7. Quantum mechanics

So far we have considered quantum field theory with 0-dimensional spacetime (to make a joke, one may say that the dimension of the space is -1). In this section, we will move closer to actual physics: we will consider 1-dimensional spacetime, i.e. the dimension of the space is 0. This does not mean that we will study motion in a 0-dimensional space (which would be really a pity) but just means that we will consider only point-like quantum objects (particles) and not extended quantum objects (fields). In other words, we will be in the realm of quantum mechanics.

7.1. **The path integral in quantum mechanics.** Let $U(q)$ be a smooth function on the real line (the potential). We will assume that $U(0) = 0$, $U'(0) = 0$, and $U''(0) = m^2$, where $m > 0$.

Remark 7.1. In quantum field theory the parameter m in the potential is called the *mass* parameter. To be more precise, in classical mechanics it has the meaning of frequency ω of oscillations. However, in quantum theory thanks to Einstein frequency is identified with energy ($E = \hbar\omega/2\pi$), while in relativisitic theory energy is identified with mass (again thanks to Einstein, $E = mc^2$).

We want to construct the theory of a quantum particle moving in the potential field $U(q)$. According to what we discussed before, this means that we want to give sense to and to evaluate the normalized correlation functions

$$\langle q(t_1) \dots q(t_n) \rangle := \frac{\int q(t_1) \dots q(t_n) e^{\frac{iS(q)}{\hbar}} Dq}{\int e^{\frac{iS(q)}{\hbar}} Dq},$$

where $S(q) = \int \mathcal{L}(q) dt$, and $\mathcal{L}(q) = \frac{\dot{q}^2}{2} - U(q)$.

As we discussed, such integrals cannot be handled rigorously by means of measure theory if $\hbar$ is a positive number; so we will only define these path integrals "in perturbation theory", i.e. as formal series in $\hbar$.

Before giving this (fully rigorous) definition, we will explain the motivation behind it. We warn the reader that this explanation is heuristic and involves steps which are mathematically non-rigorous (or "formal" in the language of physicists).

7.2. **Wick rotation.** In Section 1 we discussed path integrals with imaginary exponential (quantum mechanics), as well as real exponential (Brownian motion). If $\hbar$ is a number, then the integrals with imaginary exponential cannot be defined measure-theoretically. Therefore, people study integrals with real exponential (which can be rigorously defined), and then perform a special analytic continuation procedure called the *Wick rotation*.

In our formal setting ($\hbar$ is a formal parameter), one can actually define the integrals in both the real and the imaginary case. Still, the real case is a bit easier, and thus the Wick rotation is still useful. Besides, the Wick rotation is very important conceptually. Therefore, while it is not technically necessary, we start with introducing the Wick rotation here.

Namely, let us denote $\langle q(t_1)...q(t_n)\rangle$ by $\mathcal{G}_n^M(t_1,...,t_n)$, and "formally" make a change of variable $\tau = it$ in the formula for $\mathcal{G}_n^M(t_1,...,t_n)$. Let $q(t) := q_*(\tau)$. Then, taking into account that $d\tau = idt$, $\frac{dq}{dt} = i\frac{dq_*}{d\tau}$, we get

$$\mathcal{G}_n^M(t_1,...,t_n) = \frac{\int q_*(\tau_1)\dots q_*(\tau_n)e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}{\int e^{-\frac{1}{\hbar}\int(\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*))d\tau}Dq_*}.$$

This shows that

$$\mathcal{G}_n^M(t_1,...,t_n) = \mathcal{G}_n^E(it_1,...,it_n),$$

where

$$\mathcal{G}_n^E(t_1,...,t_n) := \frac{\int q(t_1)\dots q(t_n)e^{-\frac{S_E(q)}{\hbar}}Dq}{\int e^{-\frac{S_E(q)}{\hbar}}Dq}.$$

with $S_E(q) = \int \mathcal{L}_E(q)d\tau$, and $\mathcal{L}_E(q) = \frac{\dot{q}^2}{2} + U(q)$ (i.e. $\mathcal{L}_E$ is obtained from $\mathcal{L}$ by replacing U with $-U$).

This manipulation certainly does not make rigorous sense, but it motivates the following definition.

Definition 7.2. The function $\mathcal{G}_n^M(t_1,...,t_n)$ ($t_i \in \mathbb{R}$) is the analytic continuation of the function $\mathcal{G}_n^E(\tau_1,...,\tau_n)$ from the point $(t_1,...,t_n)$ to the point $(it_1,...,it_n)$ along the path $\theta \mapsto e^{i\theta}(t_1,...,t_n)$, $0 \le \theta \le \pi/2$.

Of course, this definition will only make sense if we define the function $\mathcal{G}_n^E(t_1,...,t_n)$ and show that it admits the required analytic continuation. This will be done below.

Remark 7.3. (On the terminology.) The function $\mathcal{G}_n^M(t_1,...,t_n)$ is called the *Minkowskian* (time ordered) correlation function, while the function $\mathcal{G}_n^E(t_1,...,t_n)$ is called the *Euclidean* correlation function (hence the notation). This terminology will be explained later, when we consider relativistic field theory.

From now on, we will mostly deal with Euclidean correlation functions, and therefore will omit the superscript E when there is no danger of confusion.

7.3. **Definition of Euclidean correlation functions.** Now our job is to define the Euclidean correlation functions $\mathcal{G}_n(t_1, ..., t_n)$. Our strategy (which will also be used in field theory) will be as follows. Recall that if our integrals were finite dimensional then by Feynman's theorem the expansion of the correlation functions in $\hbar$ would be given by a sum of amplitudes of Feynman diagrams. So, in the infinite dimensional case, we will use the sum over Feynman diagrams as a *definition* of correlation functions.

More specifically, because of the conditions on U we have an action functional without constant and linear terms in q, so that the correlation function $\mathcal{G}_n(t_1, ..., t_n)$ should be given by the sum

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\Gamma \in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \ldots, \ell_n), \tag{7.1}$$

where $G^*_{\geq 3}(n)$ is defined in Remark 3.7. Thus, we should make sense of (=define) the amplitudes F_Γ in our situation. For this purpose, we need to define the following objects.

1. The space V.
2. The form B on V which defines B^{-1} on V^*.
3. The tensors corresponding to non-quadratic terms in the action.
4. The covectors ℓ_i.

It is clear how to define these objects naturally. Namely, V should be a space of functions on $\mathbb{R}$ with some decay conditions. There are many choices for V, which do not affect the final result; for instance, a good choice (which we will make) is the space $C_0^\infty(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$. Thus V^* is the space of generalized functions on $\mathbb{R}$. Note that V is equipped with the inner product $(f, g) = \int_\mathbb{R} f(x)g(x)dx$.

The form B, by analogy with the finite dimensional case, should be twice the quadratic part of the action. In other words,

$$B(q, q) = \int (\dot{q}^2 + m^2 q^2) dt = (Aq, q),$$

where A is the operator

$$A = -\frac{d^2}{dt^2} + m^2.$$

This means that $B^{-1}(f, f) = (A^{-1}f, f)$.

The operator A^{-1} is an integral operator, with kernel

$$K(x, y) = G(x - y),$$

where $G(x)$ is the Green's function of A, i.e. the fundamental (decaying at infinity) solution of the differential equation

$$(AG)(x) = \delta(x).$$

It is straightforward to find that

$$G(x) = \frac{e^{-m|x|}}{2m}.$$

(thus B^{-1} is actually defined not on the whole V^* but on a dense subspace of V^*).

Remark 7.4. Here we already see the usefulness of the Wick rotation. Namely, the spectrum of A (interpreted as usual as a self-adjoint unbounded operator on $L^2(\mathbb{R})$) is $[m^2, +\infty)$, so it is invertible and the inverse is bounded. However, if we did not make a Wick rotation, we would deal with the operator $A' = -\frac{d^2}{dt^2} - m^2$, whose spectrum is $[-m^2, +\infty)$, i.e. contains 0, so this operator does not have a bounded inverse.

To make sense of the cubic and higher terms in the action as tensors, consider the decomposition of U in the (asymptotic) Taylor series at $x = 0$:

$$U(x) = \frac{m^2x^2}{2} - \sum_{n\geq 3} \frac{g_n x^n}{n!}.$$

This shows that cubic and higher terms in the action have the form

$$B_r(q, q, , ..., q) = \int q^r(t)dt.$$

Thus $B_r(q_1, ..., q_r)$ is an element of $(S^rV)^*$ given by the generalized function $\delta_{t_1=...=t_r}$ (the delta function of the diagonal).

Finally, the functionals ℓ_i are given by $\ell_i(q) = q(t_i)$, so $\ell_i = \delta(t - t_i)$.

This leads to the following Feynman rules of defining the amplitude of a diagram Γ.

1. To the i-th external vertex of Γ assign the number t_i.
2. To each internal vertex j of Γ, assign a variable s_j.
3. On each internal edge connecting vertices j and j', write the Green's function $G(s_j - s_{j'})$.
4. On each external edge connecting i and j write $G(t_i - s_j)$.
5. On each external edge connecting i and i' write $G(t_i - t_{i'})$.
6. Let $G_\Gamma(\mathbf{t}, \mathbf{s})$ be the product of all these functions.
7. Let $F_\Gamma(\ell_1, ..., \ell_n) := \prod_j g_{v(j)} \int G_\Gamma(\mathbf{t}, \mathbf{s})d\mathbf{s}$, where $v(j)$ is the valency of j.

We are finally able to give the following definition.

Definition 7.5. The function $\mathcal{G}_n(t_1, ..., t_n)$ is defined by formula (7.1).

Remark 7.6. Note that the integrals defining F_Γ are convergent since the integrand always decays exponentially at infinity. It is, however, crucial that we consider only graphs without components having no external vertices; for example, if Γ has a single 4-valent vertex connected to itself by two loops (Fig.21) then the amplitude integral involves $\int_{\mathbb{R}} G(0)^2 ds$, which is obviously divergent.

With this definition, the function $\mathcal{G}_n(t_1, ..., t_n)$ is a Laurent series in $\hbar$ whose coefficients are symmetric functions of $t_1, ..., t_n$ given by linear combinations of explicit (and convergent) finite dimensional integrals. Furthermore, it is easy to see that these integrals are in fact computable in elementary functions, i.e. are (in the region $t_1 \geq ... \geq t_n$) linear combinations of products of functions of the form $t_i^r e^{at_i}$. This implies the existence of the analytic continuation required in the Wick rotation procedure.

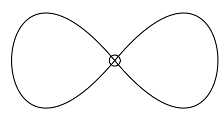

Figure 21.

Remark 7.7. As in the finite dimensional case, an alternative setting for making this definition is to assume that g_i are formal parameters. In this case, $\hbar$ can be given a numerical value, e.g. $\hbar = 1$, and the function $\mathcal{G}_n$ will be a well defined power series in $g_3, g_4, ...$.

As an example consider a free massive theory, i.e., a *harmonic oscillator*: $U(q) = \frac{m^2q^2}{2}$. In this case, there are no internal vertices, hence we get

Proposition 7.8. *(Wick's theorem) One has* $\mathcal{G}_n(t_1, ..., t_n) = 0$ *if* n *is odd, and*

$$\mathcal{G}_{2k}(t_1, ..., t_{2k}) = \hbar^k \sum_{\sigma \in \Pi_k} \prod_{i \in \{1,...,2k\}/\sigma} G(t_i - t_{\sigma(i)}).$$

In particular, $\mathcal{G}_2(t_1, t_2) = \hbar G(t_1 - t_2)$. In other words, $\mathcal{G}_2(t_1, t_2)$ is (proportional to) the Green's function. Motivated by this, physicists often refer to all correlation functions of a quantum field theory as *Green's functions.*

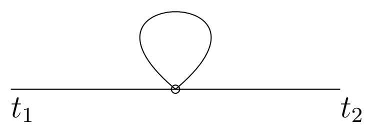

FIGURE 22.

Example 7.9. Consider the potential $U(q) = \frac{m^2q^2}{2} - \frac{gq^4}{24}$, and set $\hbar = 1$. In this case, let us calculate the 2-point correlation function modulo g^2. In other words, we have to compute the coefficient of g in this function. Thus we have to consider Feynman diagrams with two external edges and one internal vertex. Such a diagram Γ is unique: it consists of one edge with a loop attached in the middle (Fig. 22). This diagram has automorphism group $\mathbb{Z}/2$. The amplitude of this diagram is

$$F_\Gamma = g\int_{\mathbb{R}} G(s,t_1)G(s,t_2)G(s,s)ds = \frac{g}{8m^3}\int_{\mathbb{R}} e^{-m(|s-t_1|+|s-t_2|)}ds.$$

Because of symmetry in t_1 and t_2, we may assume that $t_1 \geq t_2$. Splitting the integral in a sum of three integrals, over $(-\infty, t_2]$, $[t_2, t_1]$, and $[t_1, \infty)$, respectively we get:

$$F_\Gamma = \frac{g}{8m^3}\left(2\int_0^\infty e^{-m(2s+|t_1-t_2|)}ds + |t_1-t_2|e^{-m|t_1-t_2|}\right) =$$
$$\frac{g}{8m^4}e^{-m|t_1-t_2|}(1+m|t_1-t_2|).$$

Thus

$$\mathcal{G}_2(t_1,t_2) = \widetilde{G}(t_1-t_2),$$

where

$$\widetilde{G}(t) := \tfrac{1}{2m}e^{-m|t|} + \tfrac{g}{16m^4}e^{-m|t|}(1+m|t|) + O(g^2).$$

This expression is called the *1-loop approximation* to the 2-point function, because it comes from 0-loop and 1-loop Feynman diagrams.

Remark 7.10. Here we are considering quantum mechanics of a single 1-dimensional particle. However, everything generalizes without difficulty to the case of an n-dimensional particle or system of particles (i.e., to path integrals over the space of vector-valued, rather than scalar, functions of one variable). Indeed, if q takes values in a Euclidean space V then the quadratic part of the Lagrangian is of the form $\frac{1}{2}(\dot{q}^2 - M(q))$, where M is a positive definite quadratic form on V. Diagonalizing M, we may assume that the quadratic part of the

Lagrangian looks like $\frac{1}{2}\sum_i(\dot{q_i}^2 - m_i^2 q_i^2)$, which corresponds to a system of independent harmonic oscillators. Thus in quantum theory the propagator will be the diagonal matrix with diagonal entries $\frac{e^{-m_i|t-s|}}{2m_i}$, and the correlation functions can be defined by the usual Feynman diagram procedure.

7.4. **Connected correlation functions.** Let $\mathcal{G}_n^c(t_1, ..., t_n)$ be the *connected correlation (or Green) functions*, defined by the sum of the same amplitudes as $\mathcal{G}_n(t_1, ..., t_n)$ but taken over connected Feynman diagrams only. It is clear that

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\{1,...,n\}=S_1\sqcup...\sqcup S_k} \prod \mathcal{G}^c_{|S_i|}(t_j; j \in S_i).$$

For example, $\mathcal{G}_2(t_1, t_2) = \mathcal{G}_2^c(t_1, t_2) + \mathcal{G}_1^c(t_1)\mathcal{G}_1^c(t_2)$, etc. Thus, to know the correlation functions, it is sufficient to know the connected correlation functions.

Example 7.11. In a free theory ($U = \frac{m^2 q^2}{2}$, the harmonic oscillator), all connected Green's functions except $\mathcal{G}_2$ vanish.

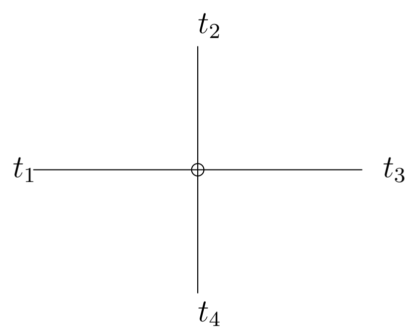

FIGURE 23.

Example 7.12. Let us compute the connected 4-point function in the theory associated to the quartic potential $U = \frac{m^2q^2}{2} - \frac{gq^4}{4}$ as above, modulo g^2. This means, we should compute the contribution of connected Feynman diagrams with one internal vertex and 4 external edges. Such a diagram Γ is unique – it is the cross (with one internal vertex), Fig. 23. This diagram has no nontrivial automorphisms. Thus,

$$\mathcal{G}_4^c(t_1, t_2, t_3, t_4) = g \int_{\mathbb{R}} G(t_1 - s)G(t_2 - s)G(t_3 - s)G(t_4 - s)ds + O(g^2).$$

It is elementary to compute this integral; we leave it as an exercise.

7.5. **The clustering property.** Note that the Green's function $G(t)$ goes to zero at infinity. This implies the following *clustering property* of the correlation functions of the free theory:

$$\lim_{z\to\infty} \mathcal{G}_n(t_1, ..., t_r, t_{r+1} + z, ..., t_n + z) = \mathcal{G}_r(t_1, ..., t_r)\mathcal{G}_{n-r}(t_{r+1}...t_n).$$

Moreover, it is easy to show that the same is true in the interacting theory (i.e. with potential) in each degree with respect to $\hbar$ (check it!). The clustering property can be more simply expressed by the equation

$$\lim_{z\to\infty} \mathcal{G}_n^c(t_1, ..., t_r, t_{r+1} + z, ..., t_n + z) = 0.$$

This property has a physical interpretation: processes distant from each other are almost statistically independent. Thus it can be viewed as a necessary condition of a quantum field theory to be "physically meaningful".

Remark 7.13. Nevertheless, there exist theories (e.g. so called *topological quantum field theories*) which do not satisfy the clustering property but are interesting both form a physical and mathematical point of view (see Subsection 10.2 below).

7.6. **The partition function.** Let $J(t)dt$ be a compactly supported measure on the real line. Consider the "partition function with external current J", which is the formal expression

$$Z(J) = \int e^{\frac{-S_E(q)+(J,q)}{\hbar}} Dq.$$

Then we have a formal equality

$$\frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int_{\mathbb{R}^n} \mathcal{G}_n(t_1, ..., t_n)J(t_1)...J(t_n)dt_1...dt_n,$$

which, as before, we will use as the definition of $Z(J)/Z(0)$. So the knowledge of $Z(J)/Z(0)$ is equivalent to the knowledge of all the Green's functions (in other words, $Z(J)/Z(0)$ is their generating function). Furthermore, as in the finite dimensional case, we have

Proposition 7.14. *One has*

$$W(J) := \log \frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^c(t_1, ..., t_n)J(t_1)...J(t_n)dt_1...dt_n$$

(i.e. W is the generating function of connected Green's functions)

The proof of this proposition is the same as in the finite dimensional case.

Remark 7.15. The statement of the proposition is equivalent to the relation between usual and connected Green's functions given in the previous subsection.

Remark 7.16. The fact that we can only define amplitudes of graphs whose all components have at least one 1-valent vertex (see above) means that we actually cannot define either $Z(0)$ or $Z(J)$ but can only define their ratio $Z(J)/Z(0)$.

Like in the finite dimensional case, we have an expansion

$$W(J) = \hbar^{-1}W_0(J) + W_1(J) + \hbar W_2(J) + ...,$$

where W_j are the j-loop contributions (in particular, W_0 is given by a sum over trees). Furthermore, we have explicit formulas for W_0 and W_1, analogously to the finite dimensional case.

Proposition 7.17. *One has*

$$W_0(J) = -S_E(q_J) + (q_J, J),$$

where q_J is the extremal of the functional $S_E^J(q) := S_E(q) - (q, J)$ which decays at infinity. Furthermore,

$$W_1(J) = -\frac{1}{2}\log\det L_J,$$

where L_J is the linear operator on V such that

$$d^2S_E^J(q_J)(f_1, f_2) = d^2S_E^0(0)(L_Jf_1, f_2).$$

The proof of this proposition, in particular, involves showing that q_J is well defined and that $\det L_J$ exists. It is analogous to the proof of the same result in the finite dimensional case which is given in Subsection 3.7 (to be precise, we gave a proof only in the 0-loop case; but in the 1-loop case, the proof is similar). Therefore we will not give this proof; rather, we will illustrate the statement by an example.

Example 7.18. Let U be the above quartic potential $\frac{m^2q^2}{2} + \frac{gq^4}{2}$ (in which for convenience we change the sign and normalization of the quartic term) and $J(t) = a\delta(t)$. In this case,

$$S_E^J(q) = \int(\tfrac{\dot{q}^2}{2} + U(q))dt - aq(0).$$

The Euler-Lagrange equation has the form

$$\ddot{q} = m^2q + 2gq^3 - a\delta(t).$$

Thus, the function q_J is continuously glued from two solutions q_+, q_- of the nonlinear differential equation

$$\ddot{q} = m^2q + 2gq^3$$

on $(-\infty, 0]$ and $[0, \infty)$, with jump of derivative at 0 equal to $-a$.

The solutions q_+, q_- are required to decay at infinity, so they must be solutions of zero energy:

$$E = \frac{\dot{q_\pm}^2}{2} - U(q_\pm) = 0.$$

Thus, by the standard formula for solutions of Newton's equation, they are defined by the equality

$$t - t_\pm = \int \frac{dq}{\sqrt{2U(q)}} = \int \frac{dq}{mq\sqrt{1+\frac{gq^2}{m^2}}} = \frac{1}{2m} \log \frac{\sqrt{1+\frac{gq^2}{m^2}} - 1}{\sqrt{1+\frac{gq^2}{m^2}} + 1}.$$

After a calculation one gets

$$q_J(t) = \frac{2mg^{-\frac{1}{2}}}{C^{-1}e^{m|t|} - Ce^{-m|t|}},$$

where C is the solution of the equation

$$\frac{C(1+C^2)}{(1-C^2)^2} = \frac{ag^{\frac{1}{2}}}{4m^2}$$

which is given by a power series in a with zero constant term. From this it is elementary (but somewhat lengthy) to compute $W_0 = -S_E^J(q_J)$.

Now, the operator L_J is given by the formula

$$L_J = 1 + \frac{gA^{-1} \circ q_J(t)^2}{2},$$

where $A = -\frac{d^2}{dt^2} + m^2$. Thus $\det L_J$ makes sense. Indeed, the operator $A^{-1} \circ q_J(t)^2$ is an integral operator given by the kernel

$$K_J(x, y) := \frac{e^{-m|x-y|} q_J(y)^2}{2m},$$

which decays exponentially at infinity; hence the determinant of the operator $1 + \frac{gA^{-1} \circ q_J(t)^2}{2}$ is well defined.

Remark 7.19. In these computations, g, a were formal variables, but the above computations in fact make sense for real numerical values of these variables as long as $ga^2 + m^2 > 0$.

7.7. **1-particle irreducible Green's functions.** Let $\mathcal{G}_n^{1PI}(t_1, ..., t_n)$ denote *1-particle irreducible Green's functions*, i.e. those defined by the sum of the same amplitudes as the usual Green's functions, but taken only over 1-particle irreducible Feynman graphs. Define also the *amputated* 1-particle irreducible Green's function: $\mathcal{G}_n^{1PIa} := A^{\otimes n}\mathcal{G}_n^{1PI}$

(it is defined by the same sum of amplitudes, except that instead of $G(t_i - s_j)$ for external edges, we write $\delta(t_i - s_j)$).

Let $S_{\text{eff}}(q)$ be the generating function of $\mathcal{G}_n^{1PIa}$ i.e.,

$$S_{\text{eff}}(q) = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^{1PIa}(t_1, ..., t_n) q(t_1)...q(t_n) dt_1...dt_n.$$

Proposition 7.20. *The function $W(J) = \log(Z(J)/Z(0))$ is the Legendre transform of $S_{\text{eff}}(q)$, i.e. it equals $-S_{\text{eff}}(\widetilde{q}_J) + (J, \widetilde{q}_J)$, where $\widetilde{q}_J$ is the extremal of $-S_{\text{eff}}(q) + (J, q)$ decaying at infinity.*

The proof of this proposition is the same as in the finite dimensional case. The proposition shows that in order to know the Green's functions, it "suffices" to know amputated 1-particle irreducible Green's functions (the generating function of usual Green's functions can be reconstructed from that for 1PI Green's functions by taking the Legendre transform and exponentiation). Which is a good news, since there are a lot fewer 1PI diagrams than general connected diagrams.

7.8. **Momentum space integration.** We saw that the amplitude of a Feynman diagram is given by an integral over the space of dimension equal to the number of internal vertices. This is sometimes inconvenient, since even for tree diagrams such integrals can be rather complicated. However, it turns out that if one passes to Fourier transforms then Feynman integrals simplify and in particular the number of integrations for a connected diagram becomes equal to the number of loops (so for tree diagrams we have no integrations at all).

Namely, we will proceed as follows. Instead of the time variable t we will consider the dual energy variable E. A function $q(t)$ with compact support will be replaced by its Fourier transform $\widehat{q}(E)$. Then, by Plancherel's theorem, for real functions q_1, q_2, we have

$$(q_1, q_2) = \int_{\mathbb{R}} q_1(t) q_2(t) dt = \int_{\mathbb{R}} \widehat{q}_1(E) \overline{\widehat{q}_2(E)} dE = \int_{\mathbb{R}} \widehat{q}_1(E) \widehat{q}_2(-E) dE.$$

This implies that the propagator is given by

$$B^{-1}(f, f) = \int_{\mathbb{R}} \frac{1}{E^2 + m^2} \widehat{f}(E) \widehat{f}(-E) dE.$$

The vertex tensors standing at k-valent vertices were $\delta_{s_1 = ... = s_k}$, so they will be replaced by $\delta_{Q_1 + ... + Q_k = 0}$, where Q_i are dual variables to s_i.

Remark 7.21. (On terminology) Physicists refer to the time variables t_i, s_j as *position variables*, and to energy variables E_i, Q_k as *momentum variables*, since in relativistic mechanics (which is the setting we will deal with when we study field theory) there is no distinction between time and position and between energy and momentum (due to the action of the Lorentz group).

This shows that the Feynman rules "in momentum space" for a given connected Feynman diagram Γ with n external vertices are as follows.

1. Orient the diagram Γ, so that all external edges are oriented inwards.

2. Assign variables E_i to external edges, and variables Q_j to internal ones. These variables are subject to the linear equations of "the first Kirchhoff law": at every internal vertex, the sum of the variables corresponding to the incoming edges equals the sum of those corresponding to the outgoing edges. Let $Y(\mathbf{E})$ be the space of solutions $\mathbf{Q}$ of these equations (it depends on Γ, but we will not write the dependence explicitly). It is easy to show that this space is nonempty only if $\sum_i E_i = 0$, and in that case $\dim Y(\mathbf{E})$ equals the number of loops of Γ (show this!).

3. On each external edge, write $\frac{1}{E_i^2+m^2}$, and on each internal edge, write $\frac{1}{Q_k^2+m^2}$. Let $\phi_\Gamma(\mathbf{E},\mathbf{Q})$ be the product of all these functions.

4. Define the *momentum space amplitude* of Γ to be the distribution $\widehat{F}_\Gamma(\mathbf{E})$:

$$\widehat{F}_\Gamma(E_1,...,E_n) = \prod_j g_{v(j)} \int_{Y(\mathbf{E})} \phi_\Gamma(\mathbf{E},\mathbf{Q})d\mathbf{Q}\cdot\delta(E_1+...+E_n)d\mathbf{E},$$

supported on the hyperplane $\sum_i E_i = 0$. It is clear that this distribution is independent on the orientation of Γ.

Remark 7.22. Here we must specify the normalization of the (translation-invariant) Lebesgue measure $d\mathbf{Q}$ on the space $Y(\mathbf{E})$. It is defined in such a way that the volume of $Y(\mathbf{E})/Y_{\mathbb{Z}}(0)$ is 1, where $Y_{\mathbb{Z}}(0)$ is the set of integer elements in $Y(0)$. So if $T\subset\Gamma$ is a spanning tree then in the coordinates $\{Q_e, e\notin T\}$ on $Y(\mathbf{E})$, we have $d\mathbf{Q}=\prod_{e\notin T} dQ_e$.

Now we have

Proposition 7.23. *The Fourier transform of the function* $F_\Gamma(\delta_{t_1},...,\delta_{t_n})$ *is* $\widehat{F}_\Gamma(E_1,...,E_n)$*. Hence, the Fourier transform of the connected Green's function is*

$$\widehat{\mathcal{G}}_n^c(E_1,...,E_n) = \sum_{\Gamma\in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\widehat{F}_\Gamma(E_1,\ldots,E_n). \quad (7.2)$$

The proof of the proposition is straightforward.

To illustrate the proposition, consider an example.

Example 7.24. The connected 4-point function for the quartic potential modulo g^2 in momentum space looks like:

$$\widehat{\mathcal{G}}_4^c(E_1, E_2, E_3, E_4) = g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta(\sum_i E_i) d\mathbf{E} + O(g^2).$$

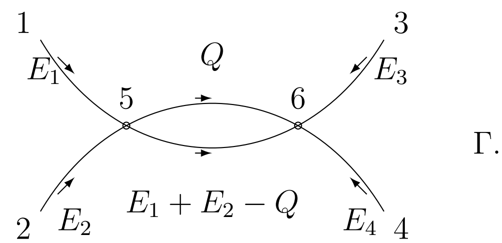

FIGURE 24.

Example 7.25. Let us compute the 1PI 4-point function in the same problem, but now modulo g^3. Thus, in addition to the above, we need to compute the g^2 coefficient, which comes from 1-loop diagrams. There are three such diagrams, differing by permutation of external edges. One of these diagrams is as follows: it has external vertices $1, 2, 3, 4$ and internal ones $5, 6$ such that $1, 2$ are connected to 5, $3, 4$ to 6, and 5 and 6 are connected by two edges (Fig.24). This diagram has the symmetry group $\mathbb{Z}/2$, so its contribution is

$$\frac{g^2}{2} \left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 + m^2)((E_1 + E_2 - Q)^2 + m^2)} \right) \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta(\sum_i E_i) d\mathbf{E}.$$

The integral inside is easy to compute, for example, by residues. This yields

$$\widehat{\mathcal{G}}_4^c(E_1, E_2, E_3, E_4) =$$

$$g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \left(1 + \frac{\pi g}{m} \sum_{i=2}^{4} \frac{1}{(E_1 + E_i)^2 + 4m^2} \right) \delta(\sum_i E_i) d\mathbf{E} + O(g^3)$$

(this is symmetric in the E_1, E_2, E_3, E_4 since when $\sum_i E_i = 0$ then for distinct i, j, k, ℓ one has $(E_i + E_j)^2 = (E_k + E_\ell)^2$).

7.9. **The Wick rotation in momentum space.** To obtain the correlation functions of quantum mechanics, we should, after computing them in the Euclidean setting, Wick rotate them back to the Minkowski setting. Let us do it at the level of Feynman integrals in momentum space. (We could do it in position space as well, but it is instructive for the future to do it in momentum space, since in higher dimensional field theory which we will discuss later, the momentum space representation is more convenient).

Consider the Euclidean propagator

$$\frac{1}{E^2+m^2} = \int_{\mathbb{R}} G(t)e^{iEt}dt,$$

where G is the Green's function. When we do analytic continuation back to the Minkowski setting, we must replace in the correlation functions the time variable t with $e^{i\theta}t$, where θ varies from 0 to $\frac{\pi}{2}$. In particular, the Green's function $G(t)$ must be replaced by $G(e^{i\theta}t)$. So we must consider

$$\int_{\mathbb{R}} G(e^{i\theta}t)e^{iEt}dt = e^{-i\theta}\int_{\mathbb{R}} G(t)e^{ie^{-i\theta}Et}dt = \frac{e^{-i\theta}}{e^{-2i\theta}E^2+m^2}.$$

As $\theta \to \frac{\pi}{2}$, this function tends (as a distribution) to the function $\lim_{\varepsilon\to 0+}\frac{i}{E^2-m^2+i\varepsilon}$. For brevity the limit sign is usually dropped and this distribution is written as $\frac{i}{E^2-m^2+i\varepsilon}$.

We see that in order to compute the correlation functions in momentum space in the Minkowski setting, we should use the same Feynman rules as in the Euclidean setting except that the propagator put on the edges should be

$$\frac{i}{E^2-m^2+i\varepsilon}.$$

For instance, the contribution of the diagram in Fig.24 is

$$-\frac{g^2}{2}\left(\int_{\mathbb{R}}\frac{dQ}{(Q^2-m^2+i\varepsilon)((E_1+E_2-Q)^2-m^2+i\varepsilon)}\right)\prod_{j=1}^{4}\frac{1}{E_j^2-m^2+i\varepsilon}\delta(\sum_i E_j)d\mathbf{E}.$$

7.10. **Quantum mechanics on the circle.** It is reasonable (at least mathematically) to consider Euclidean quantum mechanical path integrals in the case when the time axis has been replaced with a circle of length L, i.e. $t\in\mathbb{R}/L\mathbb{Z}$ (this corresponds to a Brownian particle in a potential field conditioned to return to the original position in a certain time L). In this case, the theory is the same, except the Green's function $G(t)$ is replaced by the periodic solution $G_L(t)$ of the equation $(-\frac{d^2}{dt^2}+m^2)f=\delta(t)$ on the circle. This solution has the form

$$G_L(t) = \sum_{k\in\mathbb{Z}} G(t-kL) = \frac{e^{-m(t-\frac{L}{2})}+e^{-m(\frac{L}{2}-t)}}{2m(e^{\frac{mL}{2}}-e^{-\frac{mL}{2}})},\ 0\le t\le L. \tag{7.3}$$

We note that in the case of a circle, there is no problem with graphs without external edges (as integral over the circle of a constant function is convergent), and hence one may define not only correlation functions (i.e. $Z(J)/Z(0)$), but also $Z(0)$ itself. Namely, let

$$U(q) = \frac{m^2 q^2}{2} + \sum_{n\geq 3} \frac{g_n q^n}{n!},$$

and let $m^2 = m_0^2 + g_2$ (where g_i are formal parameters). Then we can make sense of the ratio $Z_{m_0,\mathbf{g},L}(0)/Z_{m_0,0,L}(0)$ (where $Z_{m,\mathbf{g},L}(0)$ denotes the partition function for the specified values of parameters; from now on the argument 0 will be dropped). Indeed, this ratio is defined by the formula

$$\frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = \sum_{\Gamma\in G_{\geq 2}(0)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma$$

(where $G_{\geq 2}(0)$ is the set of Feynman graphs without external vertices and all vertices of valency ≥ 2), which is a well-defined expression.

It is instructive to compute this expression in the case

$$g_2 = a,\ g_3 = g_4 = ... = 0.$$

In this case, we have only 2-valent vertices, so the only connected Feynman diagrams are N-gons, which are 1-loop. Hence,

$$\log \frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = W_1 = -\frac{1}{2}\log\det M,$$

where

$$M = 1 + a(-\tfrac{d^2}{dt^2} + m_0^2)^{-1}.$$

This determinant may be computed by looking at the eigenvalues. Namely, the eigenfunctions of $-\frac{d^2}{dt^2} + m_0^2$ in the space $C^\infty(\mathbb{R}/L\mathbb{Z})$ are $e^{\frac{2\pi i n t}{L}}$, with eigenvalues $\frac{4\pi^2 n^2}{L^2} + m_0^2$. So,

$$\det M = \prod_{n\in\mathbb{Z}} \left(1 + \frac{a}{\frac{4\pi^2n^2}{L^2} + m_0^2}\right).$$

Hence, using the Euler product formula

$$\sinh(z) = z\prod_{n\geq 1}\left(1 + \frac{z^2}{\pi^2 n^2}\right),$$

we get

$$\frac{Z_{m_0,\mathbf{g},L}}{Z_{m_0,0,L}} = \frac{\sinh(\frac{m_0 L}{2})}{\sinh(\frac{mL}{2})}.$$

(Double-check this using summation over Feynman diagrams!)

Remark 7.26. More informally speaking, we see that the partition function Z for the theory with $U = \frac{m^2q^2}{2}$ has the form $\frac{C}{\sinh(\frac{mL}{2})}$, where C is a constant of our choice. Our choice from now on will be $C = \frac{1}{2}$; we will see later (in Example 8.25) why such a choice is preferable.

7.11. **The massless case.** Consider now the massless case, $m = 0$. In this case the propagator should be obtained by inverting the operator $-\frac{d^2}{dt^2}$, i.e. it should be the integral operator with kernel $G(t-s)$, where $G(t)$ is an even function satisfying the differential equation

$$-G''(t) = \delta(t).$$

There is a 1-parameter family of such solutions,

$$G(t) = -\frac{1}{2}|t| + C.$$

Using this function (for any choice of C), one may define the correlation functions of the free theory by the Wick formula.

Note that the function G does not decay at infinity. Therefore, this theory will not satisfy the clustering property (i.e. is not "physically meaningful").

We will also have difficulties in defining the corresponding interacting theory (i.e. one with a non-quadratic potential), as the integrals defining the amplitudes of Feynman diagrams will diverge. Such divergences are called *infrared divergences*, since they are caused by the failure of the integrand to decay at large times (or, in momentum space, its failure to be regular at low frequencies).

7.12. **Circle-valued quantum mechanics.** Consider now the theory with the same Lagrangian in which $q(t)$ takes values in the circle of radius r, $\mathbb{R}/2\pi r\mathbb{Z}$ (the "sigma-model"). We can do this at least classically, since the Lagrangian $\frac{\dot{q}^2}{2}$ makes sense in this case.

Let us define the corresponding quantum theory. The main difference from the line-valued case is that since $q(t)$ is circle-valued, we should consider not the usual correlators $\langle q(t_1)...q(t_n)\rangle$, but rather correlation functions of exponentials $\langle e^{\frac{ip_1q(t_1)}{r}}...e^{\frac{ip_nq(t_n)}{r}}\rangle$, where p_j are integers. They should be defined by the path integral

$$\int e^{\frac{ip_1q(t_1)}{r}}...e^{\frac{ip_nq(t_n)}{r}}e^{-\frac{S(q)}{\hbar}}Dq, \tag{7.4}$$

where $S(q) := \frac{1}{2}\int \dot{q}^2dt$ and $\int e^{-\frac{S(q)}{\hbar}}Dq$ is agreed to be 1. Note that it suffices to consider only the case $\sum_j p_j = 0$, otherwise the group of translations along the circle acts nontrivially on the integrand, hence under any reasonable definition the integral should be zero.

Now let us define the integral (7.4). Since the integral is invariant under shifts along the target circle, we may as well imagine that we are integrating over $q : \mathbb{R} \to \mathbb{R}$ with $q(0) = 0$. Now let us use the finite-dimensional analogy. Following this analogy, by completing the square we would get

$$\int e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} e^{-\frac{S(q)}{\hbar}} Dq = e^{-\frac{\hbar}{2r^2} B^{-1}(\sum_j p_j q(t_j), \sum_j p_j q(t_j))} =$$

$$e^{-\frac{\hbar}{2r^2}\sum_{j,\ell} p_\ell p_j G(t_\ell - t_j)} = e^{\frac{\hbar}{2r^2}\sum_{\ell<j} p_\ell p_j |t_\ell - t_j|},$$

where $B(q,q) := \int \dot{q}^2 dt$. Thus, it is natural to define the correlators by the formula

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_k)}{r}} \rangle = e^{\frac{\hbar}{2r^2}\sum_{\ell<j} p_\ell p_j |t_l - t_j|}.$$

We note that this theory, unlike the line-valued one, *does* satisfy the clustering property. Indeed, if $\sum p_j = 0$ (as we assumed), then (assuming $t_1 \geq t_2 \geq ... \geq t_n$), we have

$$\sum_{\ell<j} p_\ell p_j (t_\ell - t_j) = \sum_{j=1}^{n-1} (t_j - t_{j+1})(p_{j+1} + ... + p_n)(p_1 + ... + p_j) =$$

$$-\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2,$$

so the clustering property follows from the fact that $(p_1 + ... + p_j)^2 \geq 0$.

7.13. **Massless quantum mechanics on the circle.** Consider now the theory with Lagrangian $\frac{\dot{q}^2}{2}$, where q is a function on the circle of length L. In this case, according to the Feynman yoga, we must invert the operator $-\frac{d^2}{dt^2}$ on the circle $\mathbb{R}/L\mathbb{Z}$, or equivalently solve the differential equation $-G''(t) = \delta(t)$. Here we run into trouble: the operator $-\frac{d^2}{dt^2}$ is not invertible, since it has an eigenfunction 1 with eigenvalue 0; correspondingly, the differential equation in question has no solutions, as $\int G'' dt$ must be zero, so $-G''(t)$ cannot equal $\delta(t)$ (one may say that the quadratic form in the exponential is degenerate, and therefore the Gaussian integral turns out to be meaningless). This problem can be resolved by the following technique of "killing the zero mode". Namely, let us invert the operator $-\frac{d^2}{dt^2}$ on the space $\{q \in C^\infty(\mathbb{R}/L\mathbb{Z}) : \int q dt = 0\}$ (this may be interpreted as integration over this codimension one subspace, on which the quadratic form is non-degenerate). This means that we must find the solution of the differential equation $-G''(t) = \delta(t) - \frac{1}{L}$, such that $\int G dt = 0$. Such solution is indeed unique, and it equals

$$G(t) = \frac{(t-\frac{L}{2})^2}{2L} - \frac{L}{24}, \tag{7.5}$$

$t \in [0, L]$. Thus, for example $\langle q(0)^2 \rangle = \frac{L}{12}$.

Higher correlation functions are defined in the usual way. Moreover, one can define the theory with an arbitrary potential using the standard procedure with Feynman diagrams.

7.14. **Circle-valued quantum mechanics on the circle.** Finally, let us consider the circle-valued version of the same theory. Thus, our integration variable is a map $q : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}/2\pi r\mathbb{Z}$. So we have a new feature - there are different homotopy classes of maps labeled by degree. Let us first consider integration over degree zero maps. Then we should argue in the same way as in the case $t \in \mathbb{R}$, and make the definition

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle_0 := e^{-\frac{\hbar}{2r^2}\sum_{\ell,j} p_\ell p_j G(t_\ell - t_j)},$$

where $\sum_j p_j = 0$. (Here subscript 0 stands for degree zero maps). Assuming that $0 \le t_1, ..., t_n \le L$, we find after a short calculation using (7.5):

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle_0 = e^{\frac{\hbar}{2r^2}(\sum_{\ell<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum_j p_j t_j)^2}{L})}$$

(the second summand disappears as $L \to \infty$, and we recover the answer on the line).

It is, however, more natural (as we will see later) to integrate over all maps q, not only degree zero. Namely, let N be an integer. Then all maps of degree N have the form $q(t) + \frac{2\pi r N t}{L}$, where q is a map of degree zero. Thus, if we want to integrate over maps of degree N, we should compute the same integral as in degree zero, but with shift $q \mapsto q + \frac{2\pi r N t}{L}$. But it is easy to see that this shift results simply in rescaling of the integrand by the factor $e^{\frac{2\pi i N}{L}\sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}$. Thus, the integral over all maps should be defined by the formula

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle =$$

$$e^{\frac{\hbar}{2r^2}(\sum_{l<j} p_\ell p_j |t_\ell - t_j| + \frac{(\sum p_j t_j)^2}{L})} \frac{\sum_{N\in\mathbb{Z}} e^{\frac{2\pi i N}{L}\sum_j p_j t_j - \frac{2\pi^2 r^2 N^2}{\hbar L}}}{\sum_{N\in\mathbb{Z}} e^{-\frac{2\pi^2 r^2 N^2}{\hbar L}}}. \tag{7.6}$$

Introduce the elliptic theta-function

$$\theta(u, T) := \sum_{N\in\mathbb{Z}} e^{2\pi i u N - \pi T N^2}.$$

Then for $L \geq t_1 \geq ... \geq t_n \geq 0$ formula (7.6) can be rewritten in the form

(7.7)

$$\langle e^{\frac{ip_1 q(t_1)}{r}} ... e^{\frac{ip_n q(t_n)}{r}} \rangle = e^{\frac{\hbar}{2r^2}(\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2 + \frac{(\sum_j p_j t_j)^2}{L})} \frac{\theta(\frac{\sum_j p_j t_j}{L}, \frac{2\pi r^2}{\hbar L})}{\theta(0, \frac{2\pi r^2}{\hbar L})}.$$

Exercise 7.27. *Calculate the 1-particle irreducible 2-point function for a quantum particle with potential $U(q) := \frac{m^2 q^2}{2} - \frac{g q^4}{4!}$ modulo g^3 in momentum space, for $\hbar = 1$. (We have done this modulo g^2 in position space).*

Exercise 7.28. *Let $U(q) := \frac{m^2 q^2}{2} - \frac{g q^3}{3}$.*

(i) Calculate the leading term of the 1-point function $\mathcal{G}_1(t)$ (with respect to g).

(ii) Calculate the connected 2-point function modulo g^3.

Exercise 7.29. *Consider the potential $U(x) := \frac{m^2 \sinh^2(gx)}{2g^2}$. Find a formula for $W_0(J)$ (the tree part of $\log(Z(J)/Z(0))$) as explicitly as you can, when $J(t) = a\delta(t)$.*