13. Two-dimensional conformal field theory

13.1. **Classical free massless scalar in two dimensions.** Consider a free massless scalar boson ϕ on $\mathbb{R}^2$ with Lagrangian $\mathcal{L} = \frac{1}{2}(d\phi)^2$. In this case the local functional $\phi(t,x)$ satisfies the 2-dimensional wave (=string) equation

$$\phi_{tt} - \phi_{xx} = 0,$$

so it splits into a sum of two functionals

$$\phi = \tfrac{1}{\sqrt{2}}\phi_L + \tfrac{1}{\sqrt{2}}\phi_R,$$

where

$$\phi_L(t,x) = \psi_L(x+t),\ \phi_R(t,x) = \psi_R(x-t),$$

which for obvious reasons are called the *left-mover and right-mover*. In other words, we have

$$(\partial_t - \partial_x)\phi_L = 0,\ (\partial_t + \partial_x)\phi_R = 0.$$

So we get

$$\phi_x + \phi_t = \sqrt{2}\psi_L'(x+t),\ \phi_x - \phi_t = \sqrt{2}\psi_R'(x-t).$$

So the Poisson bracket of ψ_L', ψ_R' is given by

$$\{\psi_L'(x), \psi_L'(y)\} = \delta'(x-y),\ \{\psi_R'(x), \psi_R'(y)\} = -\delta'(x-y),$$

$$\{\psi_L'(x), \psi_R'(y)\} = 0.$$

Thus upon Wick rotation, which replaces t with it and makes ϕ complex-valued, setting $u := x + it$, we have

$$\overline{\partial}_u\phi_L = 0,\ \partial_u\phi_R = 0,$$

i.e., $\phi_L = \psi_L(u)$ is holomorphic and $\phi_R = \psi_R(\overline{u})$ is antiholomorphic.

Now consider the case when x runs over the circle $\mathbb{R}/2\pi\mathbb{Z}$, with Lebesgue measure normalized to have volume 1. Then, if we still want to have a decomposition of ϕ into a left-mover and a right-mover, we should "kill the zero mode" by requiring that $\int_0^{2\pi}\phi(t,x)dx = 0$ (otherwise we have a solution $\phi(t,x) = t$ of the string equation which cannot be written as a sum of a left-moving and right-moving periodic wave). Then we may introduce the coordinate $z = e^{iu}$ which takes values in $\mathbb{C}^\times$, and ϕ_L, ϕ_R become holomorphic, respectively antiholomorphic fields on $\mathbb{C}^\times$, which we'll denote by φ, φ^*. So we have Laurent expansions

$$\varphi(z) = \sum_{n\in\mathbb{Z}}\varphi_n z^{-n},\ \varphi^*(\overline{z}) = \sum_{n\in\mathbb{Z}}\varphi_n^*\overline{z}^{-n},$$

with $\varphi_0 = \varphi_0^* = 0$. When z is on the unit circle, these are just the Fourier expansions of $\phi_L(0,x)$, $\phi_R(0,x)$, and for

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} := i\partial_z\varphi(z),\ a^*(\overline{z}) = \sum_{n\in\mathbb{Z}} a_n^* \overline{z}^{-n-1} := -i\overline{\partial}_z\varphi^*(\overline{z}),$$

where $a_0 = a_0^* = 0$, we have

$$za = \partial_u\phi_L = \psi_L'(u),\ \overline{z}a^* = \overline{\partial}_u\phi_R = \psi_R'(\overline{u}).$$

Thus for $z = e^{iu}, w = e^{iv}$ we get

$$\{za(z), wa(w)\} = \delta'(u-v). \tag{13.1}$$

Note that

$$\delta'(u-v) = i\sum_{n\in\mathbb{Z}} nz^n w^{-n}.$$

So setting

$$\delta(w-z) := \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$$

(Fourier expansion of the distribution $\delta(w-z)$ on $(S^1)^2$, where $|z| = |w| = 1$), we can write (13.1) as

$$\{a(z), a(w)\} = -i\delta'(w-z). \tag{13.2}$$

In components, this takes the form

$$\{\sum_{m\in\mathbb{Z}} a_m z^{-m}, \sum_{n\in\mathbb{Z}} a_{-n} w^n\} = -i\sum_{n\in\mathbb{Z}} nz^{-n}w^n.$$

Thus we get

$$\{a_n, a_m\} = -in\delta_{n,-m}. \tag{13.3}$$

Similarly,

$$\{a_n^*, a_m^*\} = in\delta_{n,-m}, \tag{13.4}$$

and

$$\{a_n, a_m^*\} = 0, \tag{13.5}$$

which in terms of generating functions can be written as

$$\{a^*(z), a^*(w)\} = i\delta'(w-z), \{a(z), a^*(w)\} = 0. \tag{13.6}$$

Finally, let us write down the hamiltonian of the theory in terms of the Fourier (=Laurent) modes a_n. Recall that in the original notation it has the form

$$H = \frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}} (\phi_t^2 + \phi_x^2)dx.$$

Thus we have

$$H=\frac{1}{4}\int_{\mathbb{R}/2\pi\mathbb{Z}}((\overline{z}a^*-za)^2+(\overline{z}a^*+za)^2)dx=\frac{1}{2}\int_{\mathbb{R}/2\pi\mathbb{Z}}(\overline{z}^2a^{*2}+z^2a^2)dx,$$

i.e.,

$$H=\sum_{n>0}(a_{-n}a_n+a^*_{-n}a^*_n). \tag{13.7}$$

It satisfies the relations

$$\{a_m,H\}=-ima_m,\ \{a^*_m,H\}=ima^*_m.$$

13.2. **Free quantum massless scalar on $\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}$ with killed zero mode.** Consider now the free QFT of a massless scalar boson ϕ on $\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}$ with Minkowskian metric dt^2-dx^2, with killed zero mode, i.e., a quantization of the classical field theory described in Subsection 13.1. Since this is not a theory on a vector space, it won't satisfy Wightman axioms. However, we can naturally quantize the commutation relations (13.3),(13.4),(13.5) (with $\hbar=1$), by replacing them with

$$[a_n,a_m]=n\delta_{n,-m},\ [a^*_n,a^*_m]=-n\delta_{n,-m},\ [a_n,a^*_m]=0.$$

In other words, for $a(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}$, $a^*(\overline{z})=\sum_{n\in\mathbb{Z}}a^*_n\overline{z}^{-n-1}$ we have

$$[a(z),a(w)]=\delta'(w-z),\ [a^*(z),a^*(w)]=-\delta'(w-z),\ [a(z),a^*(w)]=0,$$

which quantize equations (13.2),(13.6) (this is a field-theoretic generalization of the analysis of Subsection 8.5, with an infinite sequence of harmonic oscillators labeled by positive integers). Thus we see that the Euclidean space-locality property is satisfied.

This shows that we have an infinite system of independent harmonic oscillators. To restate this algebraically, consider the infinite dimensional Heisenberg Lie algebra $\mathcal{A}$ with basis $a_n, n\neq 0$ and K (central) with commutation relations

$$[a_n,a_m]=n\delta_{n,-m}K.$$

Then we see that some dense subspace of the Hilbert space $\mathcal{H}$ of our theory should carry a pair of commuting actions of $\mathcal{A}$ (by left-movers and right-movers), with K acting by 1 and -1, respectively (we'll denote the second copy of $\mathcal{A}$ by $\mathcal{A}^*$).

Let us now describe the Hilbert space $\mathcal{H}$. Note that the Lie algebra $\mathcal{A}$ has an irreducible *Fock representation* $\mathcal{F}$ generated by Ω with defining relations

$$a_n\Omega=0,\ n>0,\quad K\Omega=\Omega.$$

As a vector space, $\mathcal{F}$ is the *Fock space*

$$\mathcal{F} = \mathbb{C}[X_1, X_2, ...]$$

(with $\Omega = 1$), on which the operators a_{-n} for $n > 0$ act by multiplication by X_n and a_n act by $n\frac{\partial}{\partial X_n}$.

Now, the hamiltonian of the system (which we rescale for convenience by a factor of 2) should satisfy the commutation relations

$$[\widehat{H}, a_n] = -na_n, \ [\widehat{H}, a_n^*] = na_n^*.$$

Thus we see that if we want the spectrum of $\widehat{H}$ to be bounded below and if $\Omega \in \mathcal{H}$ is the lowest eigenvector of $\widehat{H}$ then we must have

$$a_n\Omega = 0, \ a_{-n}^*\Omega = 0$$

for $n > 0$. But in this case the space $\mathcal{D}$ generated from Ω by the action of a_n, a_n^* has to be the irreducible representation $\mathcal{F} \otimes \mathcal{F}^*$ of the Lie algebra $\mathcal{A} \oplus \mathcal{A}^*$, where $\mathcal{F}^* := \mathbb{C}[X_1^*, X_2^*, ...]$ with a_n^* acting by multiplication by X_n^* and $a_{-n}^* \mapsto n\frac{\partial}{\partial X_n^*}$ for $n > 0$.

Thus the space $\mathcal{D}$ is the tensor product of polynomial algebras $\mathbb{C}[X_j]$ and $\mathbb{C}[X_j^*]$. Each of the algebras $\mathbb{C}[X_j]$ carries a positive inner product with X_j^n being an orthogonal basis and $||X_j^n||^2 = j^n n!$, and similarly for $\mathbb{C}[X_j^*]$. This yields a positive inner product $\langle \, , \rangle$ on $\mathcal{F}, \mathcal{F}^*$ and $\mathcal{D}$, with respect to which $a_i^\dagger = a_{-i}$ and $a_i^{*\dagger} = a_{-i}^*$. The Hilbert space $\mathcal{H}$ is the completion of $\mathcal{D}$ with respect to $\langle \, , \rangle$.

This implies that the quantum Hamiltonian has to be given by the formula

$$\widehat{H} = \sum_{n>0}(a_{-n}a_n + a_n^*a_{-n}^*) + C$$

obtained by quantizing the classical hamiltonian (13.7) (note that the annihilation operators are written on the right to make sure the infinite sum makes sense). We may write $\widehat{H}$ as the sum of left-moving and right-moving parts:

$$\widehat{H} = \widehat{H}_L + \widehat{H}_R,$$

where

$$\widehat{H}_L := \sum_{n>0} a_{-n}a_n + \tfrac{C}{2}, \ \widehat{H}_L := \sum_{n>0} a_n^*a_{-n}^* + \tfrac{C}{2}.$$

13.3. **ζ-function regularization.** At the moment it is not clear what the right value of C should be. To answer this question, recall that the hamiltonian of a single harmonic oscillator is $z\partial_z + \frac{1}{2}$ acting on $\mathbb{C}[z]$. This suggests that the formula for $\widehat{H}$ should be

$$\widehat{H} = \sum_{n>0}(a_{-n}a_n + a_n^* a_{-n}^* + n) = \frac{1}{2}\sum_{n\neq 0}(a_{-n}a_n + a_n^* a_{-n}^*),$$

which is a more symmetric and natural formula for quantization of H. This formula, however, does not make sense, since the series

$$1 + 2 + 3 + ...$$

is divergent. We may, however, regularize it using *ζ-function regularization.*

Namely, recall that the Riemann ζ-function is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

It is well known that this function extends meromorphically to the entire complex plane with a unique (simple) pole at $s = 1$ and satisfies the functional equation, which says that the function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is symmetric under the change $s \mapsto 1 - s$:

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\zeta(s).$$

Now, it is natural to define

$$C = 1 + 2 + 3 + ... := \zeta(-1).$$

But the functional equation for $s = 2$ implies that

$$\zeta(-1) = \frac{\pi^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}\zeta(2) = -\frac{\pi^{-\frac{3}{2}}}{2\pi^{\frac{1}{2}}}\frac{\pi^2}{6} = -\frac{1}{12}.$$

So from this point of view it is natural to set

$$C := -\frac{1}{12}$$

Remark 13.1. Recall that for integer $\mathrm{g} \geq 1$

$$\zeta(2\mathrm{g}) = (-1)^{\mathrm{g}+1}2^{2\mathrm{g}-1}\frac{B_{2\mathrm{g}}}{(2\mathrm{g})!}\pi^{2\mathrm{g}}.$$

So the functional equation for ζ implies that

$$\zeta(1-2\mathrm{g}) = \pi^{\frac{1}{2}-2\mathrm{g}}\frac{\Gamma(\mathrm{g})}{\Gamma(\frac{1}{2}-\mathrm{g})} \cdot (-1)^{\mathrm{g}+1}2^{2\mathrm{g}-1}\frac{B_{2\mathrm{g}}}{(2\mathrm{g})!}\pi^{2\mathrm{g}} = -\frac{B_{2\mathrm{g}}}{2\mathrm{g}}.$$

Thus the Harer-Zagier theorem can be interpreted as the statement that the Euler characteristic of the moduli space of curves of genus g is $\zeta(1-2\mathrm{g})$. In particular, for $\mathrm{g}=1$ we get the Euler characteristic of $SL_2(\mathbb{Z})$, which is $-\frac{1}{12}$.

13.4. **Modularity of the partition function.** The value $-1/12$ turns out indeed to be the most natural value of C. To see this, let us return to Euclidean signature $|du|^2 = dt^2 + dx^2$ and put our theory on the complex torus $E = E_\tau = \mathbb{C}^\times/q^{\mathbb{Z}} \cong \mathbb{R}/2\pi T\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, where

$$T > 0,\ \tau = iT,\ q = e^{2\pi i\tau} = e^{-2\pi T} \in (0,1).$$

In this case, as we know from quantum mechanics, we should consider the partition function

$$Z(\tau) := \mathrm{Tr}(e^{2\pi i\tau\widehat{H}}).$$

Note that

$$\widehat{H}(P\otimes Q) = (\deg P + \deg Q + C)P\otimes Q,$$

where $P \in \mathcal{F}$ and $Q \in \mathcal{F}^*$, and the degree is given by

$$\deg(X_n) = \deg(X_n^*) = n.$$

Thus we have

$$Z(\tau) = \frac{e^{2\pi i\tau(C+\frac{1}{12})}}{\eta(\tau)^2},$$

where

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind η-function. Now recall that $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$, namely,

$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau}\cdot\eta(\tau).$$

So the partition function Z has a nice modular property for a unique value of C, which is exactly $-\frac{1}{12}$.

Let us explain why we should expect $Z(\tau)$ to have a modular property.

For this, note that the Lagrangian of the theory

$$\mathcal{L}(\phi) = \frac{1}{4\pi}\int_E (d\phi)^2 = \frac{1}{4\pi}\int_E d\phi\wedge *d\phi$$

is *conformally invariant*, as it is written purely in terms of the Hodge *-operator which depends only on the conformal structure on E. The same is true for the equation of motion, which is the Laplace's equation $\Delta\phi = 0$. In other words, our classical field theory is *conformal*. Thus we could hope that the corresponding quantum theory is conformal as well. This should mean that $Z(-\frac{1}{\tau}) = Z(\tau)$, since the complex tori $E_{-\frac{1}{\tau}}$ and E_τ are conformally equivalent.

This said, we note that this modular property is only satisfied up to a linear factor in τ: in fact, we have

$$Z(-\tfrac{1}{\tau}) = -i\tau Z(\tau).$$

This is because we have killed the zero mode, which we should, in fact, have included (after all, the space cycle in the torus E_τ is not $SL_2(\mathbb{Z})$-invariant, hence neither is the condition that the integral of ϕ over this cycle vanishes). This is done in the next subsection.

13.5. **Including the zero mode.** The zero mode corresponds to the periodic solutions $\phi(t,x) = \alpha + \mu t$ of the string equation ($\alpha, \mu \in \mathbb{R}$), which for nonzero μ cannot be split into a left-moving and right-moving periodic wave. So putting back the zero mode corresponds to replacing the Hilbert space $\mathcal{H}$ with $\mathcal{H}_{\text{full}} := \mathcal{H} \otimes L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is the Hilbert space of a quantum-mechanical free massless particle, and the Hamiltonian $\widehat{H}$ by

$$\widehat{H}_{\text{full}} := \widehat{H} + \widehat{\mu}^2,$$

where $\widehat{\mu}$ is the quantum momentum operator for this quantum mechanical particle, acting on $L^2(\mathbb{R})$ by multiplication by the momentum μ. Thus we may write

$$\mathcal{H}_{\text{full}} = \int_{\mathbb{R}} \mathcal{H}_\mu d\mu,$$

with $\mathcal{H}_\mu = \mathcal{F}_\mu \otimes \mathcal{F}_\mu^*$ where $\mathcal{F}_\mu = \mathcal{F}$ but with $a_0 = \mu$ instead of $a_0 = 0$, and similarly $\mathcal{F}_\mu^* = \mathcal{F}^*$ but with $a_0^* = \mu$ instead of $a_0^* = 0$. Then we still have

$$\widehat{H} = \widehat{H}_L + \widehat{H}_R,$$

where

$$\widehat{H}_L = \tfrac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n - \tfrac{1}{24},\ \ \widehat{H}_R = \tfrac{1}{2}a_0^{*2} + \sum_{n>0} a_n^* a_{-n}^* - \tfrac{1}{24}.$$

According to Remark 8.29, the partition function of such a particle when time runs over $\mathbb{R}/L\mathbb{Z}$ is, up to scaling, $L^{-\frac{1}{2}}$. Thus the full partition function should be

$$\mathcal{Z}(\tau) = (-i\tau)^{-\frac{1}{2}} Z(\tau).$$

And then we have the genuine modular property:

$$\mathcal{Z}(-\tfrac{1}{\tau}) = \mathcal{Z}(\tau).$$

We note that the function $\mathcal{Z}(\tau)$ has a natural extension to arbitrary $\tau \in \mathbb{C}_+$ (not necessarily purely imaginary), which is just the path integral over a "non-rectangular" complex torus E_τ. To explain this, note that we have a natural action of the translation group $\mathbb{R}/2\pi\mathbb{Z}$ on our spacetime, hence we should expect its action on the Hilbert space $\mathcal{H}$. The infinitesimal generator D of this group should satisfy the commutation relations

$$[D, a_n] = na_n, \ [D, a_n^*] = na_n^*$$

(which differs from the corresponding relations for $\widehat{H}$ by the sign in the first relation). As $D\Omega = 0$, it follows that

$$D(P \otimes Q) = (\deg P - \deg Q) P \otimes Q,$$

i.e.,

$$D = \widehat{H}_L - \widehat{H}_R.$$

Let $s \in \mathbb{R}$ and $\tau := iT + s$. Then a twisted version of the Feynman-Kac formula implies that given $s \in \mathbb{R}$, we have

$$Z(\tau) = \mathrm{Tr}(e^{-2\pi T\widehat{H}} e^{2\pi i s D}) = |q|^{-\frac{1}{12}} \mathrm{Tr}(q^{\widehat{H}_L} \overline{q}^{\widehat{H}_R}),$$

where $q = e^{-2\pi(T+is)} = e^{2\pi i\tau}$. Thus we still have

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2}.$$

Hence

$$\mathcal{Z}(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2},$$

which is a (real analytic) modular function for $SL(2, \mathbb{Z})$, i.e., invariant under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. Thus here we have a genuine quantum conformal symmetry (as the moduli of complex tori E_τ is exactly $\mathbb{C}_+/SL_2(\mathbb{Z})$). Indeed, this function is obviously symmetric under $\tau \mapsto \tau + 1$, and we've seen that it is invariant under $\tau \mapsto -1/\tau$, but these two transformations generate $SL_2(\mathbb{Z})$.

13.6. **Correlation functions on the cylinder and torus.** We may also consider correlation functions of the quantum fields a and a^*. They are computed separately in $\mathcal{F}$ and $\mathcal{F}^*$ and can be easily found using representation theory. For example, we have $a_n a_{-n}\Omega = n\Omega$ for $n > 0$, so the 2-point function is given by

$$\langle\, \Omega, a(z)a(w)\Omega \,\rangle = \sum_{n=1}^{\infty} n z^{-n-1} w^{n-1} = \frac{1}{(z-w)^2}.$$

More precisely, the series converges only for $|w| < |z|$, but the function analytically continues to all $z \neq w$. Since our theory is free, the higher correlation functions are given by Wick's formula:

Proposition 13.2. *We have*

$$\langle \Omega, a(z_1)....a(z_{2k})\Omega \rangle = \sum_{\sigma\in\Pi_{2k}} \frac{1}{\prod_{j\in[1,2k]/\sigma}(z_j - z_{\sigma(j)})^2},$$

and the $2k+1$-point correlation functions are zero.

We note that since $\mathcal{F}$ is generated by Ω as an $\mathcal{A}$-module, these functions determine $a(z)$ as a local operator (=quantum field). More generally, they determine the operators $a(z_1)...a(z_r)$ when $z_i \neq z_j$, which are symmetric in $z_1, .., z_r$ due to space locality. However, these operators are not well defined (have poles) on the diagonals $z_i = z_j$.

Exercise 13.3. *Give a direct algebraic proof of Proposition 13.2.*

Exercise 13.4. *Compute the normalized 2-point correlation function of the quantum field $\widetilde{a}(z) := za(z)$ on the torus $E := \mathbb{R}/2\pi T\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$ in terms of theta functions.*

Hint. *This correlation function is given by*

$$\frac{\langle \widetilde{a}(z)\widetilde{a}(w) \rangle_E}{\langle \emptyset \rangle_E} = \mathrm{Tr}_{\mathcal{F}}(\widetilde{a}(z)\widetilde{a}(w)e^{-2\pi T\widehat{H}_L}).$$

13.7. **Infinitesimal conformal symmetry: the Virasoro algebra.** We have already pointed out that the theory of a free massless scalar in two dimensions is classically conformally invariant and saw some manifestations of the fact that this invariance survives at the quantum level (modular invariance of the partition function on the torus). However, to study conformal symmetry systematically, we need to consider *infinitesimal conformal symmetry*, given by "infinitesimal conformal mappings", i.e., holomorphic vector fields on $\mathbb{C}^\times$.

For simplicity we consider polynomial vector fields $P(z)\partial_z$ where P is a Laurent polynomial (this is sufficient since polynomial fields are dense in all holomorphic vector fields in an appropriate topology). Such vector fields form a Lie algebra called the *Witt algebra* (or *centerless Virasoro algebra* in the physics literature), and we'll denote it by W. A convenient basis of W is $\{L_n = -z^{n+1}\partial_z, n \in \mathbb{Z}\}$ which satisfies the commutation relations

$$[L_n, L_m] = (n-m)L_{m+n},\ m, n \in \mathbb{Z}.$$

The Lie algebra W acts by symmetries of the classical field theory of a free massless scalar, since its Lagrangian is conformally invariant. In fact, importantly, this action is only $\mathbb{R}$-linear and not $\mathbb{C}$-linear, which is a good thing - this means that we have an action of the complexification $W_{\mathbb{C}} = W \oplus W^*$, where W^* is the Lie algebra of antiholomorphic vector fields; in other words, we have two commuting actions of W.

If our theory is quantum-mechanically conformally invariant, then the Lie algebra $W \oplus W^*$ should act on the space $\mathcal{D}$ in a way compatible with the action of $\mathcal{A} \oplus \mathcal{A}^*$, i.e., so that

$$[L_n, a(z)] = z^{n+1}a'(z) + (n+1)z^n a(z),$$
$$[L_n^*, a^*(\overline{z})] = \overline{z}^{n+1}a^{*\prime}(\overline{z}) + (n+1)\overline{z}^n a_*(\overline{z}),$$
$$[L_n^*, a(z)] = [L_n, a^*(\overline{z})] = 0,$$

or in components

$$[L_n, a_m] = -ma_{m+n},\ [L_m^*, a_n^*] = -ma_{m+n}^*,\ [L_n, a_m^*] = [L_n^*, a_m] = 0.$$

Is there such an action? To figure this out, first note that the operators L_0, L_0^* satisfy the same commutation relations with a, a^* as $\widehat{H}_L, -\widehat{H}_R$ respectively. Since $\mathcal{D}$ is an irreducible $\mathcal{A} \oplus \mathcal{A}^*$-module, this means that by Schur's lemma we must have

$$L_0 = \widehat{H}_L + C_L,\ L_0^* = -\widehat{H}_R + C_R$$

for some constants C_L, C_R. This shows that L_n has to shift the grading in $\mathcal{F}$ by n, and similarly for L_n^* and $\mathcal{F}^*$.

Now by analogy with the formula

$$L_0 = \sum_{k \geq 1} a_{-k}a_k + \text{const},$$

define for $n \neq 0$

$$L_n := \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{-k}a_{k+n}. \tag{13.8}$$

It is easy to check that this operator on $\mathcal{F}$ (and hence on $\mathcal{D} = \mathcal{F} \otimes \mathcal{F}^*$) is well defined, and satisfies the desired commutation relations

$$[L_n, a(z)] = -z^{n+1}a'(z) + (n+1)z^n a(z),\ [L_n, a^*(z)] = 0.$$

Again using irreducibility of $\mathcal{D}$ and Schur's lemma, we see that if the desired action of W exists at all, then L_n **must** be given by formula (13.8) (note that here we can't add a constant since L_n must shift the degree). So it remains to check if the constructed operators satisfy the commutation relations of W.

First assume $n \neq -m$. In this case using the Jacobi identity, we see that the operator $[L_n, L_m] - (n-m)L_{m+n}$ commutes with a, a^*, so again by Schur's lemma it must be a constant; however, since it shifts degree, we get the desired relation

$$[L_n, L_m] - (n-m)L_{m+n} = 0.$$

So it remains to consider the case $n = -m > 0$. In this case the same argument shows that

$$[L_n, L_{-n}] - 2nL_0 = C(n),$$

where $C(n) \in \mathbb{C}$, and we have an action of W if $C(n) = 0$ for all n. So let us compute $C(n)$. To this end, note that the eigenvalue by which $[L_n, L_{-n}]$ acts on Ω is $2nC_L + C(n)$. So it suffices to compute this eigenvalue, i.e., the vector $L_n L_{-n}\Omega$.

In terms of the polynomial realization, we have

$$L_{-n}\Omega = \tfrac{1}{2} \sum_{0<j<n} X_j X_{n-j}.$$

Thus

$$L_n L_{-n}\Omega = \tfrac{1}{4} \sum_{0<j<n} j(n-j) \tfrac{\partial^2}{\partial X_j \partial X_{n-j}} \sum_{0<j<n} X_j X_{n-j} = \tfrac{1}{2} \sum_{0<j<n} j(n-j) = \frac{n^3 - n}{12}.$$

So

$$C(n) = \frac{n^3 - n}{12}.$$

Thus we see that we almost have an action of W, but not quite - no matter how we choose C_L, the cubic term in n will be present (a quantum anomaly)! Instead, we have a *projective* representation of W, which is, in fact, a representation of a *central extension* of W. Such projective actions are, in fact, common in quantum mechanics, since quantum states correspond not to actual unit vectors in the space of states, but rather to vectors up to a phase factor, on which (as well as on quantum observables) there is a genuine action of the symmetry group. Prototypical examples of this are the *Heisenberg uncertainty relation* $[\widehat{p}, \widehat{x}] = -i\hbar$, when the classical 2-dimensional group (or Lie algebra) of translations of the phase plane is replaced in quantum theory by the 3-dimensional Heisenberg group (Lie algebra), and the *phenomenon of spin*, when the classical rotational symmetry group $SO(3)$ is replaced in quantum theory by its double cover $SU(2)$.

This motivates the following definition.

Definition 13.5. The *Virasoro algebra* is the 1-dimensional central extension of the Witt algebra W with basis $L_n, n \in \mathbb{Z}$ and C (a central element) with commutation relations

$$[L_n, L_m] = (n - m)L_{m+n} + \frac{n^3 - n}{12}\delta_{n,-m}C.$$

Thus we have a 1-dimensional central ideal $\mathbb{C}C \subset \mathrm{Vir}$ spanned by C, and $\mathrm{Vir}/\mathbb{C}C \cong W$.

So we obtain

Theorem 13.6. *The formulas*

$$L_0 = \sum_{k\geq 1} a_{-k}a_k,\ \ L_n = \tfrac{1}{2}\sum_{k\in\mathbb{Z}} a_{-k}a_{k+n}, n \neq 0$$

define an action of Vir *on* $\mathcal{F}$ *with* C *acting by* 1*.*

It is easy to check that the same theorem holds more generally on the space $\mathcal{F}_\mu$ where $a_0 = \mu$. The only change is that L_0 acquires an additional summand $\frac{1}{2}\mu^2$:

$$L_0 = \tfrac{1}{2}\mu^2 + \sum_{k\geq 1} a_{-k}a_k.$$

If C acts on a representation $\mathbb{V}$ of Vir by a scalar c (as it will, for instance, on every irreducible representation) then one says that $\mathbb{V}$ has *central charge* c. Thus $\mathcal{F}_\mu$ is a representation of Vir of central charge $c = 1$.

Similarly, the formulas

$$L_0^* = -\tfrac{1}{2}\mu^2 - \sum_{k\geq 1} a_k^* a_{-k}^*,\ \ L_n^* = -\tfrac{1}{2}\sum_{k\in\mathbb{Z}} a_k^* a_{-k+n}^*, n \neq 0$$

define an action of Vir on $\mathcal{F}_\mu^*$ with the central element C^* acting by -1 (i.e., of central charge $c = -1$).

Thus we obtain two commuting projective actions of W on the space $\mathcal{D} = \mathcal{F} \otimes \mathcal{F}^*$ which define usual linear actions only for the central extension Vir of W. Still, the corresponding adjoint action of W on quantum observables is a genuine linear action, so this quantum field theory is *conformal*.

We note that the Virasoro action preserves the positive Hermitian form on $\mathcal{F}_\mu$ in the sense that

$$L_n^\dagger = L_{-n}.$$

Thus $\mathcal{F}_\mu$ is a *positive energy unitary representation of* Vir (positive energy means that L_0 is diagonalizable with spectrum bounded below).

More generally, we may consider the theory of ℓ massless scalars $\phi_1, ..., \phi_\ell$. In this case $\mathcal{D} = \mathcal{F}^{\otimes \ell} \otimes \mathcal{F}^{*\otimes \ell}$, and $\mathcal{F}^{\otimes \ell}$ is a positive energy unitary Vir-module with central charge $c = \ell$ (the tensor product of ℓ copies of $\mathcal{F}$).

Exercise 13.7. *1. Show that* Vir *is a non-trivial central extension of* W *(i.e., not isomorphic to* $W \oplus \mathbb{C}$ *as a Lie algebra).*

2. Show that Vir *is a universal central extension of* W*, i.e., every non-trivial central extension of* W *by* $\mathbb{C}$ *is isomorphic to* Vir*.*

13.8. **Normal ordering, composite operators and operator product expansion in conformal field theory.** Let us now summarize the theory of normal ordering, composite operators and operator product expansion from Subsection 11.11 in the case of conformal field theory, for the running example of a quantum massless scalar boson. We have seen that the operator product $a(z)a(w)$ is well defined only if $w \neq z$ and has a pole when $w = z$, leading to the local operator $a(z)^2$ not being well defined. So let us expand this operator product in a Laurent series near $w = z$ and identify the singular part involving negative powers of $w - z$. For this purpose consider the difference

$$: a(z)a(w) := a(z)a(w) - \frac{1}{(z-w)^2}.$$

The formula for the correlation functions for $a(z)$ implies that

$$\langle\, \Omega, a(z_1)...a(z_{i-1}) : a(z_i)a(z_{i+1}) : a(z_{i+2})...a(z_n)\Omega\, \rangle =$$

$$\sum_{\sigma \in \Pi_{2k}:\sigma(i)\neq i+1} \frac{1}{\prod_{j\in \Pi_{2k}/\sigma}(z_j - z_{\sigma(j)})^2}.$$

Note that this function is regular at $z_i = z_{i+1}$, hence the operator $: a(z)a(w) :$ is regular at $z = w$, i.e., defined for all $z, w \in \mathbb{C}^\times$. This operator is called the *normally ordered product* of $a(z)$ and $a(w)$. In particular, although the square $a(z)^2$ is not defined, we have a well defined normally ordered square $: a(z)^2 :$.

In terms of Laurent coefficients,

$$: a(z)a(w) := \sum_{m,n\in\mathbb{Z}} : a_n a_m : z^{-n-1} w^{-m-1},$$

where $: a_n a_m := a_n a_m$ if $m \geq n$ and $: a_n a_m := a_m a_n$ if $m < n$ (normal ordering of modes). Of course, this ordering only matters if $m + n = 0$. In particular, we see that

$$\tfrac{1}{2} : a(z)^2 := T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

the generating function of the Virasoro modes L_n. This operator is called the *(quantum) energy-momentum tensor*.

Thus we see that the Virasoro modes L_n may be viewed as Noether charges for the corresponding infinitesimal conformal symmetries, in the holomorphic sector of the theory. The corresponding Noether currents are $z^{n+1}T(z)$, as

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) dz.$$

The Noether charges for the full theory are then $L_n+\overline{L}_n$, with currents $z^{n+1}T(z)+\overline{z^{n+1}T(z)}$. In particular, the Hamiltonian H, up to adding a constant, is $L_0+\overline{L_0}$, which agrees with formula (11.5).

Similarly, we may define the normal ordered products of more than two factors, $: a(z_1)....a(z_n) :$. This can be done by induction in n. Namely, we have

(13.9)

$$: a(z_0)a(z_1)...a(z_n) := a(z_0) : a(z_1)...a(z_n) : - \sum_{k\in[1,n]} \frac{: \prod_{j\neq k} a(z_j) :}{(z_0-z_k)^2}$$

It is easy to see that the operator $: a(z_1)....a(z_n) :$ has no singularities and is well defined for all values $z_1,...,z_n \in \mathbb{C}^\times$. Thus for every $r_1,..,r_n$ we have the operator

$$: a^{(r_1)}(z_1)...a^{(r_n)}(z_n) := \partial_{z_1}^{r_1}...\partial_{z_n}^{r_n} : a(z_1)....a(z_n) :$$

Setting $z_1=...=z_n$, we can then define the local operator $: P(a)(z) :$ for any differential polynomial P in $a(z)$. This local operator, called a *composite operator*, is a quantization of the corresponding local functional $P(a)(z)$ in classical field theory.

Exercise 13.8. *(The state-operator correspondence) Show that the map* $P \mapsto P(a)(z)\Omega|_{z=0}$ *is well defined and gives an isomorphism between the space* $\mathcal{V}$ *of (polynomial) local operators and the Fock space* $\mathcal{F}$.

More generally, repeatedly using (13.9), we have

$$: a(z_1)...a(z_n) : \cdot : a(w_1)...a(w_m) := \sum_{I\subset[1,n],J\subset[1,m],s:I\cong J} \frac{: \prod_{i\notin I} a(z_i) \prod_{j\notin J} a(w_j) :}{\prod_{i\in I}(z_i-w_{s(i)})^2}.$$

So setting $z_i=z, w_j=w$, we obtain

$$: a(z)^n :: a(w)^m := \sum_{k=0}^{\min(m,n)} k! \binom{n}{k}\binom{m}{k} \frac{: a(z)^{n-k}a(w)^{m-k} :}{(z-w)^{2k}}.$$

E.g. for $n=m=1$ we get the familiar identity

$$a(z)a(w) = \frac{1}{(z-w)^2} + : a(z)a(w) := \frac{1}{(z-w)^2} + \text{regular terms}.$$

More generally, for $n=1$ and any m we get

$$a(z) : a(w)^m := \frac{m : a^{m-1}(w) :}{(z-w)^2} + : a(z)a(w)^m :$$

$$= \frac{m : a^{m-1}(w) :}{(z-w)^2} + \text{regular terms}.$$

For $m = 2$ this can be written as

$$a(z)T(w) = \frac{a(w)}{(z-w)^2} + \text{regular terms},$$

which encodes the commutation relations between a_i and L_j.

For $n = 2$, $m = 2$ we get

$$:a(z)^2::a(w)^2: = \frac{2}{(z-w)^4} + \frac{4:a(z)a(w):}{(z-w)^2} + :a(z)^2a(w)^2: =$$

$$\frac{2}{(z-w)^4} + \frac{4:a(w)^2:}{(z-w)^2} + \frac{4:a(w)a'(w)}{z-w} + \text{regular terms}.$$

This can also be written as

$$T(z)T(w) = \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms},$$

which encodes the commutation relations between L_i. More generally, at central charge c this relation would look like

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{regular terms}.$$

These are the simplest examples of the *operator product expansion*. In fact, we have the following theorem, whose proof we will leave to the reader:

Theorem 13.9. *For any local operators $P, Q \in \mathcal{V}$, there exist a unique finite sequence of local operators $R_1, ..., R_N \in \mathcal{V}$ such that*

$$P(a)(z)Q(a)(w) = \sum_{j=1}^{N} R_j(a)(w)(z-w)^{-j} + \text{regular terms},$$

where $(z-w)^{-j} := \sum_{k\geq 0} \binom{k+j-1}{j-1} z^{-j-k}w^k$.

Note that the space locality property implies that $Q(a)(w)P(a)(z)$ is given by the same formula, but with $(z-w)^{-j}$ expanded in the opposite direction, i.e., $(z-w)^{-j} := -\sum_{k<0} \binom{k+j-1}{j-1} z^{-j-k}w^k$. Thus. we have

$$[P(a)(z), Q(a)(w)] = \sum_{j=1}^{N} \frac{1}{(j-1)!} R_j(a)(w)\delta^{(j-1)}(w-z).$$

Thus Theorem 13.9 gives us information about commutators between the modes of P and Q. For example, as we have seen above,

$$[a(z), a(w)] = \delta'(w-z),$$

and also

$$[a(z), T(w)] = a(w)\delta'(w-z),$$

$$[T(z), T(w)] = \frac{c}{12}\delta'''(w-z) + 2T(w)\delta'(w-z) + T'(w)\delta(w-z),$$

where in our example $c = 1$.

Moreover, it is clear that one can uniquely continue the expansion of Theorem 13.9 to also include terms of nonnegative degree; namely, we simply need to expand the regular terms into a Taylor series with respect to $z - w$ for fixed w. For example, we have an asymptotic expansion

$$a(z)a(w) \sim \frac{1}{(z-w)^2} + \sum_{k=0}^{\infty} :a^{(k)}(w)a(w): \frac{(z-w)^k}{k!}$$

So in general we have

$$P(a)(z)Q(a)(w) \sim \sum_{j=-\infty}^{N} R_j(a)(w)(z-w)^{-j}.$$

This formula is called the *operator product expansion* of the product of P and Q. The operator product expansion satisfies certain axioms, which means that it defines on the space $\mathcal{V} \cong \mathcal{F}$ an algebraic structure called a *vertex algebra* (which we will not discuss here, however).

13.9. **Vertex operators.** *Vertex operators* are obtained by quantizing the local functional $e^{i\lambda\varphi(z)}$, where

$$\varphi(z) = -i\int a(z)dz = -i(a_0 \log z + \sum_{n\neq 0} \tfrac{a_{-n}}{n} z^n + a_0^\vee)$$

and $a_0^\vee$ is a constant of integration (dual variable to a_0). In other words, we have

$$e^{i\lambda\varphi(z)} = e^{\lambda\int a(z)dz} = e^{\lambda(a_0 \log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} e^{\lambda a_0^\vee}.$$

A natural quantization of this functional is the operator

$$X(\lambda, z) :=: e^{\lambda(a_0 \log z + \sum_{n\neq 0}\frac{a_{-n}}{n}z^n)} : e^{\lambda a_0^\vee} =$$

$$= e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}} z^{\lambda\mu} e^{\lambda\partial_\mu},$$

which, due to the last factor, acts from $\mathcal{F}_\mu$ to $\mathcal{F}_{\mu+\lambda}$ by $X_0(\lambda, z)z^{\lambda\mu}$, where

$$X_0(\lambda, z) := e^{\lambda\sum_{n>0}\frac{a_{-n}}{n}z^n} e^{-\lambda\sum_{n>0}\frac{a_n}{n}z^{-n}}.$$

Here we work over the group algebra of $\mathbb{C}$ with basis $z^\alpha, \alpha \in \mathbb{C}$.

Now note that if $[A, B]$ commutes with A, B then by the Campbell-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]},$$

and that

$$[\sum_{n>0} \frac{a_n}{n} z^{-n}, \sum_{n>0} \frac{a_{-n}}{n} w^n] = \sum_{n>0} \frac{z^{-n}w^n}{n} = -\log(1 - \tfrac{w}{z}).$$

Thus

$$X_0(\lambda, z)X_0(\nu, w) = (1 - \tfrac{w}{z})^{\lambda\nu} : X_0(\lambda, z)X_0(\nu, w) :$$

for $|w| < |z|$. So we get

$$X(\lambda, z)X(\nu, w) = (z - w)^{\lambda\nu} : X(\lambda, z)X(\nu, w) :$$

for $|w| < |z|$, where the normal ordering puts ∂_μ to the right of μ. More generally, we see that

$$X(\lambda_1, z_1)...X(\lambda_n, z_n) = \prod_{1\le j<k\le n} (z_j - z_k)^{\lambda_j\lambda_k} : X(\lambda_1, z_1)...X(\lambda_n, z_n) :$$

for $|z_1| > ... > |z_n|$. In particular, denoting the highest weight vector of $\mathcal{F}_\mu$ by Ω_μ, we have

$$\langle\, \Omega_{\mu+\lambda}, X(\lambda_1, z_1)...X(\lambda_n, z_n)\Omega_\mu \,\rangle = \prod_{j=1}^{n} z_j^{\lambda_j\mu} \prod_{1\le j<k\le n} (z_j - z_k)^{\lambda_j\lambda_k}.$$

for $|z_1| > ... > |z_n|$.

We see that this correlation function admits analytic continuation to the complement of the diagonals $z_i \neq z_j$, but this continuation is not, in general, single valued. In other words, the fields $X(\lambda, z)$ in general do not satisfy space locality. Instead, we have

$$X(\lambda, z)X(\nu, w) = e^{\pi i\lambda\nu} X(\nu, w)X(\lambda, z), \tag{13.10}$$

which is understood in the sense of analytic continuation along a path where $v := w/z$ passes from the region $|v| < 1$ to the region $|v| > 1$ along positive reals, avoiding the point $v = 1$ from above. In particular,

$$X(\lambda, z)X(\lambda, w) = e^{\pi i\lambda^2} X(\lambda, w)X(\lambda, z),$$

i.e., $X(\lambda, z)$ has "statistics $\lambda^2/2$" (where statistics $\alpha \in \mathbb{R}/\mathbb{Z}$ means that switching the order produces a phase factor $e^{2\pi i\alpha}$; e.g. statistics 0 corresponds to bosons and statistics 1/2 to fermions).

Note that if we apply commutation relation (13.10) twice, we obtain a multiplier $e^{2\pi i\lambda\nu}$, which corresponds to the fact that the operator product $X(\lambda, z)X(\nu, w)$ is multivalued in general.

This is an example of appearance of a *braiding* in conformal field theory. Namely, relation (13.10) is called *braided space-locality* (or *braided commutativity*), since it can be viewed as commutativity in a suitable braided monoidal category.

Note also that

$$X'(\lambda, z) = \lambda : a(z)X(\lambda, z) :$$

where $X' := \partial_z X$, and

$$[a_n, X(\lambda, z)] = \lambda z^n X(\lambda, z).$$

Hence

$$[L_n, X(\lambda, z)] = z^{n+1}X'(\lambda, z) + \frac{\lambda^2}{2}(n+1)z^n X(\lambda, z),$$

which implies that $X(\lambda, z)$ has spin $\lambda^2/2$. Thus we have the spin-statistics property for $X(\lambda, z)$, which generalizes the usual one: spin modulo $\mathbb{Z}$ equals statistics.

As noted in Remark 11.2, such quantum fields are called "anyons" (as they can have any spin and statistics) and can exist only in two dimensions. The most general spin-statistics property for these anyons says that if X, Y are anyons of spins $s_X, s_Y \geq 0$ then

$$X(z)Y(w) = e^{2\pi i\sqrt{s_X s_Y}}Y(w)X(z).$$

In particular, we see that if $\lambda^2 \in \mathbb{Z}$ is odd then $X(n\lambda, z)$ behave like fermions for odd n and like bosons for even n with respect to each other (i.e., the corresponding operators $X(n_1\lambda, z)$ and $X(n_2\lambda, z)$ commute if $n_1 n_2$ is even and anticommute if $n_1 n_2$ is odd), while for even λ^2 they all behave like bosons (i.e., the operators commute).

13.10. **The circle-valued theory.** Now consider the theory of a massless scalar on $\mathbb{C}^\times$ with values in the circle $\mathbb{R}/2\pi r\mathbb{Z}$. This theory is the same as the line-valued one, except for the zero mode, which entails the following circle-valued solutions of the string equation:

$$\phi(t, x) = \alpha + \mu t + Nrx,$$

where $\alpha \in \mathbb{R}/2\pi r\mathbb{Z}$, $\mu \in \mathbb{R}$, and N is an integer (the winding number). The space of such solutions is a disjoint union of cylinders T^*S^1 labeled by values of N. So in quantum theory we get the Hilbert space

$$\mathcal{H}_r^\circ = \bigoplus_{N,\ell\in\mathbb{Z}} \mathcal{H}_r^\circ(N, \ell),$$

where $\mathcal{H}_r^\circ(N, \ell)$ is the completion of $\mathcal{F}_{\frac{1}{\sqrt{2}}(\ell r^{-1}+Nr)} \otimes \mathcal{F}^*_{\frac{1}{\sqrt{2}}(\ell r^{-1}-Nr)}$. Thus we obtain the following formula for the partition function on the torus E_τ:

$$\mathcal{Z}_r^\circ(\tau) = |\eta(\tau)|^{-2}\vartheta_r(\tau, \overline{\tau}),$$

where

$$\vartheta_r(\tau, \overline{\tau}) := \sum_{\ell,N\in\mathbb{Z}} e^{\frac{1}{2}\pi i\tau(\ell r^{-1}+Nr)^2 - \frac{1}{2}\pi i\overline{\tau}(\ell r^{-1}-Nr)^2} =$$

$$\sum_{\ell,N\in\mathbb{Z}} e^{-\pi(\ell^2 r^{-2}+N^2r^2)\mathrm{Im}\tau+2\pi i\ell N\mathrm{Re}\tau}.$$

This shows an interesting duality $\mathcal{Z}_r^\circ(\tau)=\mathcal{Z}_{r^{-1}}^\circ(\tau)$; in fact, we see that the whole theory with parameter r is equivalent to the one with parameter r^{-1}. This duality is called *T-duality*, and it plays an important role in string theory.

Also we note that ϑ_r is a real modular form of weight 1:

$$\vartheta_r(-\tfrac{1}{\tau},-\tfrac{1}{\overline{\tau}})=|\tau|\vartheta_r(\tau,\overline{\tau}),$$

which leads to modular invariance of the function $\mathcal{Z}_r(\tau)$, as expected in a conformal field theory. To see this, it is enough to note that in the exponential we have a quadratic form on $\mathbb{Z}^2$ with matrix

$$Q(\tau)=\begin{pmatrix} r^2\mathrm{Im}\tau & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & r^{-2}\mathrm{Im}\tau\end{pmatrix}$$

So

$$Q(\tau)^{-1}=|\tau|^{-2}\begin{pmatrix} r^{-2}\mathrm{Im}\tau & i\mathrm{Re}\tau \\ i\mathrm{Re}\tau & r^{2}\mathrm{Im}\tau\end{pmatrix}=\begin{pmatrix} r^{-2}\mathrm{Im}\tau' & -i\mathrm{Re}\tau' \\ -i\mathrm{Re}\tau' & r^{2}\mathrm{Im}\tau'\end{pmatrix}=SQ(\tau')S,$$

where $\tau':=-\frac{1}{\tau}$ and $S=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Thus the result follows from the Poisson summation formula.

We see that if $r^2=\frac{p}{q}\in\mathbb{Q}$ (in lowest terms), this conformal field theory has a special property called *rationality*: the Hilbert space $\mathcal{H}_r^\circ$ is the completion of a finite sum of "sectors" $\oplus_{i=1}^n\mathcal{V}_i\otimes\mathcal{V}_i^*$, where the left-moving fields act on $\mathcal{V}_i$ and right-moving ones in $\mathcal{V}_i^*$, so that $\vartheta_r(\tau,\overline{\tau})$ and hence $\mathcal{Z}_r^\circ(\tau)$ are finite sums of products of a holomorphic and an antiholomorphic function (in fact, it is easy to see that $n=2pq$). For example, the vacuum vector Ω is contained in the tensor product $\mathcal{V}(pq)\otimes\mathcal{V}(pq)^*$ where for $s\in\mathbb{Z}_{>0}$ we defined $\mathcal{V}(s):=\oplus_{m\in\mathbb{Z}}\mathcal{F}_{m\sqrt{2s}}$. The space $\mathcal{V}(s)$ is a vertex algebra called the *lattice vertex algebra* attached to the even lattice $\sqrt{2s}\mathbb{Z}$. This algebra is generated by the vertex operators $X(m\sqrt{2s},z)$ (which, as we know, satisfy the bosonic version of space locality).

Example 13.10. Consider the case $r=1$. In this case we have two sectors, the vacuum sector $\mathcal{V}(2)\otimes\mathcal{V}(2)^*$ and another one, $\mathcal{W}\otimes\mathcal{W}^*$, where $\mathcal{W}=\oplus_{n\in 2\mathbb{Z}+1}\mathcal{F}_{\frac{n}{\sqrt{2}}}$. The particles corresponding to $\mathcal{F}_{\frac{n}{\sqrt{2}}}$ for odd n are anyons with statistics $\frac{1}{4}$, so they satisfy the braided commutativity relation of the form $X(z)Y(w)=iY(w)X(z)$.

It is not difficult to show that the Fourier modes of the vertex operators $X(\sqrt{2},z)$ and $X(-\sqrt{2},z)$ generate a projective action of the Lie algebra $\mathfrak{sl}_2[z, z^{-1}]$ on $\mathcal{V}(2) = \mathbb{L}_0$ and on $\mathcal{W} = \mathbb{L}_1$, which are exactly the irreducible integrable representations of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}K$ (the universal central extension of $\mathfrak{sl}_2[t, t^{-1}]$ at level $k = 1$ (i.e., K acts by 1), namely $X(\sqrt{2}, z)$, $X(-\sqrt{2}, z)$, $\sqrt{2}a(z)$ give the currents $e(z)$, $f(z)$ and $h(z)$, where for $b \in \mathfrak{sl}_2$

$$b(z) := \sum_n (b \otimes t^n) z^{-n-1}.$$

This is the so called *Frenkel-Kac vertex operator construction* of level 1 irreducible integrable modules (defined for any finite dimensional simply-laced simple Lie algebra) in the simplest special case $\mathfrak{g} = \mathfrak{sl}_2$. Thus the circle-valued theory of a free boson for $r = 1$ is the so-called *Wess-Zumino-Witten model* in the simplest example of the Lie algebra $\mathfrak{sl}_2$ and level 1.

Example 13.11. Let $r = \sqrt{2}$. In this case we have four sectors: $\mathcal{V}_j \otimes \mathcal{V}_{-j}^*$, $j = 0, 1, 2, 3$, where $\mathcal{V}_j = \oplus_{n \in 4\mathbb{Z}+j} \mathcal{F}_{\frac{n}{2}}$. In particular, $\mathcal{V}_0 = \mathcal{V}(4)$ and particles in $\mathcal{V}_2 = \mathcal{F}_1$ are fermions arising in the boson-fermion correspondence.

13.11. **Free massless fermions.** In a similar way to free massless bosons, one can describe the theory of a free massless fermion $\xi(z)$. As explained in Subsection 11.4, in two dimensions it makes sense to consider chiral spinors taking values in the tautological representation of $\mathrm{Spin}(2) = U(1)$ with kinetic term $(\xi, \mathbf{D}\xi)$. So we have a single quantum field

$$\xi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \xi_n z^{-n-\frac{1}{2}}$$

and the conjugate quantum field $\xi_*(\overline{z})$. The modes of $\xi(z)$ satisfy the relation

$$[\xi(z), \xi(w)]_+ = \delta(z - w),$$

where $[,]_+$ is the supercommutator. This yields the Clifford algebra relations

$$\xi_n \xi_m + \xi_m \xi_n = \delta_{m,-n}$$

for $m, n \in \mathbb{Z}$. This algebra has a unique irreducible positive energy representation $\Lambda = \wedge(\xi_{-1/2}, \xi_{-3/2}, ...)$ on which ξ_j acts by multiplications for $j < 0$ and by differentiations for $j > 0$. There is an invariant positive Hermitian inner product on Λ in which the Clifford monomials in ξ_j, $j > 0$ form an orthonormal basis (invariance means that $\xi_j^\dagger = \xi_{-j}$). Thus the Hilbert space of the theory is the completion of $\mathcal{D} := \Lambda \otimes \Lambda^*$, where Λ^* is the dual of Λ corresponding to antiholomorphic fields.

The hamiltonian H is supposed to satisfy commutation relations

$$[H, \xi_n] = -\xi_n, \ [H, \xi_n^*] = \xi_n^*,$$

So we have

$$H = H_L + H_R,$$

where

$$H_L = \sum_{n>0} n\xi_{-n}\xi_n$$

and similarly for H_R. The Virasoro algebra is defined by

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} n : \xi_n \xi_{-n+m} :,$$

i.e., $H_L = L_0$.

Exercise 13.12. *Show that these operators L_n satisfy the Virasoro commutation relations with central charge $c = \frac{1}{2}$.*

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Spring 2023