12. Perturbative expansion for interacting QFT

12.1. **General strategy of quantization.** We now pass to non-free field theories defined by the action $S(\phi) := \int \mathcal{L}(\phi)dx$ in Minkowski space $V \cong \mathbb{R}^d$, where $\mathcal{L}(\phi)$ is a local Poincaré-invariant Lagrangian. The general strategy of quantization of such theories is as follows.

Step 1. Write down the Euclidean path integral correlators for the theory:

$$\langle \phi(x_1)...\phi(x_n) \rangle = \int \phi(x_1)...\phi(x_n) e^{-\frac{S_E(\phi)}{\hbar}} D\phi.$$

Compute the corresponding formal expansion in $\hbar$ using the Feynman rules (as we have done in the case of quantum mechanics, $d = 1$).

Step 2. Perform Borel summation of this formal series, to obtain actual functions defined for small enough $\hbar > 0$.

Step 3. Perform the Wick rotation of these functions to Minkowski space to obtain Wightman correlation functions W_n.

Step 4. Use the functions W_n to define a Wightman QFT, i.e., extract the Hilbert space $\mathcal{H}$, the representation π of the (double cover of the) Poincaré group on $\mathcal{H}$, the vacuum vector Ω and the field map ϕ.

All these steps are non-trivial, and while Step 1 can be performed fully rigorously, starting from Step 2 a rigorous implementation is only known for a handful of theories treated in constructive field theory (and for many Lagrangians the ultimate Wightman QFT, in fact, does not exist). For most physically interesting theories, doing these steps rigorously is still an open problem. In this section, we will only discuss Step 1.

12.2. **The ϕ^3 theory.** As a running example, we will use the theory of a scalar boson ϕ with Euclidean Lagrangian

$$\mathcal{L}_E(\phi) := \frac{1}{2}((d\phi)^2 + m^2\phi^2) + \frac{g}{6}\phi^3,$$

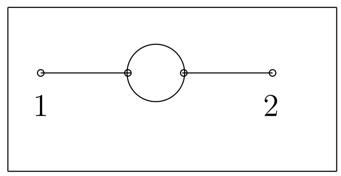
which we will call the ϕ^3-*theory.* This theory is a deformation of the theory of free scalar boson obtained by adding a single interaction term $\frac{g}{6}\phi^3$, which in Feynman calculus corresponds to a 3-valent vertex. Physically this vertex corresponds to an interaction in which two particles collide and transform into a third one.

We will set $\hbar = 1$ and consider the formal expansion in powers of g (which is equivalent to Step 1 by rescaling ϕ).

Let us compute the 1-loop correction to the 2-point correlation function of the free theory

$$\widehat{G}_0(p) = \frac{1}{p^2+m^2}$$

in the momentum space presentation. It is easy to see that this correction is given by a single Feynman diagram



The amplitude of this Feynman diagram is

$$A(p) = \frac{g^2}{2(p^2+m^2)^2}\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)}.$$

If $d < 4$, this integral is convergent and can be computed explicitly. To this end, we may use the following lemma from multivariable calculus, which is known in physics literature as the *Feynman famous formula*:

Lemma 12.1. *Let Δ_n be the $n-1$-dimensional simplex defined in $\mathbb{R}^n$ by the equation*

$$y_1 + ... + y_n = 1,$$

and dy be the Lebesgue measure on Δ_n of volume 1. Then for positive numbers $a_1, ..., a_n$ we have

$$\int_{\Delta_n} \frac{dy}{(a_1y_1+...+a_ny_n)^n} = \frac{1}{a_1...a_n}.$$

Proof. We have

$$\frac{1}{(a_1y_1+...+a_ny_n)^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-(a_1y_1+...+a_ny_n)t}dt.$$

So we get

$$\int_{\Delta_n} \frac{dy}{(a_1y_1+...+a_ny_n)^n} = \frac{1}{(n-1)!}\int_{\Delta_n}\int_0^\infty t^{n-1}e^{-(a_1y_1+...+a_ny_n)t}dtdy$$

$$= \frac{1}{(n-1)!}\int_{t\Delta_n}\int_0^\infty e^{-a_1z_1+...+a_nz_n}dtdz$$

$$= \int_{z_1,\dots,z_n \geq 0} e^{-a_1 z_1 + \dots + a_n z_n} dz = \prod_{j=1}^{n} \int_0^\infty e^{-a_j z_j} dz_j = \frac{1}{a_1 \dots a_n}.$$

□

Applying the Feynman famous formula to our integral and making a change of variable $q \mapsto q + (1-y)p$, we have

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \int_0^1 \int_V \frac{dq}{((1-y)q^2 + y(p-q)^2 + m^2)^2} dy =$$

$$\int_0^1 \int_V \frac{dq}{(q^2 + M^2(y,p))^2} dy,$$

where

$$M^2(y,p) := y(1-y)p^2 + m^2.$$

Now, using spherical coordinates

$$\int_V \frac{dq}{(q^2+M^2)^2} = C_d \int_0^\infty \frac{r^{d-1} dr}{(r^2+M^2)^2},$$

where C_d is the area of the unit sphere in $\mathbb{R}^d$. Thus for $d = 2$

$$\int_V \frac{dq}{(q^2+M^2)^2} = 2\pi \int_0^\infty \frac{r dr}{(r^2+M^2)^2} = \pi \int_0^\infty \frac{ds}{(s+M^2)^2} = \frac{\pi}{M^2}.$$

It follows that

$$\int_V \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} = \pi \int_0^1 \frac{dy}{y(1-y)p^2+m^2}$$

$$= \frac{2\pi}{p^2 \sqrt{\frac{4m^2}{p^2}+1}} \text{arccotanh} \sqrt{\frac{4m^2}{p^2}+1}.$$

The case $d = 3$ can be computed similarly.

However, for $d \geq 4$ we encounter our first difficulty: the integral diverges (as the integrand behaves at infinity as $|q|^{-4}$). More specifically, for a cutoff $\Lambda > 0$, define

$$A_\Lambda(p) := \frac{g^2}{2(p^2+m^2)^2} \int_{|q| \leq \Lambda} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)},$$

the integral over the ball in V of radius Λ. Then

$$A_\Lambda(p) \sim \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}), \ \Lambda \to \infty$$

for $d = 4$ and

$$A_\Lambda(p) \sim C_d \frac{g^2}{2(d-4)(p^2+m^2)^2} \Lambda^{d-4}, \ \Lambda \to \infty$$

if $d > 4$. A way to remedy this difficulty is to add a Λ-dependent term in the Lagrangian, called a *counterterm*, which blows up as $\Lambda \to \infty$ but which will cancel this divergence, in the sense that when integration is performed over the ball $|q| \le \Lambda$ then the integral has a finite limit as $\Lambda \to \infty$.

For example, consider $d = 4$. In this case modulo g^3 the momentum space 2-point function computed with cutoff Λ looks like

$$\widehat{G}_{\Lambda,m^2}(p) = \frac{1}{p^2+m^2} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

(here we explicitly indicate dependence of $\widehat{G}$ on m^2 since we are about to vary it). Let us try to fix the divergence by replacing the parameter m^2 by $m^2 + Kg^2 \log(\frac{\Lambda}{m})$ for a constant K. So we have

$$\widehat{G}_{\Lambda,m^2+Kg^2\log(\frac{\Lambda}{m})}(p) = \frac{1}{p^2+m^2+Kg^2\log(\frac{\Lambda}{m})} + \pi^2 \frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

$$= \frac{1}{p^2+m^2} + (\pi^2 - K)\frac{g^2}{(p^2+m^2)^2} \log(\tfrac{\Lambda}{m}) + ...$$

where we ignore terms of order higher than g^2. Thus to cancel the divergence, we should take $K = \pi^2$, i.e., replace the Lagrangian with

$$\mathcal{L}_{E,\Lambda} := \frac{1}{2}((d\phi)^2 + (m^2 + \pi^2 g^2 \log(\tfrac{\Lambda}{m}))\phi^2) + \frac{g}{6}\phi^3.$$

For this Lagrangian, if integration is performed with cutoff Λ, then the 2-point function modulo g^2 will have a finite limit as $\Lambda \to \infty$, given by

$$\widehat{G}(p) = \frac{1}{p^2+m^2} + \frac{g^2}{2(p^2+m^2)^2} I(p),$$

where

$$I(p) = \lim_{\Lambda\to\infty} \left(\int_{\mathbb{R}^4} \frac{dq}{(q^2+m^2)((p-q)^2+m^2)} - 2\pi^2 \log(\tfrac{\Lambda}{m}) \right).$$

This limit is easy to compute using the Feynman famous formula. Namely, computing similarly to the $d < 4$ case, we get

$$I(p) = \int_0^1 I(p,y)dy,\ I(p,y) := \lim_{\Lambda\to\infty} \left(\int_0^\Lambda \frac{r^3 dr}{(r^2 + M^2(y,p))^2} - 2\pi^2 \log(\tfrac{\Lambda}{m}) \right).$$

So

$$I(p,y) = 2\pi^2(\log m - \frac{1}{2}(1 + \log(y(1-y)p^2 + m^2))).$$

Thus

$$I(p) = 2\pi^2 \left(\frac{1}{2} + \sqrt{\frac{4m^2}{p^2}+1} \cdot \operatorname{arccotanh}\sqrt{\frac{4m^2}{p^2}+1} \right).$$

For $d > 4$ the calculation becomes more elaborate. Namely, while for $d = 5$ we have

$$A_\Lambda(p) \sim C_5 \frac{g^2}{2(p^2+m^2)^2}\Lambda + O(1), \lambda \to \infty,$$

so the procedure is the same, with mass parameter modification $m^2 \mapsto m^2 + K\Lambda$, already for $d = 6$ we will have to take a deeper expansion of the divergent integral:

$$A_\Lambda(p) \sim \frac{g^2}{4(p^2+m^2)^2}(C_6\Lambda^2 + Cp^2\log(\tfrac{\Lambda}{m}) + O(1)),\ \Lambda \to \infty$$

We can cancel the most singular term $C_6\Lambda^2$ by mass modification $m^2 \mapsto m^2 + K\Lambda^2$, but after that we will still have logarithmic divergence, $Cp^2\log(\frac{\Lambda}{m})$, which depends on p. To kill this divergence, we must modify the coefficient of $\frac{1}{2}(d\phi)^2$ in the Lagrangian by a counterterm, changing it from 1 to $1 + C'g^2\log(\frac{\Lambda}{m})$ for an appropriate constant C'. Also we find that the 1-loop correction to the 3-point function is logarithmically divergent: the corresponding contribution in momentum presentation is, up to scaling,

$$\frac{g^3}{\prod_{j=1}^3(p_j^2+m^2)}J(p_1,p_2,p_3)\delta(p_1+p_2+p_3),$$

where

$$J(p_1,p_2,p_3) = \int_V \frac{dq}{(q^2+m^2)((q-p_1)^2+m^2)((q-p_1-p_2)^2+m^2)}$$

for $p_1 + p_2 + p_3 = 0$, which is divergent and behaves like $\log\Lambda$ when computed over the ball of radius Λ. So to kill this divergence, we must change the coefficient of $\frac{1}{6}\phi^3$ in the Lagrangian by a counterterm, changing it from g to $g + C''g^3\log(\frac{\Lambda}{m})$.

We are starting to see the main idea of *renormalization theory*, which allows us to regularize divergent integrals coming from Feynman diagrams in all orders of perturbation series. This idea is that the coefficients of the Lagrangian are actually **not** meaningful physical quantities, — they are just mathematical parameters depending on the scale (cutoff) Λ at which we are doing the computation, and may blow up when $\Lambda \to \infty$ (called the *ultraviolet limit*, as Λ has the meaning of frequency of oscillation). Rather, the meaningful quantity is the answer, the correlation functions $\langle\, \phi(x_1)...\phi(x_n)\,\rangle$ (or their Fourier transforms, if we work in the momentum realization). This answer depends on some parameters, which are the actual parameters of the theory. So the coefficients in the Lagrangian must be adjusted in such a way that the answer has a finite limit as $\Lambda \to \infty$. The specific answer we will get will depend on the adjustment procedure, but in good cases (called renormalizable) will lie in a nice universal family (often, but not always depending on finitely many parameters).

12.3. **Super-renormalizable, renormalizable, and non-renormalizable theories.** Let us discuss this more systematically. Consider a theory of a scalar boson with a general Lagrangian. Given a Feynman diagram Γ, we have the corresponding Feynman integral I_Γ in momentum space realization, which is an integral of a rational volume form over a real vector space. We can define the *superficial degree of divergence* $D(\Gamma)$ to be the degree of the numerator of this form (where the differentials of coordinates have degree 1) minus the degree of its denominator. It is clear that if $D(\Gamma) \geq 0$ then the integral diverges. Note that the converse is false: if $d(\Gamma) < 0$, the integral may still diverge.

Let us compute $D(\Gamma)$. The degree of the denominator is easy to compute: it is just $2e(\Gamma)$ where $e(\Gamma)$ is the number of internal edges of Γ (indeed, every edge contributes a propagator, which is the inverse of a quadratic function). On the other hand, the number of integrations over V is the number of loops, i.e, $d(e(\Gamma) - v(\Gamma) + 1)$, where $v(\Gamma)$ is the number of internal vertices. Finally, the terms in the Lagrangian containing derivatives of ϕ contribute the number of such derivatives to the degree of the numerator. It follows that

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d + N,$$

where N is the total number of derivatives in vertex monomials. In particular, when there are no derivatives, we have

$$D(\Gamma) = (d-2)e(\Gamma) - dv(\Gamma) + d.$$

This shows that we may compute $D(\Gamma)$ as a sum of contributions over vertices, defining the degree $D(\Phi)$ of a differential monomial Φ standing at a fully internal vertex (one whose all edges are internal) as the contribution of this vertex to $D(\Gamma)$. Indeed, every Φ contributes

$$D(\Phi) = \frac{d-2}{2}e(\Phi) - d + N_\Phi,$$

where $e(\Phi)$ is the number of edges of Φ (i.e., its degree with respect to ϕ) and N_Φ is the number of derivatives in Φ.

We see that a more natural invariant is

$$[\Phi] := D(\Phi) + d,$$

as it is multiplicative:

$$[\Phi_1\Phi_2] = [\Phi_1][\Phi_2].$$

This is not surprising since Φ comes with a volume factor dx, so $D(\Phi)$ is actually the scaling dimension of Φdx; thus to get the scaling dimension of Φ, we need to add d (as the scaling dimension of dx is $-d$). This motivates

Definition 12.2. The number $[\Phi]$ is called the *classical scaling dimension* of the differential monomial Φ.

Thus for a Feynman diagram Γ we have

$$D(\Gamma) = d - \frac{k(d-2)}{2} + \sum_{\Phi} D(\Phi), \tag{12.1}$$

where k is the number of external vertices of Γ.

For example, for $\Phi = \phi^n$ we get

$$D(\phi^n) = \tfrac{n}{2}(d-2) - d = (\tfrac{n}{2} - 1)d - n,$$

Each derivative adds a 1 to the degree, so for instance

$$D(\phi^{n-2}(d\phi)^2) = (\tfrac{n}{2} - 1)(d-2).$$

So for the 1-loop Feynman diagram Γ for the k-point function (a cycle with k legs), we have

$$D(\Gamma) = d - \frac{k}{2}(d-2) + kD(\phi^3) = d - \frac{k}{2}(d-2) + \frac{k}{2}(d-6) = d - 2k.$$

Definition 12.3. Let Φ be a differential monomial in ϕ. We will say that Φ is *super-renormalizable* if $D(\Phi) < 0$, *renormalizable* (or critical) if $D(\Phi) = 0$, and *non-renormalizable* if $D(\Phi) > 0$.

Thus super-renormalizable terms improve convergence, renormalizable ones do not affect it, and non-renormalizable ones worsen it.

Example 12.4. 1. The kinetic term $(d\phi)^2$ has $D = 0$, so is renormalizable; in fact, this is so by definition in any QFT. Note that this can be used to easily compute the classical scaling dimensions of monomials. Namely, we have $[(d\phi)^2] = D((d\phi)^2) + d = d$, so $2[\phi] + 2 = d$, i.e. $[\phi] = \frac{d-2}{2}$. Using multiplicativity, we now immediately compute $[\Phi]$ for any Φ.

2. The mass term ϕ^2 has $D = -2$, so it is super-renormalizable. The term ϕ^3 has $D = \frac{1}{2}d - 3$, so it is super-renormalizable for $d < 6$, renormalizable for $d = 6$ and non-renormalizable for $d > 6$.

Definition 12.5. A Lagrangian is called

• super-renormalizable if all its terms except the kinetic term are super-renormalizable;

• renormalizable (or critical) if all its terms are at worst renormalizable and there is at least one renormalizable non-kinetic (i.e., interacting) term;

• non-renormalizable if it contains non-renormalizable terms.

Clearly, every Lagrangian is of exactly one of these three types.

Proposition 12.6. *(i) If a Lagrangian is super-renormalizable then the degree of superficial divergence of the corresponding Feynman diagrams is bounded above, and there are finitely many superficially divergent diagrams with any given number of external edges; moreover, if $d > 2$ then there are finitely many superficially divergent diagrams altogether.*

(ii) If a Lagrangian is renormalizable, then there are infinitely many superficially divergent diagrams with a fixed number of external edges, but the degree of superficial divergence of these diagrams is still bounded above.

(iii) If a Lagrangian is non-renormalizable, then the degree of superficial divergence of diagrams with a fixed number of external edges is unbounded above.

Proof. This is clear from formula (12.1). □

This means that for a non-renormalizable Lagrangian, regularization of divergent integrals will definitely get out of control. Namely, if we want to regularize diagrams with unbounded above degree of superficial divergence, then we will have to introduce counterterms with unlimited number of derivatives, and our renormalized Lagrangian will no longer depend on a finite number of derivatives of ϕ.

On the other hand, if the Lagrangian is renormalizable, then for $d > 2$ there are only finitely many terms that we will need to modify in the renormalization procedure; namely, these are the possible super-renormalizable and renormalizable terms in the Lagrangian. The fact that this procedure works to all orders of perturbation theory is a rather non-trivial fact which we will not prove here; but the result is a finite-parametric family of perturbative QFT.

In two dimensions, there is an additional feature - there are infinitely many (super)renormalizable terms in the Lagrangian; but they all have at most two derivatives.

Finally, in the super-renormalizable case the renormalization procedure is completed in finitely many steps.

12.4. **Critical dimensions of some important QFT.** For interacting QFT defined by Lagrangians, the theory is only (super-)renormalizable in small dimensions, and becomes non-renormalizable when dimension grows. If a theory is renormalizable in some dimension d and non-renormalizable for bigger dimensions, we say that d is the *critical dimension* of the theory.

12.4.1. *Scalar bosons.* For example, since $D(\phi^n) = (\frac{n}{2} - 1)d - n$, for a scalar boson, a term ϕ^n is (super-)renormalizable iff $d \le \frac{2n}{n-2}$. So in a (super-)renormalizable theory, the term ϕ^3 can be present only for $d \le 6$, ϕ^4 only for $d \le 4$, ϕ^5 and ϕ^6 only for $d \le 3$. Also, since $D(\phi^{n-2}(d\phi)^2) = (\frac{n}{2} - 1)(d - 2)$, such terms with $n > 2$ cannot be present in a (super-)renormalizable theory unless $d = 2$. With more derivatives things get even worse. So we obtain

Proposition 12.7. *For the scalar bosonic field ϕ, the most general (super-)renormalizable non-quadratic Poincaré-invariant Lagrangian is (up to scaling):*

- $d > 6$*: none;*
- *:* $d = 5, 6$*:* $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_3(\phi)$*;*
- *:* $d = 4$*:* $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_4(\phi)$*;*
- *:* $d = 3$*:* $\mathcal{L} = \frac{1}{2}(d\phi)^2 + P_6(\phi)$*;*
- *:* $d = 2$*:* $\mathcal{L} = \frac{1}{2}g(\phi)(d\phi)^2 + U(\phi)$*,*

where P_m is a polynomial of degree m, and U and g are arbitrary (real analytic) functions.

Note that without loss of generality, one may assume that P_m are missing the constant and linear terms. Thus the number of parameters for the theory with Lagrangian $\frac{1}{2}(d\phi)^2 + P_m(\phi)$ is $m-1$ (the coefficients of P_m).

12.4.2. *Fermions.* Recall that for a fermionic field ψ the kinetic term looks like $(\psi, \mathbf{D}\psi)$. This implies that

$$2[\psi] + 1 = d,$$

i.e.,

$$[\psi] = \frac{d-1}{2},$$

which is always positive. So for mass terms $(\psi, M\psi)$ we have $D = -1$ and they are super-renomalizable. Beyond quadratic, we see that the only possibly (super-)renormalizable terms in ψ for $d \ge 2$ are of the general shape ψ^{2k}, and

$$D(\psi^{2k}) = 2k[\psi] - d = k(d-1) - d = (k-1)(d-1) - 1.$$

The only case when this is (super-)renormalizable is $d = 2$ and $k = 2$, i.e., the term ψ^4, in which case $D = 0$ (critical). Such terms indeed occur in the so-called *Gross-Neveu model.*

For $d > 2$, any fermionic term in a renormalizable Lagrangian must therefore be quadratic in the fermions. But it can contain other (bosonic) fields as factors. For example, $[\phi^n\psi^2] = n\frac{d-2}{2} + d - 1$, so

$$D(\phi^n\psi^2) = n\frac{d-2}{2} - 1.$$

This shows that in 3 dimensions we can have a term $\phi\psi^2$ (Yukawa interaction) and $\phi^2\psi^2$, while in 4 dimensions we can have only the Yukawa term $\phi\psi^2$, and for $d > 4$ there are no possible (super-)renromalizable terms.

12.4.3. *Gauge theory.* A similar result holds when ϕ is vector-valued, i.e., has any number of components. This allows us to treat another important example, which is *gauge theory.*

Recall from Subsection 11.13 that to define a gauge theory, we fix a compact Lie group G (for example, $U(n)$) and the field is a connection ∇ on a principal G-bundle P on V. Since all such bundles are trivial, we may think of ∇ as a 1-form A with values in $\mathfrak{g} = \mathrm{Lie}G$; i.e. $\nabla_A = d + A$. The curvature of ∇_A is given by the formula

$$F_A = dA + \frac{1}{2}[A, A],$$

and the Lagrangian of the pure gauge theory is

$$\mathcal{L} := \int_V |F_A|^2 dx.$$

As mentioned in Subsection 11.13, he subtlety here is that A is only considered up to gauge transformations $\nabla_A \mapsto g^{-1}\nabla_A g$, i.e., $A \mapsto g^{-1}dg + g^{-1}Ag$, where $g : V \to G$ is a smooth function with prescribed behavior at infinity, but this is irrelevant for the discussion of critical dimension.

If G is abelian (e.g. $G = U(1)$) then the Lagrangian is quadratic and this theory is free (this is the quantum electrodynamics without matter, i.e., quantization of Maxwell equations). This theory satisfies Wightman axioms in all dimensions, and its Wightman functions can be explicitly computed similarly to the case of scalar boson.

However, if G is non-abelian (e.g. $G = SU(2)$ for weak interactions and $G = SU(3)$ for strong interactions in the standard model) then the Lagrangian is not quadratic and the equations of motion are not linear (they are the *Yang-Mills equations*). Treating A as a (vector-valued) boson, we see that the non-quadratic terms in the Lagrangian are of schematic form A^2dA and A^4. The degrees of these terms are $\frac{1}{2}(d-4)$ and $d-4$, so we see that this theory is critical in dimension 4 (the physical case!) and super-renormalizable in lower dimensions, but non-renormalizable for $d > 4$. Note that the fact that we have a vector boson rather than a collection of scalar bosons (under the action of $\mathbf{P}$) does not matter for the dimension count.

Note also that in $d \leq 4$ dimensions we can also consider renormalizable Lagrangians with terms $(\nabla_A \phi)^2$ or $(\psi, \mathbf{D}_A \psi)$, where ϕ is a scalar and ψ a spinor with values in the associated bundle $P \times_G \rho$, where ρ is a finite dimensional representation of G (it is easy to check that all occurring monomials have $D \leq 0$). Such terms do occur in the standard model; the simplest case is $(\psi, \mathbf{D}_A \psi)$ where A is a $U(1)$-connection and ψ is a spinor valued in the tautological representation of $U(1)$, corresponding to an electron.

12.4.4. *σ-model.* The *σ-model* is a theory of a scalar boson taking values in a Riemannian manifold M. Thus the field is a map $\phi : V \to M$, and the Lagrangian is $\mathcal{L} = \frac{1}{2}(d\phi)^2$, which in local coordinates has the form

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{\dim M} g_{ij}(\phi) d\phi^i d\phi^j,$$

where g_{ij} is the Riemannian metric on M. We may also add a potential $U(\phi)$, where U is a smooth function on M. By the above computations, this Lagrangian for a non-constant metric is renormalizable only in dimension $d = 2$, but in this case g_{ij} and U can be arbitrary.

12.4.5. *Gravity.* The theory of gravity (general relativity) is a theory of a bosonic field $h(x)$ taking values in symmetric tensors S^2V^*; i.e., the Minkowskian metric on V is perturbed by setting $g = g_0 + h$, where g_0 is the standard Minkowskian metric. The Lagrangian of general relativity is

$$\mathcal{L} = R(g)$$

where R is the scalar curvature of the metric g. Since curvature is expressed in terms of second derivatives of the metric, up to scaling this can be schematically written in terms of h as

$$\mathcal{L} = (dh)^2 + ...$$

where the dots stand for terms having at most two derivatives in h. Thus the general shape of this Lagrangian (for the purposes of computing classical scaling dimensions) is the same as for the σ-model; so this theory is only renormalizable in two dimensions. This is one of the main reasons why it has not yet been possible to incorporate gravity into the standard model, which lives in 4 spacetime dimensions.

Remark 12.8. We have seen in Subsection 11.11 that even in a free quantum field theory, the composite operators like $\phi^2(x)$ are not automatically defined, and require a normal ordering procedure to regularize them. This is all the more so in an interacting QFT.

It turns out that the normal ordering procedure, composite operators, and operator product expansion in a critical perturbative QFT can be defined analogously to the free case, using renormalization theory. We will not discuss it here and refer the reader to [QFS], vol. 1, p. 452.

18.238 Geometry and Quantum Field Theory
Spring 2023