Solutions to Problems for 3D Heat and Wave Equations

18.303 Linear Partial Differential Equations

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1 Problem 1

A rectangular metal plate with sides of lengths L, H and insulated faces is heated to a uniform temperature of u_0 degrees Celsius and allowed to cool with its edges maintained at 0^o C. You may use dimensional coordinates, with PDE

$$u_t = \kappa \nabla^2 u, \qquad 0 \le x \le L, \qquad 0 \le y \le H.$$

(i) Find the smallest eigen-value λ. For large time t, the temperature is given approximately by the term with $e^{-\lambda\kappa t}$. Show that this term is

$$u(x,y,t) = A \exp\left(-\pi^2 \left(\frac{1}{L^2} + \frac{1}{H^2}\right) \kappa t\right) \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{H}\right).$$

Find the value of A. For fixed $t = t_0 \gg 0$, sketch the level curves $u =$ constant in the xy-plane.

Solution: We use the method of separation of variables,

$$u(x,y,t) = v(x,y)\, T(t)$$

The PDE becomes

$$\frac{T'}{\kappa T} = \frac{\nabla^2 v}{v}$$

Since the left hand side depends on time t only, and the right hand side depends only on (x,y), both must be equal to a constant,

$$\frac{T'}{\kappa T} = \frac{\nabla^2 v}{v} = -\lambda$$

The boundary conditions on $u(x,y,t)$ imply that $v(x,y)=0$ on the boundary.

Given λ, the solution for $T(t)$ is

$$T(t)=ce^{-\lambda\kappa t}$$

for some constant c. The problem for $v(x,y)$ is the Sturm-Liouville problem on the rectangle,

$$\nabla^2 v+\lambda v=0; \qquad v=0 \text{ on boundary}$$

We solve for $v(x,y)$ as in class, by separation of variables. The eigen-solutions are

$$v_{mn}(x,y)=\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right), \qquad \lambda_{mn}=\pi^2\left(\frac{m^2}{L^2}+\frac{n^2}{H^2}\right).$$

The corresponding solution $u_{mn}(x,y,t)$ is

$$u_{mn}(x,y,t)=A_{mn}v_{mn}(x,y)\,T_{mn}(t)=A_{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)e^{-\kappa\lambda_{mn}t}$$

To satisfy the initial condition, we sum all the individual solutions (of the PDE and BCs),

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,t)$$

Imposing the IC gives

$$\begin{aligned}u_0 &= u(x,y,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}v_{mn}(x,y)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)\end{aligned}$$

Multiplying both sides by $v_{lk}(x,y)$ and integrating over the rectangle gives

$$\begin{aligned}&u_0\int_0^L\int_0^H\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{k\pi y}{H}\right)dydx\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\int_0^L\int_0^H\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{k\pi y}{H}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dydx\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\int_0^L\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\int_0^H\sin\left(\frac{k\pi y}{H}\right)\sin\left(\frac{n\pi y}{H}\right)dy\end{aligned}$$

Using our well known orthogonality relations gives

$$u_0\int_0^L\int_0^H\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{k\pi y}{H}\right)dydx=A_{lk}\frac{LH}{4}$$

Thus, replacing l with m and k with n (it's ok, these are just dummy indices), we have

$$\begin{aligned} A_{nm} &= \frac{4u_0}{LH}\int_0^L \sin\left(\frac{m\pi x}{L}\right)dx\int_0^H \sin\left(\frac{n\pi y}{H}\right)dy \\ &= \frac{4u_0}{LH}\frac{\cos(m\pi)-1}{m\pi/L}\frac{\cos(n\pi)-1}{n\pi/H} \\ &= \frac{4u_0}{\pi^2 mn}\left((-1)^m-1\right)\left((-1)^n-1\right) \end{aligned}$$

Thus $A_{2m,2n}=0$ and

$$A_{2n-1,2m-1}=\frac{16u_0}{\pi^2(2m-1)(2n-1)}$$

Finally, the full solution is

$$u(x,y,t)=\frac{16u_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp(-\kappa\lambda_{2m-1,2n-1}t)}{(2m-1)(2n-1)}\sin\left(\frac{(2m-1)\pi x}{L}\right)\sin\left(\frac{(2n-1)\pi y}{H}\right)$$

The smallest eigenvalue is

$$\lambda_{11}=\pi^2\left(\frac{1}{L^2}+\frac{1}{H^2}\right)$$

Note that for $m>1$, $n>1$,

$$\begin{aligned} \left|\frac{u_{mn}(x,y,t)}{u_{11}(x,y,t)}\right| &\le \left|\frac{A_{mn}}{A_{11}}\right|\left|\frac{\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)}{\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)}\right|\left|\frac{e^{-\kappa\lambda_{mn}t}}{e^{-\kappa\lambda_{11}t}}\right| \\ &\le \frac{1}{(2m-1)(2n-1)}\left|\frac{mn\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)}{\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)}\right|e^{-\kappa(\lambda_{mn}-\lambda_{11})t} \\ &\le \frac{mn}{(2m-1)(2n-1)}e^{-\kappa(\lambda_{mn}-\lambda_{11})t} \\ &\le e^{-\kappa(\lambda_{mn}-\lambda_{11})t} \end{aligned}$$

Since the next terms in the series are u_{31} and u_{13}, then

$$\begin{aligned} \left|\frac{u_{13}(x,y,t)}{u_{11}(x,y,t)}\right| &\le e^{-\kappa(\lambda_{13}-\lambda_{11})t}=\exp\left(-\frac{8\kappa\pi^2}{H^2}t\right) \\ \left|\frac{u_{31}(x,y,t)}{u_{11}(x,y,t)}\right| &\le e^{-\kappa(\lambda_{31}-\lambda_{11})t}=\exp\left(-\frac{8\kappa\pi^2}{L^2}t\right) \end{aligned}$$

Hence for $\kappa t>\max\{L,H\}/\pi^2$, we have

$$\left|\frac{u_{13}(x,y,t)}{u_{11}(x,y,t)}\right|,\left|\frac{u_{31}(x,y,t)}{u_{11}(x,y,t)}\right|\le\exp(-8)<3.4\times10^{-4}$$

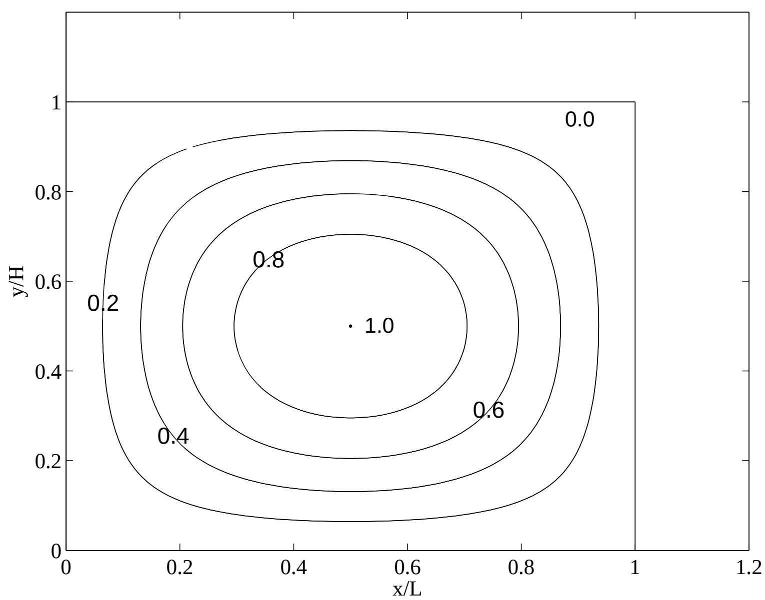

Figure 1: Level curves of $\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)$.

Thus, after a short time the other terms in the series become small relative to the first term, and

$$u(x,y,t) \approx u_{11}(x,y,t) = \frac{16u_0}{\pi^2}\exp\left(-\pi^2\left(\frac{1}{L^2}+\frac{1}{H^2}\right)\kappa t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)$$

Thus $A = 16u_0/\pi^2$.

To sketch the level curves, note that the rectangular plate is symmetric about $x = L/2$ and $y = H/2$. Thus these are heat flow lines at which the level curves meet at right angles. Also, for large time, the level curves are very close to those of the first term approximation, $u_{11}(x,y,t)$, which are effectively the level curves of $\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)$ (see Figure 1). The function $\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{H}\right)$ has a maximum of 1 and is symmetric about the curves $x = L/2$ and $y = H/2$.

(ii) Of all rectangular plates of equal area, which will cool the slowest? Hint: for each type of plate, the smallest eigenvalue gives the rate of cooling.

Solution: For each plate, the smallest eigenvalue gives the rate of cooling,

$$\lambda_{11} = \pi^2\left(\frac{1}{L^2}+\frac{1}{H^2}\right)$$

For plates of equal area $A_0 = LH$, the smallest eigenvalue can be written

$$\lambda_{11} = \pi^2\left(\frac{1}{L^2}+\frac{L^2}{A_0^2}\right)$$

We think of λ_{11} as a function of L. To find the smallest eigenvalue, we minimize λ_{11} with respect to the side length L:

$$\frac{d\lambda_{11}}{dL} = \pi^2 \left(-\frac{2}{L^3} + \frac{2L}{A_0^2} \right)$$

Setting $d\lambda_{11}/dL = 0$ gives $L = \sqrt{A_0}$. Since $d\lambda_{11}/dL < 0$ for $0 < L < \sqrt{A_0}$ and $d\lambda_{11}/dL > 0$ for $L > \sqrt{A_0}$, then $L = \sqrt{A_0}$ is a local minimum. Thus $H = A_0/L = \sqrt{A_0} = L$ and the shape with the smallest eigenvalue and smallest rate of cooling is a square.

(iii) Will a square plate, side length L, cool more or less rapidly than a rod of length L, with insulated sides, and with ends maintained at 0^o C? You may use the results we derived in class for the rod, without derivation.

Solution: Recall that for the rod of length 1 whose ends were kept at 0 degrees and whose sides were insulated, the smallest eigenvalue is

$$\lambda_{rod} = \pi^2$$

In dimensional form,

$$\lambda'_{rod} = \frac{\lambda_{rod}}{L^2} = \frac{\pi^2}{L^2}.$$

The smallest eigenvalue for the square plate of side length L is

$$\lambda'_{11} = \frac{\pi^2}{L^2}(1+1) = \frac{2\pi^2}{L^2} = 2\lambda'_{rod}$$

Thus, the square plate cools faster than the rod (at twice the rate). The physical reason is that the plate can cool from all sides, wereas the rod can only cool from its ends.

2 Problem 2

A rectangular metal plate with sides of lengths L, H and insulated faces has two parallel sides maintained at 0^o C, one side at 100^o C, and one side insulated.

(i) Find the equilibrium (steady-state) temperature $u_E(x, y)$ of the plate.

Solution: Let the origin be at the corner between one of the sides kept at 0 and the other kept at 100 side. The steady-state temperature satisfies Laplace's equation on the plate,

$$\nabla^2 u_E = 0$$

with boundary conditions

$$\begin{aligned} u_E(x, 0) &= u_E(x, L) = 0, \qquad \{0 \leq x \leq L\}, \\ u_E(0, y) &= 100, \qquad u_{Ex}(L, y) = 0, \qquad \{0 \leq y \leq H\}. \end{aligned}$$

The condition $u_{Ex}(L, y) = 0$ indicates that the side $x = L$ of the plate is insulated, i.e., the heat flux normal to the side is zero, $-K_0 \nabla u \cdot \mathbf{n} = 0$. Thus the heat moves along the side, not across it. Separating variables, we have

$$u_E(x, y) = X(x) Y(y).$$

The PDE implies

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

and since the l.h.s. depends only on x and the r.h.s. depends only on y, both must equal a constant,

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda. \tag{1}$$

The BCs imply

$$\begin{aligned} 0 &= u_E(x, 0) = X(x) Y(0) \\ 0 &= u_E(x, H) = X(x) Y(H) \\ 0 &= \frac{\partial u_E}{\partial x}(L, y) = X'(L) Y(y) \end{aligned} \tag{2}$$

For non-trivial solutions, $X(x)$ and $Y(y)$ cannot be identically zero. Thus, to satisfy the BCs,

$$Y(0) = Y(H) = X'(L) = 0.$$

The problem for $Y(y)$ is

$$Y''(y) + \lambda Y(y) = 0; \qquad Y(0) = Y(H) = 0.$$

This is simply the 1D Sturm-Liouville problem, with eigen-solutions (you know this by heart),

$$Y_n(y) = \sin\left(\frac{n\pi y}{H}\right), \qquad \lambda = \lambda_n = \left(\frac{n\pi}{H}\right)^2, \qquad n = 1, 2, 3, ... \tag{3}$$

Therefore, the constant λ in Eq. (1) must be positive and take the values $\lambda = \lambda_n$, $n = 1, 2, 3, ...$

The problem for $X(x)$ is, from (1),

$$X'' - \lambda X = 0; \qquad X'(L) = 0.$$

Since $\lambda = \lambda_n > 0$, $X(x)$ is the sum of exponentials,

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x} \\ &= c_3 \sinh\left(\sqrt{\lambda}(L - x)\right) + c_4 \cosh\left(\sqrt{\lambda}(L - x)\right) \end{aligned}$$

We have written the solution in a more convenient form, since $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$. Imposing the BC $X'(L) = 0$ gives $c_3 = 0$. Thus for each n, we have a solution

$$X_n(x) = a_n \cosh\left(\sqrt{\lambda_n}(L - x)\right) \tag{4}$$

Combining (3) and (4) gives a solution $u_n(x, y)$ to Laplace's equation and the three BCs in (2),

$$u_n(x, y) = X_n(x) Y_n(y) = A_n \cosh\left(\frac{n\pi}{H}(L - x)\right) \sin\left(\frac{n\pi y}{H}\right) \tag{5}$$

To satisfy the BC along the side with $u = 100$, we must sum over all n. The general solution is

$$u_E(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{H}(L - x)\right) \sin\left(\frac{n\pi y}{H}\right)$$

where the constants A_n are determined by imposing that $u = 100$ along the side $\{x = 0, 0 \le y \le H\}$,

$$100 = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

Multiplying by $\sin(m\pi y/H)$, integrating from $y = 0$ to $y = H$, and using orthogonality gives

$$\begin{aligned} 100 \int_0^H \sin\left(\frac{m\pi y}{H}\right) dy &= \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right) \int_0^H \sin\left(\frac{m\pi y}{H}\right) \sin\left(\frac{n\pi y}{H}\right) dy \\ &= A_m \cosh\left(\frac{m\pi L}{H}\right) \frac{H}{2} \end{aligned}$$

Therefore,

$$A_m \cosh\left(\frac{m\pi L}{H}\right) = \frac{200}{H}\int_0^H \sin\left(\frac{m\pi y}{H}\right) dy = \frac{200}{H}\frac{1-\cos(m\pi)}{m\pi/H} = 200\frac{1-(-1)^m}{m\pi}$$

Thus $A_{2n} = 0$ and

$$A_{2n-1} = \frac{400}{\pi(2n-1)\cosh((2n-1)\pi L/H)}$$

The complete steady-state solution is therefore

$$\begin{aligned} u_E(x,y) &= \sum_{n=1}^{\infty} A_{2n-1}\cosh\left(\frac{(2n-1)\pi(L-x)}{H}\right)\sin\left(\frac{(2n-1)\pi y}{H}\right) \\ &= \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{\cosh((2n-1)\pi(L-x)/H)}{\cosh((2n-1)\pi L/H)}\sin\left(\frac{(2n-1)\pi y}{H}\right) \quad (6) \end{aligned}$$

(ii) If $H = 2L$, approximate the temperature at

(a) the hottest point on the insulated edge, and

(b) at the center of the plate (use first-term approx).

Find an upper bound for the error in (a) and (b) by getting an upper bound on the ratio of the second to the first term on the insulated edge.

Bonus: use a symmetry argument to find the exact answer for (a).

Solution: With $H = 2L$, $u_E(x,y)$ in (i) is symmetric about $y = L$, which implies that $u_{Ey}(x,L) = 0$. The hottest point on the insulated edge is, physically, that farthest away from the size at 0, i.e. at $(x,y) = (L,L)$. From (6),

$$u_E(L,L) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)\cosh((2n-1)\pi/2)}$$

This is an alternating series, whose truncation error is bounded by the magnitude of the first omitted term,

$$u_E(L,L) \approx \frac{400}{\pi\cosh(\pi/2)} \approx 50.74^o, \qquad |error| < \frac{400}{3\pi\cosh(3\pi/2)} \approx 0.76^o$$

At the center of the plate, $(x,y) = (L/2, L)$, (6) gives

$$u_E\left(\frac{L}{2}, L\right) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\frac{\cosh((2n-1)\pi/4)}{\cosh((2n-1)\pi/2)}$$

The first term approximation is

$$u_E(L,L) \approx \frac{400\cosh(\pi/4)}{\pi\cosh(\pi/2)} \approx 67.22^o, \qquad |error| < \frac{400\cosh(3\pi/4)}{3\pi\cosh(3\pi/2)} \approx 4.06^o$$

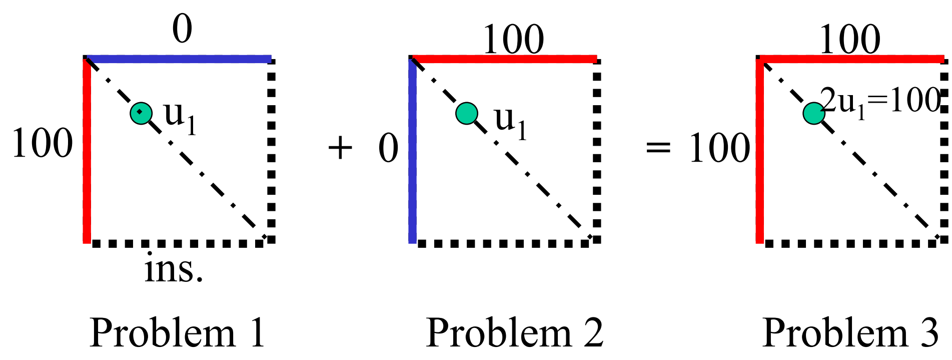

Figure 2: Illustration of symmetry argument for question 2(ii).

A two-term approximation would give

$$\begin{aligned} u_E(L,L) &\approx \frac{400}{\pi}\left[\frac{\cosh(\pi/4)}{\cosh(\pi/2)} - \frac{\cosh(3\pi/4)}{3\cosh(3\pi/2)}\right] \approx 63.16^o, \\ |error| &< \frac{400\cosh(5\pi/4)}{5\pi\cosh(5\pi/2)} \approx 0.50^o. \end{aligned}$$

The temperature of the hottest point on the insulated side $x = L$ can be found by symmetry. Since $u_{Ey} = 0$ along $y = L$ (no heat flows across this line, only parallel to it) and since u_E is symmetric about $y = L$, we need only consider the upper half of the plate. Consider the heat problems on the three plates considered in Figure 2. Plate 2 is found by rotating plate 1 about the dash-dotted diagonal. Thus, the temperature at a given point on this diagonal is the same on both plates 1 and 2. By linearity, superposing the solutions of 1 and 2 gives solution 3. Clearly, solution 3 is $u_{E3}(x,y) = 100^o$ everywhere, since the sides are either insulated or maintained at 100 degrees. Thus, the temperature along the diagonals in both plates 1 and 2 is $100/2 = 50^o$. Also, since there are no sources or sinks in the upper triangular region of plate 1 bounded by the diagonal, the insulated side $x = L$ and the top side at 0, the temperature must decrease along the side $x = L$ from $y = L$ to $y = 2L$. Hence the hottest point on the insulated side $x = L$ is at $(x,y) = (L,L)$, and since this is a point on the diagonal,

$$u_E(L,L) = 50^o.$$

(iii) Sketch typical isothermal curves and heat flow lines (the orthogonal trajectories). Where is the temperature gradient ∇u equal to zero?

Solution: Since $u_{Ex} = 0$ along $y = L$ and $u_{Ey} = 0$ along $x = L$, then $\nabla u = 0$ at $(x,y) = (L,L)$. To aid you in drawing the isothermal curves and heat flow lines, think about drawing these lines on a square of side length $2L$, whose horizontal sides are kept at 0^o and vertical sides kept at 100^o. On this square, the lines $x = L$ and

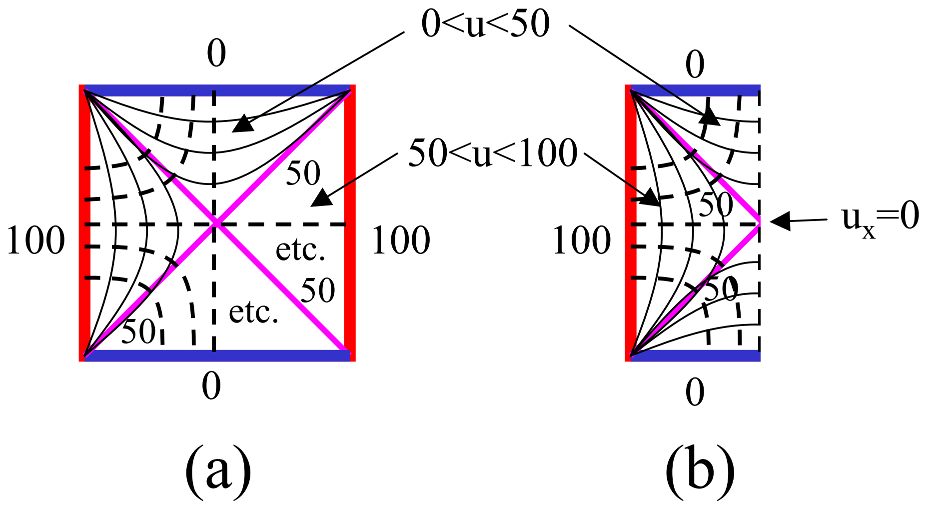

Figure 3: Level curves (solid) and heat flow lines (dashed) of $u_E(x, y)$ on (a) a square of side length $2L$ and (b) on the rectangle of problem (iii).

$y = L$ are lines of symmetry. Thus $x = L$ and $y = L$ are heat flow lines, i.e. these act as insulating boundaries. Thus our original rectangular plate is simply the left side of the $2L$ square. We have shown in (ii) that the diagonals are at 50^o. It is easiest to first draw the isothermal curves and then the heat flow lines in the upper left triangle (see Figure 3(a)). Then the desired isothermal curves on the rectangle follow (Figure 3(b)).

(iv) Consider a square plate of side length L, with one side at 100^o C, an adjacent edge at 0^o C, and the other two edges insulated. Find the steady-state temperature at points A (center of the edge opposite the side at 0^o) and B (corner joining insulated sides) without doing any more calculation, i.e., use your solution to (ii) and symmetry. Hint: lines of symmetry in the original plate (with $H = 2L$) are heat flow lines, and can effectively divide the square into smaller parts, where the lines of symmetry are insulating boundaries.

Solution: The desired plate is simply the upper half of the plate considered in parts (i) – (iii) (with $H = 2L$), since we argued that the line $y = L$ is a line of symmetry in our rectangle with $H = 2L$. Thus, the temperature at the desired points is merely the temperatures found in part (ii). At point A, the temperature is $u_E\left(L/2, L\right) \approx 63.16^o$ and at point B, the temperature is $u_E\left(L, L\right) = 50^o$.

3 Problem 3

Consider the eigenvalue problem for the Laplacian

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D, \end{aligned}$$

where $D = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3$ is a sphere of radius 1.

(i) If v is a pure radial function, $v = R(r)$, show that the eigenvalue problem reduces to

$$\frac{d^2}{dr^2}(rR) + \lambda r R = 0,$$
$$R(1) = 0, \qquad R(0) \text{ bounded}$$

Show that the radial eigen-functions and eigenvalues are

$$R_n(r) = g(n\pi r), \qquad n = 1, 2, 3, ...$$

where

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$$

Sketch the graphs of the first two eigen-functions.

You may use the fact that for $v = R(r)$, the Laplacian is

$$\nabla^2 R(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

Solution: With $v = R(r)$, the Sturm-Liouville problem becomes

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda R &= 0, \quad \text{in} \quad 0 < r < 1, \\ R &= 0, \quad \text{on} \quad r = 1. \end{aligned} \tag{7}$$

Multiplying (7) by r and rearranging gives

$$\frac{d^2}{dr^2}(rR) + \lambda r R = 0$$

Let $S(r) = rR(r)$. Then

$$\frac{d^2 S}{dr^2} + \lambda S = 0; \qquad S(0) = 0, \qquad S(1) = 0.$$

The first BC follows since $R(0)$ is bounded. If $\lambda < 0$, then

$$S = c_1 e^{-\sqrt{|\lambda|} r} + c_2 e^{\sqrt{|\lambda|} r}$$

and imposing the BCs $S = 0$ on $r = 0, 1$ gives $c_1 = c_2 = 0$. Thus $\lambda < 0$ gives a trivial solution. For $\lambda = 0$, $S = ar + b$, and imposing the BCs again gives the trivial solution. Lastly, assuming $\lambda > 0$, we have

$$S = c_3 \sin\left(\sqrt{\lambda} r\right) + c_4 \cos\left(\sqrt{\lambda} r\right)$$

Imposing the BC $S(0) = 0$ gives $c_4 = 0$, and $S(1) = 0$ implies $\sin\left(\sqrt{\lambda}\right) = 0$, or $\lambda = n^2\pi^2$ for some positive integer n. Thus

$$S_n(r) = c_n \sin(n\pi r)$$

and

$$R_n(r) = \frac{S_n(r)}{r} = \frac{\sin(n\pi r)}{n\pi r}$$

where we have chosen $c_n = 1/n\pi$. Note that the eigenfunctions are only unique up to a multiplicative constant, and when we go to form the solution to the heat or wave problem we will always multiply the n'th eigenfunction by a constant A_n. In other words, we are free to choose the constant c_n to simplify the form of the n'th eigenfunction. Lastly, note that L'Hopital's rule implies that

$$\lim_{r\to 0} R_n(r) = \lim_{r\to 0} \frac{\sin(n\pi r)}{n\pi r} = 1$$

Thus we continuously extend the function $R_n(r)$ to 1 at $r = 1$. Hence

$$R_n(r) = g(n\pi r)$$

as required. The first two eigenfunctions $R_1(r)$ and $R_2(r)$ are plotted in Figure 4.

(ii) A solid sphere of dimensionless radius 1 is heated uniformly to a temperature of u_0 degrees Celsius, placed in ice at 0^o, and allowed to cool. Show that the temperature at the center for $t > 0$ is given by

$$u(0,t) = 2u_0 \sum_{n=1}^{\infty} (-1)^{n-1} \exp\left(-n^2\pi^2 t\right)$$

Hint: use the dimensionless heat equation $u_t = \nabla^2 u$. What are the BCs? What is the IC?

Compare the central temperature with the temperature at the center of a rod of scaled length 2, and the same initial temperature. You may use the results we derived in class - you'll need to rescale the spatial coordinate x via $\hat{x} = 2x$ to make the scaled rod length 2, rather than 1.

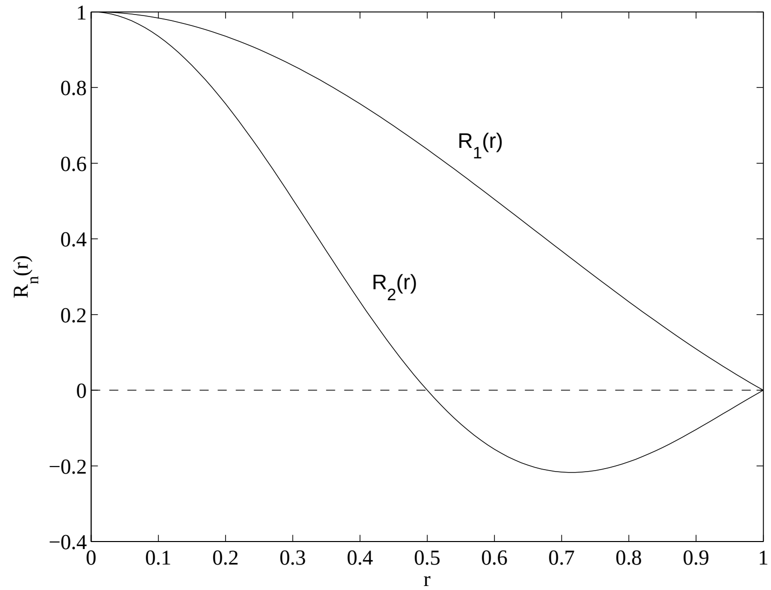

Figure 4: The eigen-functions $R_1(r)$ and $R_2(r)$.

Solution: By symmetry, the temperature distribution in the sphere is given by $u(r,t) = R(r)\,T(t)$. The heat equation $u_t = u_{xx}$ becomes

$$\frac{T'}{T} = \frac{\nabla^2 R(r)}{R}$$

Since the left hand side depends only on t, and the right hand side on r, both sides must equal a constant, $-\lambda$,

$$\frac{T'}{T} = \frac{\nabla^2 R(r)}{R} = -\lambda$$

We have shown above that

$$R = R_n(r), \qquad \lambda = \lambda_n = (n\pi)^2$$

Thus the full solution is

$$u(r,t) = \sum_{n=1}^{\infty} A_n R_n(r) \exp\left(-n^2\pi^2 t\right) = \sum_{n=1}^{\infty} A_n g(n\pi r) \exp\left(-n^2\pi^2 t\right)$$

To find the A_n's, we impose the IC,

$$u_0 = u(r,0) = \sum_{n=1}^{\infty} A_n g(n\pi r) = \sum_{n=1}^{\infty} A_n \frac{\sin(n\pi r)}{n\pi r}$$

Multiplying by $r\sin(m\pi r)$ and integrating from $r=0$ to $r=1$ gives

$$u_0 \int_0^1 r\sin(m\pi r)\,dr = \sum_{n=1}^{\infty} \frac{A_n}{n\pi} \int_0^1 \sin(n\pi r)\sin(m\pi r)\,dr = \frac{A_m}{2m\pi}$$

The second equality was obtained by orthogonality. Rearranging gives

$$
\begin{aligned}
A_m &= 2m\pi u_0 \int_0^1 r\sin(m\pi r)\,dr \\
&= \frac{2m\pi u_0}{(m\pi)^2}\left[\sin(m\pi r) - m\pi r\cos(m\pi r)\right]_0^1 \\
&= 2u_0(-1)^{m-1}
\end{aligned}
$$

Thus

$$
u(r,t) = 2u_0 \sum_{n=1}^{\infty} (-1)^{m-1} g(n\pi r)\exp\left(-n^2\pi^2 t\right)
$$

Since $g(0) = 1$, the center temperature is

$$
u(0,t) = 2u_0 \sum_{n=1}^{\infty} (-1)^{m-1} \exp\left(-n^2\pi^2 t\right)
$$

as required.

The temperature of a rod of length L, whose ends are kept at 0^o and whose initial temperature is u_0 is, in dimensional form,

$$
u'_{rod}(x',t') = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x'/L)}{(2n-1)}\exp\left(-(2n-1)^2\frac{\pi^2\kappa t'}{L}\right)
$$

For a rod of length $L = 2$, with $\kappa t' = t$ and $x = x'$, we have

$$
u_{rod}(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x/2)}{(2n-1)}\exp\left(-(2n-1)^2\frac{\pi^2 t}{2}\right)
$$

The central temperature ($x = 1$) is

$$
\begin{aligned}
u_{rod}(1,t) &= \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi/2)}{(2n-1)}\exp\left(-(2n-1)^2\frac{\pi^2 t}{2}\right) \\
&= \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\left(-(2n-1)^2\frac{\pi^2 t}{2}\right)
\end{aligned}
$$

After $t > 1/\pi^2$, the cooling of both the rod and sphere is limited by the first term approximation,

$$
u_{rod}(1,t) \approx \frac{4u_0}{\pi}\exp\left(-\frac{\pi^2 t}{2}\right), \qquad u_{sphere}(r,t) \approx 2u_0\exp\left(-\pi^2 t\right).
$$

Thus the rod cools at half the rate of the sphere.

4 Problem 4

Find the eigenvalue λ and corresponding eigen-function v for the isosceles right triangle; v and λ satisfy

$$\begin{aligned}\nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D,\\ v &= 0 \quad \text{on} \quad \partial D,\end{aligned}$$

where $D = \{(x,y) : 0 < y < x, \quad 0 < x < 1\}$.

Hint: combine the eigen-functions on the square

$$D_{sq} = \{(x,y) : 0 < x < 1, 0 < y < 1\}$$

to obtain an eigen-function on D that is positive on D. We know that the first eigen-function can be characterized (up to a non-zero multiplicative constant) as the eigen-function that is of one sign.

Solution: Let's try a linear combination of the two functions $v_{21}(x,y)$ and $v_{12}(x,y)$,

$$v_T = c_1 v_{12} + c_2 v_{21}$$

where

$$v_{mn}(x,y) = \sin(m\pi x)\sin(n\pi y).$$

Note that since $\lambda_{12} = \lambda_{21}$, both satisfy the Sturm-Liouville problem

$$\nabla^2 v + \lambda_{12} v = 0 \quad \text{in} \quad D_{sq}; \quad v = 0 \quad \text{on} \quad \partial D_{sq}. \tag{8}$$

Thus, by linearity, v_T also satisfies (8). We have

$$\begin{aligned} v_T(x,y) &= c_1 \sin(\pi x)\sin(2\pi y) + c_2 \sin(2\pi x)\sin(\pi y)\\ &= c_1 \sin(\pi x)\, 2\sin(\pi y)\cos(\pi y) + c_2 2\sin(\pi x)\cos(\pi x)\sin(\pi y)\\ &= 2\sin(\pi x)\sin(\pi y)\left(c_1\cos(\pi y) + c_2\cos(\pi x)\right) \end{aligned} \tag{9}$$

For any c_1, c_2, $v_T(x,y) = 0$ along the horizontal and vertical sides, $x = 0,1$ and $y = 0,1$. This must happen since each v_{mn} is zero on the boundary of the square. For v_T to vanish along the diagonal of the triangle, we must also have $v_T(x,x) = 0$, so that (9)

$$0 = v_T(x,x) = 2(c_1 + c_2)\sin^2(\pi x)\cos(\pi x)$$

Hence, $c_2 = -c_1$. Substituting this into (9) gives

$$v_T(x,y) = 2c_1 \sin(\pi x)\sin(\pi y)\left(\cos(\pi y) - \cos(\pi x)\right)$$

Thus $v_T = 0$ on the boundary ∂D of the triangle, and since it satisfies (8) on the square, then v_T is an eigenfunction of the Sturm-Liouville problem on the triangle,

$$\nabla^2 v + \lambda_{12} v = 0 \quad \text{in} \quad D; \quad v = 0 \quad \text{on} \quad \partial D. \tag{10}$$

However, we need $v_T > 0$ in D. This amounts to setting the sign of c_1. Note that on D, $0 < y < x < 1$, and hence

$$\cos(\pi y) > \cos(\pi x).$$

Hence we take $c_1 = 1$ (recall we only need to find v_T up to positive multiplicative constant) so that

$$v_T = v_{21} - v_{12} = 2\sin(\pi x)\sin(\pi y)(\cos(\pi y) - \cos(\pi x))$$

satisfies the Sturm-Liouville problem (10) on the triangle and is positive on the interior of the triangle. We asserted in class that the eigenvalue associated with an eigenfunction of these properties is in fact the smallest eigenvalue,

$$\lambda_{21} = \lambda_{12} = 5\pi^2$$

on the triangle $D = \{(x, y) : 0 < y < x, \quad 0 < x < 1\}$.

5 Problem 5

Consider the boundary value problem

$$\begin{aligned} \nabla^2 v &= 0, \qquad 0 < x < 1, \qquad 0 < y < 1 \\ v(0, y) &= 0, \qquad v(1, y) = 100, \qquad 0 < y < 1 \\ v(x, 0) &= 100, \qquad v(x, 1) = 0, \qquad 0 < x < 1. \end{aligned} \tag{11}$$

Give a symmetry argument to show that $v(x, x) = 50$ for $0 < x < 1$. Sketch the level curves of v.

Solution: The symmetry argument is illustrated in Figure 5. Problem 1 represents the boundary value problem (11). Problem 2 is simply problem 1 rotated by π (180 degrees). Let v_1, v_2, v_3 be the solutions to Laplace's equation with BCs illustrated in problems 1, 2, 3 in Figure 5. Then by symmetry (see Figure 5), $v_1(x, x) = v_2(x, x)$. Problem 3 is the superposition of Problems 1 and 2, thus

$$v_3(x, x) = v_1(x, x) + v_2(x, x) = 2v_1(x, x) \tag{12}$$

Clearly, though, the solution to Problem 3 is $v_3 = 100$. Thus by (12), $v_1(x, x) = 50$. The isotherms (level curves) are shown in Figure 6.

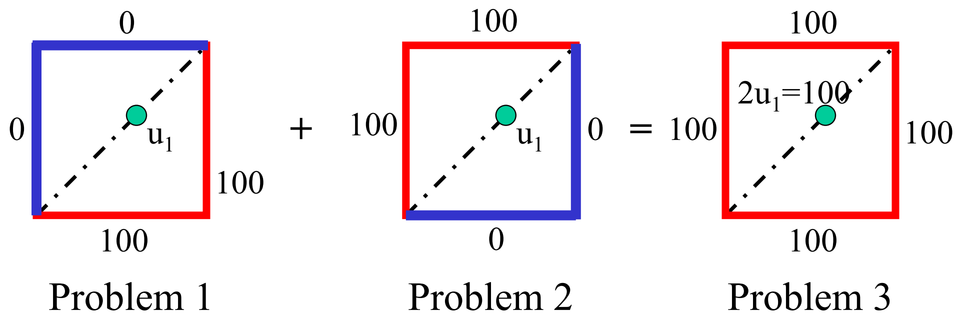

Figure 5: Illustration of symmetry argument for question 5.

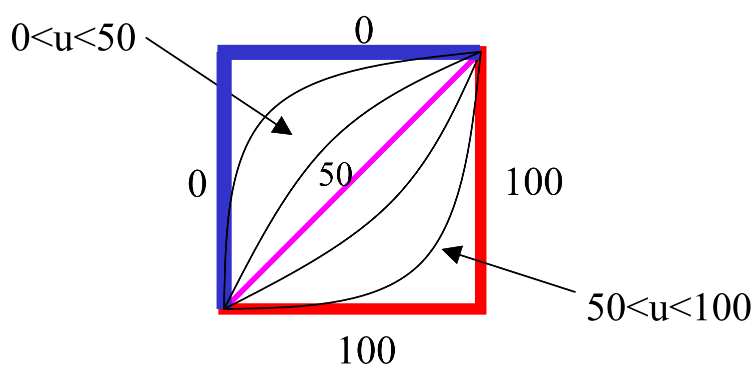

Figure 6: Isotherms (level curves of temperature) for square in question 5.