Problems for 1- D Heat Equation

3 Problem 4

Solve the inhomogeneous heat problem with type I boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = 0 = u(1,t); \qquad u(x,0) = P_\varepsilon(x)$$

where $t > 0$, $0 \leq x \leq 1$, and

$$P_\varepsilon(x) = \begin{cases} 0 & \text{if } \left|x - \frac{1}{2}\right| > \frac{\varepsilon}{2} \\ \frac{u_0}{\varepsilon} & \text{if } \left|x - \frac{1}{2}\right| \leq \frac{\varepsilon}{2} \end{cases} \tag{5}$$

Note: you already know the solution (just replace $P_\varepsilon(x)$ with $f(x)$ and write down the solution from class). Using symmetry of $P_\varepsilon(x)$ about $1/2$ can be used to simplify the calculation of the Fourier coefficients.

(a) The temperature at the midpoint of the rod, $x = 1/2$, at scaled time $t = 1/\pi^2$ is, from (6) and (8),

(b) Illustrate the solution qualitatively by sketching (i) some typical temperature profiles in the $u - t$ plane (i.e. x = constant) and in the $u - x$ plane (i.e. t = constant), and (ii) some typical level curves $u(x,t)$ = constant in the $x - t$ plane. At what points of the set $D = \{(x,t) : 0 \leq x \leq 1, t \geq 0\}$ is $u(x,t)$ discontinuous?

4 Problem 5

Consider two iron rods (thermal diffusivity $\kappa = 0.15$ cm^2 sec^{-1}) each 20 cm long and with insulated sides, one at a temperature of 100^oC and the other at 0^oC throughout. The rods are joined end to end in perfect thermal contact, and their free ends are kept at 0^oC. Show that the temperature at the interface 10 minutes after contact has been made approximately 36.5^oC. Find an upper bound for the error in your answer. Can this method be applied if the rods are made of glass (thermal diffusivity $\kappa = 0.006$ cm^2 sec^{-1})?

5 Problem 7

Consider the heat flow problem with dimensionless position and time,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}; \qquad 0 < x < 1, \qquad t > 0 \\
u(0,t) &= 0 = \frac{\partial u}{\partial x}(1,t); \qquad t > 0 \\
u(x,0) &= f(x) \qquad 0 < x < 1.
\end{aligned} \tag{16}$$

6 Problem 8

Suppose a chemical is dissolved in water, in some long thin reaction container and let ϕ (moles/cm^3) indicate its concentration. Fick's Law in chemistry states that the rate of diffusion of a solute is proportional to the negative gradient of the solute concentration. Assume that the chemical is created, due to a chemical reaction, at a rate $g(x,t)$ (moles/cm^3 sec).

(a) Derive a PDE describing the distribution of ϕ. Formulate appropriate BCs and IC and state all assumptions.

(b) Show that the solution to the initial boundary value problem derived in (a) is unique.