Problems for Quasi-Linear PDEs

18.303 Linear Partial Differential Equations

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1 Problem 1

Solve the traffic flow problem

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = f(x)$$

for an initial traffic group

$$f(x) = \begin{cases} \frac{1}{3}, & |x| \geq 1 \\ \frac{1}{2}\left(\frac{5}{3} - |x|\right), & |x| \leq 1 \end{cases}$$

(a) At what time t_s and position x_s does a shock first form?

(b) Sketch the characteristics and indicate the region in the xt-plane in which the solution is well-defined (i.e. does not break down).

(c) Sketch the density profile $u = u(x,t)$ vs. x for several values of t in the interval $0 \leq t \leq t_s$.

2 Problem 2 : Water waves

The surface displacement for shallow water waves is governed by (in scaled coordinates),

$$\left(1 + \frac{3}{2}h\right)\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} = 0$$

Here, $h = 0$ is the mean free surface of the water. Consider the initial water wave profile

$$h(x,0) = \begin{cases} \varepsilon(1+\cos x), & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}$$

(a) Find the parametric solution and characteristic curves.

(b) Show that two characteristics starting at $s = s_1$ and $s = s_2$ where $s_1, s_2 \in (0, \pi)$ intersect at time

$$t_{int} = \frac{2}{3\varepsilon}\left(-\frac{s_1 - s_2}{\cos s_1 - \cos s_2}\right)$$

Show that

$$t_{int} \geq \frac{2}{3\varepsilon}, \qquad \text{for all } s_1, s_2 \in (0, \pi)$$

and

$$t_{int} \to \frac{2}{3\varepsilon}, \qquad \text{as } s_1, s_2 \to \frac{\pi}{2}$$

Thus the solution breaks down along the characteristics starting at $s = \pi/2$, when $t = t_c = 2/(3\varepsilon)$.

(c) Calculate $\partial h/\partial x$ using implicitly differentiation (the solution cannot be found explicitly) and hence show that along the characteristic starting at $s = \pi/2$,

$$\lim_{t \to t_c^-} \frac{\partial h}{\partial x} = -\infty$$

Thus the wave slope becomes vertical.

(d) Sketch the wave profile $u(x, t_c)$, giving the x-values where the wave is vertical and where the maximum displacement occurs.

3 Problem 3

Consider the quasi-linear PDE and initial condition

$$\begin{aligned} u_t + u\,u_x + \frac{1}{2}u &= 0, \qquad t > 0, \qquad -\infty < x < \infty \\ u(x, 0) &= \varepsilon \sin x, \qquad -\infty < x < \infty \end{aligned}$$

where $\varepsilon > 0$ is constant.

(a) Find the parametric solution and characteristic curves.

(b) Give the solution u in implicit form by writing u in terms of x, t (but not r, s).

(c) For $\varepsilon = 1$, show that the solution first breaks down at $t = t_c = 2\ln 2$. Show that along the characteristic through $(x, t) = (\pi, 0)$, we have

$$\lim_{t \to t_c^-} u_x = -\infty.$$

(d) For $\varepsilon = 1$, sketch the characteristics and the solution profile at time t_c.

(e) Show that the solution exists for all time if $0 < \varepsilon \leq 1/2$.