$,

$$b(\widetilde{\Gamma}) = 2\mathrm{g}(\widetilde{\Gamma}) - 2 + \nu(\widetilde{\Gamma})$$

(minus the Euler characteristic). Thus, defining

$$\widehat{Z}_N(\hbar) := Z_N(\tfrac{\hbar}{N}),$$

we obtain

Theorem 4.3.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

This implies the following important result, due to t'Hooft.

Theorem 4.4. *(1) There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c(\mathbf{n})[0]$ *denotes the set of* **planar** *connected fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g} \in \mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected fat graphs of genus* g.

Remark 4.5. Genus zero fat graphs are said to be planar because the underlying usual graphs can be put on the 2-sphere (and hence on the plane) without self-intersections.

Remark 4.6. t'Hooft's theorem may be interpreted in terms of the usual Feynman diagram expansion. Namely, it implies that for large N, the leading contribution to $\log Z_N(\frac{\hbar}{N})$ comes from the terms in the Feynman diagram expansion corresponding to planar graphs (i.e. those that admit an embedding into the 2-sphere).

4.3. **Integration over real symmetric matrices.** One may also consider the matrix integral over the space $\mathfrak{s}_N$ of real symmetric matrices of size N. Namely, one puts

$$Z_N = \hbar^{-\frac{N(N+1)}{4}} \int_{\mathfrak{s}_N} e^{-\frac{S(A)}{\hbar}} dA,$$

where S and dA are as above. Let us generalize Theorem 4.4 to this case.

As before, consideration of the large N limit leads to consideration of fat flowers and gluing of them. However, the exact nature of gluing is now somewhat different. Namely, in the Hermitian case we had $(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il}\delta_{jk}$, which forced us to glue fat flowers preserving orientation. On the other hand, in the real symmetric case $e_i^* = e_i$, and the inner product of the functionals $e_i \otimes e_j$ on the space of symmetric matrices is given by $(e_i \otimes e_j, e_k \otimes e_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. This means that besides the usual (orientation preserving) gluing of fat flowers, we now must allow gluing with a twist of the ribbon by 180°. Fat graphs thus obtained will be called *twisted fat graphs*. That means, a twisted fat graph is a surface with boundary (possibly not orientable), together with a partition into fat flowers, and orientations on each of them (which may or may not match at the cuts, see Fig.13).

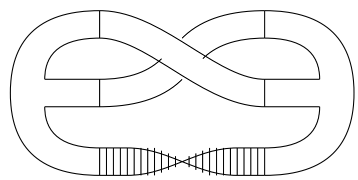


FIGURE 13. Twisted fat graph

Now one can show analogously to the Hermitian case that the $\frac{1}{N}$ expansion of $\log \widehat{Z}_N$ (where $\widehat{Z}_N := Z_N(\frac{2\hbar}{N})$) is given by the same formula as before, but with summation over the set $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})$ of twisted fat graphs:

Theorem 4.7.

$$\log \widehat{Z}_N = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Here the genus g of a (possibly non-orientable) surface is defined for closed surfaces by $\mathrm{g} := 1 - \frac{\chi}{2}$, where χ is the Euler characteristic. Thus the genus of $\mathbb{RP}^2$ is $\frac{1}{2}$, the genus of the Klein bottle is 1, and so on.

In particular, we have the following analog of t'Hooft's theorem.

Theorem 4.8. *(1) There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\log \widehat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[0]$ *denotes the set of* **planar** *connected twisted fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion

$$\frac{\log \widehat{Z}_N}{N^2} = \sum_{\mathrm{g}\in\frac{1}{2}\mathbb{Z}_{\geq 0}} a_{\mathrm{g}} N^{-2\mathrm{g}},$$

where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} \prod_i (g_i \hbar^{\frac{i}{2}-1})^{n_i} \sum_{\widetilde{\Gamma}\in\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]} \frac{1}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c^{\mathrm{tw}}(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected twisted fat graphs which have genus* g*.*

Exercise 4.9. *Consider the matrix integral over the space* $\mathfrak{q}_N$ *of quaternionic Hermitian matrices of size* N*. Show that in this case the results are the same as in the real case, except that each twisted fat graph counts with a sign equal to* $(-1)^\nu$*, where* ν *is the number of boundary components. In other words,* $\log Z_N^{\mathrm{quat}}(\hbar)$ *equals* $\log Z_{2N}^{\mathrm{real}}(\hbar)$ *with* N *replaced by* $-N$*.*

Hint: *Use that the quaternionic unitary group* $U(N,\mathbb{H})$ *is a real form of* $Sp(2N)$*, and* $\mathfrak{q}_N$ *is a real form of the representation of* $\Lambda^2 V$*, where* V *is the standard (vector) representation of* $Sp(2N)$*. Compare to the case of real symmetric matrices, where the relevant representation is* S^2V *for* $O(N)$*, and the case of complex Hermitian matrices, where it is* $V\otimes V^*$ *for* $GL(N)$*.*

4.4. **The number of ways to glue a surface from a polygon and the Wigner semicircle law.** Matrix integrals are so rich that even the simplest possible example reduces to a nontrivial counting problem. Namely, consider the matrix integral Z_N over complex Hermitian matrices with $\hbar = 1$ in the case $S(A) = \frac{\mathrm{Tr}(A^2)}{2} - s\frac{\mathrm{Tr}(A^{2m})}{2m}$, where $s^2 = 0$ (i.e. we work over the ring $\mathbb{C}[s]/(s^2)$). Then from Theorem 4.4 we get

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA = P_m(N),$$

where $P_m(N)$ is a polynomial given by the formula

$$P_m(N) = \sum_{\mathrm{g}\geq 0} \varepsilon_{\mathrm{g}}(m) N^{m+1-2\mathrm{g}},$$

and $\varepsilon_{\mathrm{g}}(m)$ is the number of ways to glue a surface of genus g from a $2m$-gon with labeled sides, i.e., to match the sides and then glue the matching ones to each other in an orientation-preserving manner. Indeed, in this case we have only one fat flower of valency $2m$, which has to be glued with itself; so a direct application of our Feynman rules leads to counting ways to glue a surface of a given genus from a polygon.

The value of this integral is given by the following non-trivial theorem.

Theorem 4.10. *(Harer-Zagier,* [HZ] *1986)*

$$P_m(x) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} \binom{m}{p} 2^p \frac{x(x-1)\dots(x-p)}{(p+1)!}.$$

The theorem is proved in the next subsections.

Looking at the leading coefficient of P_m, we get

Corollary 4.11. *The number of ways to glue a sphere from a $2m$-gon is the Catalan number $C_m = \frac{(2m)!}{m!(m+1)!} = \frac{1}{m+1}\binom{2m}{m}$.*

Corollary 4.11 actually has another (elementary combinatorial) proof, which is as follows. For each matching σ on the set of sides of the $2m$-gon, let us connect the midpoints of the matched sides by straight lines (Fig.14). It is geometrically evident that if these lines don't intersect then the gluing will give a sphere. We claim that the converse is true as well. Indeed, assume the contrary, i.e. that for cyclically ordered edges a, b, c, d, the edge a connects to c and b to d. Then it is easy to see that gluing these two pairs of edges gives a torus with a hole (or without if $m = 2$). But an (open) torus with a hole can't be embedded into a sphere (e.g. it contains a copy of K_5), contradiction.

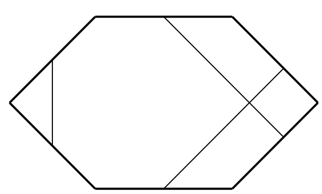


FIGURE 14. Matching of sides of a 6-gon.

Now it remains to count the number of ways to connect midpoints of sides with lines without intersections. Suppose we draw one such line, such that the number of sides on the left of it is $2k$ and on the right is $2l$ (so that $k + l = m - 1$). Then we face the problem of connecting the two sets of $2k$ and $2l$ sides without intersections. This shows that the number of gluings D_m satisfies the recursion

$$D_m = \sum_{k+l=m-1} D_k D_l, \ D_0 = 1.$$

In other words, the generating function

$$h(x) := \sum_m D_m x^m = 1 + x + \cdots$$

satisfies the equation $h(x) - 1 = xh(x)^2$. This implies that

$$h(x) = \frac{1 - \sqrt{1-4x}}{2x},$$

which yields that $D_m = C_m$. We are done.

Corollary 4.11 can be used to derive the following fundamental result from the theory of random matrices, discovered by Wigner in 1955.

Theorem 4.12. *(Wigner's semicircle law) Let f be a continuous function on $\mathbb{R}$ of at most polynomial growth at infinity. Then*

$$\lim_{N\to\infty} \frac{1}{N} \int_{\mathfrak{h}_N} \mathrm{Tr} f(\tfrac{A}{\sqrt{N}}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} = \frac{1}{2\pi} \int_{-2}^{2} f(x)\sqrt{4-x^2}dx.$$

This theorem is called the semicircle law because it says that the graph of the density of eigenvalues of a large random Hermitian matrix distributed according to the "Gaussian unitary ensemble" (i.e. with density $e^{-\frac{\mathrm{Tr}(A^2)}{2}}dA$) is a semicircle. In particular, we see that for large N almost all eigenvalues of A belong to the interval $[-2\sqrt{N}, 2\sqrt{N}]$, so the limit does not depend on the values of f outside $[-2, 2]$.

Proof. By Weierstrass' theorem on uniform approximation of a continuous function on an interval by polynomials, we may assume that f is a polynomial. (Exercise: Justify this step). Thus, it suffices to check the result if $f(x) = x^{2m}$. In this case, by Corollary 4.11, the left hand side is C_m. On the other hand, an elementary computation yields

$$\frac{1}{2\pi} \int_{-2}^{2} x^{2m}\sqrt{4-x^2}dx = C_m,$$

which implies the theorem. □

4.5. **Hermite polynomials.** The proof[11] of Theorem 4.10 given below uses Hermite polynomials. So let us recall their properties.

Hermite polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

So the leading term of $H_n(x)$ is $(2x)^n$.

We collect the standard properties of $H_n(x)$ in the following theorem.

[11] I adopted this proof from D.Jackson's notes.

Theorem 4.13. *(i) The exponential generating function of $H_n(x)$ is*

$$f(x,t) = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

(ii) $H_n(x)$ satisfy the differential equation $f'' - 2xf' + 2nf = 0$. In other words, $H_n(x)e^{-x^2/2}$ are eigenfunctions of the operator $L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$ (Hamiltonian of the quantum harmonic oscillator) with eigenvalues $n + \frac{1}{2}$.

(iii) $H_n(x)$ are orthogonal:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\delta_{mn}.$$

Moreover, the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal basis of $L^2(\mathbb{R})$.

(iv) One has

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x)dx = \frac{(2m)!}{(m-k)!}2^{2(k-m)}$$

(if $k > m$, the answer is zero).

(v) One has

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^{r} \frac{r!}{2^k k!^2 (r-k)!} H_{2k}(x).$$

Proof. (sketch) (i) Follows immediately from the fact that the operator $\sum_{n\geq 0}(-1)^n \frac{t^n}{n!}\frac{d^n}{dx^n}$ maps a function $g(x)$ to $g(x-t)$.

(ii) Follows from (i) and the fact that the function $f(x,t)$ satisfies the PDE

$$f_{xx} - 2xf_x + 2tf_t = 0.$$

(iii) The orthogonality follows from (i) by direct integration:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)e^{-x^2}dx = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2ut-(x-u-t)^2}dx = e^{2ut}.$$

Thus the functions $H_n(x)e^{-\frac{x^2}{2}}$ form an orthogonal system in $L^2(\mathbb{R})$.

To show that these functions are complete, denote by $E \subset L^2(\mathbb{R})$ the closure of their span $\mathbb{C}[x]e^{-\frac{x^2}{2}}$. By approximating the function e^{ipx} by its Taylor polynomials, it is easy to see that $e^{ipx-\frac{x^2}{2}} \in E$ for any $p \in \mathbb{R}$. Thus for any compactly supported smooth $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\phi(x)e^{-\frac{x^2}{2}} = \int_{\mathbb{R}} \widehat{\phi}(p)e^{ipx-\frac{x^2}{2}}dp \in E.$$

where $\widehat{\phi}$ is the (suitably normalized) Fourier transform of ϕ. In other words, $C_0^\infty(\mathbb{R})$ is dense in E. But $C_0^\infty(\mathbb{R})$ is clearly dense in $L^2(\mathbb{R})$, so $E = L^2(\mathbb{R})$, as claimed.

(iv) By (i), one should calculate $\int_{\mathbb{R}} x^{2m} e^{2xt-t^2} e^{-x^2} dx$. This integral equals

$$\int_{\mathbb{R}} x^{2m} e^{-(x-t)^2} dx = \int_{\mathbb{R}} (y+t)^{2m} e^{-y^2} dy == \sqrt{\pi} \sum_p \binom{2m}{2p} \frac{(2m-2p)!}{2^{m-p}(m-p)!} t^{2p}.$$

The result is now obtained by extracting individual coefficients.

(v) By (iii), it suffices to show that

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_r^2(x) H_{2k}(x) e^{-x^2} dx = \frac{2^{r+k} r!^2 (2k)!}{k!^2 (r-k)!}$$

To prove this identity, let us integrate the product of three generating functions. By (i), we have

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x,t) f(x,u) f(x,v) e^{-x^2} dx =$$

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2(ut+uv+tv)-(x-u-t-v)^2} dx = e^{2(ut+tv+uv)}.$$

Extracting the coefficient of $t^r u^r v^{2k}$, we get the result. □

4.6. **Proof of Theorem 4.10.** We need to compute the integral

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\frac{\mathrm{Tr}(A^2)}{2}} dA.$$

To do this, we note that the integrand is invariant with respect to conjugation by unitary matrices. Therefore, the integral can be reduced to an integral over the eigenvalues $\lambda_1, \dots, \lambda_N$ of A.

More precisely, consider the spectrum map $\sigma : \mathfrak{h}_N \to \mathbb{R}^N / S_N$. It is well known (due to H.Weyl) that the direct image $\sigma_* dA$ is given by the formula $\sigma_* dA = C e^{-\sum_i \frac{\lambda_i^2}{2}} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda$, where $C > 0$ is a normalization constant that will not be relevant to us. Thus, we have

$$P_m(N) = \frac{N J_m}{J_0}, \; J_m := \int_{\mathbb{R}^N} (\frac{1}{N} \sum_i \lambda_i^{2m}) e^{-\sum_i \frac{\lambda_i^2}{2}} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda.$$

To calculate J_m, we will use Hermite polynomials. Observe that since $H_n(x)$ are polynomials of degree n with highest coefficient 2^n, we have

$$\prod_{i<j} (\lambda_i - \lambda_j) = 2^{-\frac{N(N-1)}{2}} \det(H_k(\lambda_\ell)),$$

where k runs through the set $0, 1, \ldots, N-1$ and ℓ through $1, ..., N$. Thus, we find

(4.2)
$$J_m = 2^{m+\frac{N^2}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \det(H_k(\lambda_j))^2 d\lambda =$$
$$2^{m-\frac{N(N-2)}{2}} \sum_{\sigma,\tau \in S_N} (-1)^\sigma (-1)^\tau \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma(i)}(\lambda_i) H_{\tau(i)}(\lambda_i) d\lambda.$$

(Here $(-1)^\sigma$ denotes the sign of σ).

Since Hermite polynomials are orthogonal, the only terms in this sum which are nonzero are the terms with $\sigma(i) = \tau(i)$ for $i = 2, \ldots, N$. That is, the nonzero terms have $\sigma = \tau$. Thus, we have

$$J_m = 2^{m-\frac{N(N-2)}{2}} \sum_{\sigma \in S_N} \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum_i \lambda_i^2} \prod_i H_{\sigma i}(\lambda_i)^2 d\lambda =$$

(4.3)
$$2^{m-\frac{N(N-2)}{2}} (N-1)! \gamma_0 \ldots \gamma_{N-1} \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx,$$

where $\gamma_i := \int_{-\infty}^{\infty} H_i(x)^2 e^{-x^2} dx$ are the squared norms of the Hermite polynomials. Applying this for $m = 0$ and dividing NJ_m by J_0, we find

$$P_m(N) = 2^m \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx.$$

Using Theorem 4.13 (iii) and (v), we find that $\gamma_i = 2^i i! \sqrt{\pi}$, and hence

$$P_m(N) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^m x^{2m} H_{2k}(x)}{2^k k!^2 (j-k)!} e^{-x^2} dx.$$

Now, using Theorem 4.13 (iv), we get

$$P_m(N) = \frac{(2m)!}{2^m} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^k j!}{(m-k)! k!^2 (j-k)!} =$$

$$\frac{(2m)!}{2^m m!} \sum_{j=0}^{N-1} \sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k}.$$

The sum over k can be represented as the constant term of a polynomial:

$$\sum_{k=0}^{j} 2^k \binom{m}{k}\binom{j}{k} = C.T.((1+z)^m(1+2z^{-1})^j).$$

Therefore, summation over j (using the formula for the sum of the geometric progression) yields

$$P_m(N) = \frac{(2m)!}{2^m m!} C.T.\left((1+z)^m \frac{(1+2z^{-1})^N - 1}{2z^{-1}}\right) =$$

$$\frac{(2m)!}{2^m m!} \sum_{p=0}^{m} 2^p \binom{m}{p}\binom{N}{p+1}.$$

We are done.

Exercise 4.14. *Find the number of ways to glue an orientable surface of genus* $\mathrm{g} \geq 1$ *from a* $4\mathrm{g}$*-gon (the gluing must preserve orientation), and prove your answer.*

Answer: $\frac{(4\mathrm{g}-1)!!}{2\mathrm{g}+1}$.

Exercise 4.15. *Consider a random Hermitian matrix* $A \in \mathfrak{h}_N$*, distributed with Gaussian density* $e^{-\mathrm{Tr}(A^2)}dA$*. Show that the most likely eigenvalues of* A *are the roots of the* N*-th Hermite polynomial* H_N*.*

Hint. *1) Write down the system of algebraic equations for the maximum of the density on eigenvalues.*

2) Introduce the polynomial $P(z) = \prod_i (z - \lambda_i)$*, where* λ_i *are the most likely eigenvalues. Let* $f = P'/P$*. Compute* $f' + f^2$ *(look at the poles).*

3) Reduce the obtained Riccati equation for f *to a second order linear differential equation for* P*. Show that this equation is the Hermite's equation, and deduce that* $P = \frac{H_N}{2^N}$*.*